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On the Solution of a Mixed Problem for Pivots Oscillation Equation with Convolution

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Abstract. A mixed problem for a pivots oscillation equation containing a fractional time derivative of order $1 < \alpha < 2$ is considered. The solution is sought in the form of expansion in eigen functions of the corresponding spectral problem and the Cauchy problem is obtained for a fractional derivative equation that is solved by using the Laplace transform.

1. Introduction

Consider a differential equation of the form:

$$D_{0t}^{\alpha}u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} = \Phi(t) * \frac{\partial^{2r} u(x,t)}{\partial x^{2r}}, \tag{1}$$

where $x \in (0,1),\ t > 0,\ r = 0,1$ and

$$\Phi(t) = \begin{cases} t^{-\beta} \quad for\ \ t > 0,\ \ 0 < \beta < 1 \\ 0 \quad for\ \ t < 0. \end{cases}$$

The right hand side of equation (1) is the convolution of the functions $\Phi(t)$ and $\frac{\partial^{2r} u(x,t)}{\partial x^{2r}}$, $1 < \alpha < 2$, $D_{0t}^{\alpha}u(x,t)$ is a fractional derivative of the function $u(x,t)$ in the Riemann-Liouville sense and of order α, determined by the formula:

$$D_{ot}^{\alpha}f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t f(\tau)(t-\tau)^{1-\alpha} d\tau.$$

Set the initial and boundary conditions:

$$\lim_{t \to +0} D_{0t}^{\alpha-1} u(x,t) = \varphi_0(x),$$

$$\lim_{t \to +0} D_{0t}^{\alpha-2} u(x,t) = \varphi_1(x),\ 0 < x < 1 \tag{2}$$

and

$$u^k(0,t) = u^{(k)}(1,t) = 0, \quad k = 0,2, \quad t > 0. \tag{3}$$

The functions $\varphi_0(x)$, $\varphi_1(x)$ are still considered continuous in (0,1). The additional conditions of these functions are determined on grounding the formula obtained for the solution of the stated problem.

Note that the equations with convolution for hyperbolic equations were considered in various papers. The fundamental solution of wave equations with convolution that are called the memory equations, were studied in [4].

In [5], an initial-boundary value problem for diffusion equation with fractional derivative of order $0 < \alpha < 1$ and a bounded function $\Phi(t)$ was considered. Note that the above stated problems are solved by using the Laplace transform.

While solving problem (1)-(3) we use the indicated transformation. So, we look for the solution of the stated problem in the form:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \pi n x, \tag{4}$$

where $T_n(t)$is a still unknown function.

Substituting (4) in equality (1)-(3), for the function $T_n(t)$ we get the relations:

$$T_n^{\alpha}(t) + (\pi n)^4 T_n(t) = \Phi(t) * (\pi n)^{2r} T_n(t) \tag{5}$$

$$\begin{aligned} \lim_{t \to +0} T_n^{(\alpha-1)}(t) = \varphi_{0n}, \\ \lim_{t \to +0} T_n^{(\alpha-2)}(t) = \varphi_{1n}, \end{aligned} \tag{6}$$

here $\varphi_{kn} = 2\int_0^1 \varphi_k(x) \sin \pi n x dx, \quad k = 0,1$. Thus, we obtain the Cauchy problem for the unknown function $T_n(t)$ that may be solved by using the Laplace transform. If is known that the Laplace transform with fractional derivative is determined by the formula [1], [2]:

$$\int_0^{\infty} e^{-st} D_{0t}^{\alpha} f(t) dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \left[D_{0t}^{\alpha-k-1} f(t) \right]_{t=0}, \tag{7}$$

here $F(s) = \int_0^{\infty} e^{-st} f(t) dt$.

Thus, by applying formula (7), from (5) and (6) we get an algebraic equation of the form:

$$s^{\alpha} \tilde{T}_n(s) + (\pi n)^4 \tilde{T}_n(s) = (\pi n)^{2r} \tilde{T}_n(s) \cdot \tilde{\Phi}(s) + \varphi_{0n} + \varphi_{1n} s. \tag{8}$$

Here $\tilde{T}_n(s)$ and $\tilde{\Phi}(s)$ are the Laplace transforms of the functions $T_n(t)$ and $\Phi(t)$, respectively, moreover $\tilde{\Phi}(s) = \Gamma(1-\beta) s^{\beta-1}, \quad \Gamma(z)$ is the Euler's gamma function.

We rewrite equality (8) in the form:

$$\tilde{T}_n(s) = \frac{\varphi_{0n} + s\varphi_{1n}}{s^\alpha + (\pi n)^4 - (\pi n)^{2r}\tilde{\Phi}(s)} = \frac{\varphi_{0n} + s\varphi_{1n}}{s^\alpha + (\pi n)^4} \cdot \frac{1}{1 - \frac{(\pi n)^{2r}\tilde{\Phi}(s)}{s^\alpha + (\pi n)^4}}. \tag{9}$$

Taking into account that $\tilde{\Phi}(s) = \Gamma(1-\beta)s^{\beta-1}$, $0 < \beta < 1$, $0 < \alpha < 2$, $r = 0, 1$, we deduce

$$\left| \frac{(\pi n)^{2r}\tilde{\Phi}(s)}{s^\alpha + (\pi n)^4} \right| < 1.$$

Thus, we can treat the second multiplier in (9) as a sum of infinite geometrical progression with denominator $q = \frac{(\pi n)^{2r}\tilde{\Phi}(s)}{s^\alpha + (\pi n)^4}$. Allowing for what has been said, we can represent formula (8) in the form of a series:

$$\tilde{T}_n(s) = \frac{\varphi_{0n}}{s^\alpha + (\pi n)^4} \cdot \sum_{p=0}^{\infty} \left(\frac{(\pi n)^{2r}\tilde{\Phi}(s)}{s^\alpha + (\pi n)^4} \right)^p + \frac{s\varphi_{1n}}{s^\alpha + (\pi n)^4} \cdot \sum_{p=0}^{\infty} \left(\frac{(\pi n)^{2r}\tilde{\Phi}(s)}{s^\alpha + (\pi n)^4} \right)^p =$$

$$= \varphi_{0n} \sum_{p=0}^{\infty} \frac{(\pi n)^{2rp}\Gamma^p(1-\beta)s^{(\beta-1)p}}{(s^\alpha + (\pi n)^4)^{p+1}} + \varphi_{1n} \sum_{p=0}^{\infty} \frac{(\pi n)^{2rp}\Gamma^p(1-\beta)s^{(\beta-1)p+1}}{(s^\alpha + (\pi n)^4)^{p+1}} =$$

$$= \tilde{T}_{n0}(s) + \tilde{T}_{n1}(s).$$

Now calculate the inverse transformation of each addend separately. This time we'll take into account the known formula [1]:

$$\int_0^\infty e^{st}(t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm a t^\alpha))dt = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \tag{10}$$

where a is a real parameter, $E_{\alpha,\beta}(z)$ is a Mittag-Leffler function determined by the formula:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Here we accept the denotation:

$$E_{\alpha,\beta}^{(k)}(z) = \frac{d^k}{dz^k} E_{\alpha,\beta}(z). \tag{11}$$

By applying formula (10), we see that the following inverse transformation holds:

$$L^{-1}\left(\frac{s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}} \right) = \frac{1}{k!} t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)} \cdot (\pm a t^\alpha).$$

In our case $a = (\pi\, n)^4$, and therefore

$$L^{-1}\left(\frac{s^{(\beta-1)p}}{(s^{\alpha}+(\pi\, n)^4)^{p+1}}\right)=L^{-1}\left(\frac{s^{\alpha-(\alpha-(\beta-1)p)}}{(s^{\alpha}+(\pi\, n)^4)^{p+1}}\right)=$$

$$=\frac{1}{p!}t^{\alpha\, p+(\alpha-(\beta-1)p)-1}E^{(p)}_{\alpha,\alpha-(\beta-1)p}(-(\pi\, n)^4t^{\alpha}).$$

Taking this into account, for $T_{n1}(t)$ we get the representation:

$$T_{n0}(t)=\varphi_{0n}\cdot\sum_{p=0}^{\infty}\frac{(\pi\, n)^{2rp}\Gamma^p(1-\beta)}{p!}t^{\alpha\, p+(\alpha-(\beta-1)p)-1}\times$$

$$\times E^{(p)}_{\alpha,\alpha-(\beta-1)p}(-(\pi\, n)^4t^{\alpha}).$$

In the similar way:

$$T_{n1}(t)=\varphi_{1n}\cdot\sum_{p=0}^{\infty}\frac{(\pi\, n)^{2rp}\Gamma^p(1-\beta)}{p!}t^{\alpha\, p+(\alpha-(\beta-1)p)-2}\times$$

$$\times E^{(p)}_{\alpha,\alpha-(\beta-1)p-1}(-(\pi\, n)^4t^{\alpha}).$$

The sum of the last formulas defines the formal solution of the stated problem in the form:

$$u(x,t)=\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{(\pi\, n)^{2rp}\Gamma^p(1-\beta)}{p!}t^{\alpha(p+1)-(\beta-1)p-2}\times$$

$$\times(\varphi_{0n}tE^{(p)}_{\alpha,\alpha-(\beta-1)p}(-(\pi\, n)^4t^{\alpha})+\varphi_{1n}E^{(p)}_{\alpha,\alpha-(\beta-1)p-1}(-(\pi\, n)^4t^{\alpha}))=$$
$$=\textstyle\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{(\pi\, n)^{2rp}\Gamma^p(1-\beta)}{p!}t^{\alpha(p+1)-(\beta-1)p-2}\times$$
$$\times\left(2\left(\textstyle\int_0^1\varphi_0(x)\sin\pi\, nxdx\right)\cdot tE^{(p)}_{\alpha,\alpha-(\beta-1)p}(-(\pi\, n)^4t^{\alpha})+\right.$$

$$\left.+\,2\left(\int_0^1\varphi_1(x)\sin\pi\, nxdx\right)\cdot E^{(p)}_{\alpha,\alpha-(\beta-1)p-1}(-(\pi\, n)^4t^{\alpha})\right).\qquad(12)$$

Note that formula (12) defining the formal solution of problem (1)-(3) contains the derivative of the Mittag-Leffler function of order p, that may by get rid of.

We do it in the following way. In formula (11) we substitute $y=-(\pi\, n)^4t^{\alpha}$ and pass from differentiation with respect to t^{α} to the variable t.

Taking into account

$$\frac{d}{dt^{\alpha}}=\frac{t^{1-\alpha}}{\alpha}\frac{d}{dt},$$

we get that for any k times differentiable function we can write the representation:

$$\frac{d^k f(t)}{d(t^\alpha)^k} = \sum_{i=1}^{k} P_i(\alpha,t)\frac{d^i f(t)}{dt^i}, \tag{13}$$

where $P_i(\alpha,t)$ are some functions from α and t. Further, by the direct calculation we get

$$\frac{d^i f(t)}{dt^i} = \sum_{j=0}^{i} C_j(i,\beta)t^{1-\beta-j}\frac{d^{i-j}}{dt^{i-j}}(t^{\beta-1}f(t)), \tag{14}$$

here $C_j(i,\beta)$ are determined coefficients, moreover $C_0(i,\beta) = 1$.

Now, having taken in formula (13) and (14) $f(t) = E_{\alpha,\beta}$ and using the known formula [1]

$$\frac{d^m}{dt^m}(t^{\beta-1}E_{\alpha,\beta}(t^\alpha)) = t^{\beta-m-1}E_{\alpha,\beta-m}(t^\alpha),$$

in representation (12) we can pass from arbitrary $E^{(p)}_{\alpha,\beta}$ to $E_{\alpha,\beta-p}$ with some determined coefficients.

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