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# On Fredholm property of boundary problem

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**Abstract.** In this paper we consider the boundary problem for third-order equation of composite type, i.e. in considered domain equation has both the real and complex characteristics. The presence of real characteristics is the cause of incorrectness of corresponding boundary problem.

**Key Words and Phrases**: Equations of composite type, nonlocal boundary conditions, necessary conditions, singularity, regularization, Fredholm property.

## 1. Introduction

As known, if we consider the problem for linear ordinary differential equation, regardless of whether we study the Cauchy problem and boundary problem, the number of conditions (initial or boundary) coincides with the highest order derivative included in the equation in [1]. If we consider the partial differential equation, the number of initial conditions coincides with the highest order derivative with respect to time included in the equation, and the number of boundary conditions in the general case (for an arbitrary domain, when the number of spatial variables is greater than one) is equal to half the highest-order derivative with respect to the spatial variable [2]. For the Laplace equation is given one condition (Dirichlet, Neumann or Poincare) for the disharmonic equation (fourth-order equation) two conditions, etc.

If we consider the boundary value problem for linear differential equations with partial derivatives of odd order, then the following question appears: how many boundary conditions must be given? When the process in a nuclear reactor, a mathematical model, which is a linear integral-differential equation of first order in three dimensions, boundary conditions are set at half of the boundary [3]. In connection with Trikomy problem on one of the seminars of the Institute of Mathematics on the name of Steklov Bitsadze A.V. said that such problems are not well placed. According to Bitsadze, the boundary conditions for the entire boundary must be a carrier. In this regard, consider the nonlocal boundary conditions. Nonlocal boundary condition is primarily eliminated, then a misunderstanding which was in boundary problem for linear ordinary differential equations with partial

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derivatives [4], [5]. As for the nonlocal boundary conditions moves on at least two points, then that condition must be complied with Carleman [6]. In this paper we investigate the conditions of the following Fredholm boundary problem:

$$\frac{\partial^3 U(x)}{\partial x_1^3}+\frac{\partial^3 U(x)}{\partial x_2^3}, \quad x\in D, \tag{1}$$

$$\begin{cases} \sum_{k=1}^{8}\left[\alpha_{i,3k-2}(x_1)\,U(t)+\alpha_{i,3k-1}(x_1)\frac{\partial U(t)}{\partial t_1}+\alpha_{i,3k}(x_1)\frac{\partial U(t)}{\partial t_2}\right], \\ t_1 = M_k(x_1)=\varphi_i(x_1)\,, t_2=N_k(x_1) \\ \sum_{k=1}^{8}\alpha_{j+8,k}(x_1)\left(\frac{\partial^2 U(x)}{\partial t_1^2}-\frac{\partial^2 U(x)}{\partial t_1\partial t_2}+\frac{\partial^2 U(x)}{\partial t_2^2}\right)=\varphi_{j+8}(x_1)\,, t_1=M_k(x_1)\,; t_2=N_k(x_1) \end{cases} \tag{2}$$

Where

$$D=\left\{(x_1,x_2): x_1^2+x_2^2<r^2\right\},\ \ i=\overline{1,8},\ \ j=\overline{1,4}$$

$$M_1(x_1)=x_1, \quad N_1(x_1)=\sqrt{r^2-x_1^2},$$

$$M_2(x_1)=\sqrt{r^2-x_1^2}, N_2(x_1)=x_{.1},$$

$$M_2(x_1)=-\sqrt{r^2-x_1^2}, N_3(x_1)=x_1$$

$$M_4(x_1)=-x_1,\ \ N_4(x_1)=\sqrt{r^2-x_1^2},$$

$$M_5(x_1)=-x_1,\ \ N_5(x_1)=-\sqrt{r^2-x_1^2},$$

$$M_6(x_1)=-\sqrt{r^2-x_1^2},\ \ N_6(x_1)=-x_1,$$

$$M_7(x_1)=\sqrt{r^2-x_1^2},\ \ N_7(x_1)=-x_1,$$

$$M_8(x_1)=x_1, N_8(x_1)=\sqrt{r^2-x_1^2},\ \ x_1\in\left(r/\sqrt{2},r\right).$$

Here the point $(M_j,N_j)$ are the relevant parts $\Gamma_j$ of the boundary $\Gamma, j=\overline{1,8}\left(\Gamma=U_{j=1}^{8}\Gamma_j\right)$. We factorize equation (1) as follows:

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\partial^2}{\partial x_2^2}\right)U(x)=0$$

Then the problem (1), (2) splits into two problems:

$$\frac{\partial^2 U(x)}{\partial x_1^2}-\frac{\partial^2 U(x)}{\partial x_1\partial x_2}+\frac{\partial^2 U(x)}{\partial x_2^2}=U_1(x)\,,\ x\in D \tag{3}$$

$$\sum_{k=1}^{8}\left[\alpha_{i,3k-2}(x_1)U(t)+\alpha_{i,3k-1}(x_1)\frac{\partial U(t)}{\partial t_1}+\alpha_{i,3k}(x_1)\frac{\partial U(t)}{\partial t_2}\right]=\varphi_i(x_1),$$

$$t_1=M_k(x_1),t_2=N_k(x_1);\ \ i=\overline{1,8},\ \ x_1\in\left(r/\sqrt{2},r\right) \tag{3'}$$

$$\frac{\partial U_1(x)}{\partial x_1}+\frac{\partial U_1(x)}{\partial x_2}=0,\quad x\in D \tag{4}$$

$$\sum_{k=1}^{8}\alpha_{j+8,k}(x_1)U_1(t)=\varphi_{i+8}(x_1),$$

$$t_1=M_k(x_1),t_2=N_k(x_1);\ i=\overline{1,8},\ \ x_1\in\left(r/\sqrt{2},r\right) \tag{4'}$$

First solve the problems $(4)$, $\left(4'\right)$ and thus obtained a solution, we substitute the right-hand side of (3), and then prove the Fredholm property of problems $(3)$, $\left(3'\right)$. According to the scheme in [3], we can easily construct a fundamental solution of equation (4), which has the form:

$$\delta_1(x-\xi)=e(x_2-\xi_2)\,\delta(x_1-\xi_1-(x_2-\xi_2)) \tag{5}$$

Where

$$e(z)=\begin{cases}\frac{1}{2}, & z>0\\ 0, & z=0\\ -\frac{1}{2}, & z<0\end{cases}$$ - Heaviside-function ; $\delta(z)$- Dirac $\delta$ - function.

Multiplying equation (4) to (5), integrating it over D and applying Green's second formula, we obtain:

$$0=\int_D\left(\frac{\partial U_1(x)}{\partial x_1}+\frac{\partial U_1(x)}{\partial x_2}\right)\varepsilon_1(x-\xi)\,dx=$$

$$=\int_\Gamma U_1(x)\,\varepsilon_1(x-\xi)\cos\left(n^\wedge x_1\right)dl-\int_D U_1(x)\frac{\partial\varepsilon_1(x-\xi)}{\partial x_1}dx+$$

Where n-the outward normal in boundary oblast D, dl an element of . If we are using the definition of the fundamental solution, we obtain the following result:

**Theorem 1.** *Any solution of equation (4) defined in D, and its boundary values satisfy the following relations:*

$$\int_\Gamma U_1(x)\,\varepsilon_1(x-\xi)\left(\cos\left(n^\wedge x_1\right)+\cos\left(n^\wedge x_2\right)\right)dl=\begin{cases}U_1(\xi), & \xi\in D;\\ \frac{1}{2}U_1(\xi), & \xi\in\Gamma.\end{cases} \tag{6}$$

Hence, for the boundary values are obtained the following conditions:

$$U_1\left(\xi_1,\sqrt{r^2-\xi_1^2}\right)=U_1\left(-\sqrt{r^2-\xi_1^2},-\xi_1\right)$$

$$U_1\left(\sqrt{r^2-\xi_1^2},\xi_1\right)=U_1\left(-\xi_1,-\sqrt{r^2-\xi_1^2}\right)$$

$$U_1\left(-\xi_1,\sqrt{r^2-\xi_1^2}\right)=U_1\left(-\sqrt{r^2-\xi_1^2},\xi_1\right)$$
$$U_1\left(\xi_1,-\sqrt{r^2-\xi_1^2}\right)=U_1\left(\sqrt{r^2-\xi_1^2},-\xi_1\right) \qquad (7)$$
$$\xi_1\in\left(r/\sqrt{2},r\right)$$

These relationships take into account in $\left(4^{'}\right)$:

$$\left(\alpha_{j+8,1}\left(x_1\right)+\alpha_{j+8,6}\left(x_1\right)\right)U_1\left(x_1,\sqrt{r^2-x_1^2}\right)+$$
$$+\left(\alpha_{j+8,5}\left(x_1\right)+\alpha_{j+8,5}\left(x_1\right)\right)U_1\left(\sqrt{r^2-x_1^2},x_1\right)+$$
$$+\left(\alpha_{j+8,3}\left(x_1\right)+\alpha_{j+8,4}\left(x_1\right)\right)U_1\left(-\sqrt{r^2-x_1^2},x_1\right)+$$
$$+\left(\alpha_{j+8,7}\left(x_1\right)+\alpha_{j+8,8}\left(x_1\right)\right)U_1\left(x_1,-\sqrt{r^2-x_1^2}\right)=$$

$$=\varphi_{j+8}\left(x_1\right)+\left(\alpha_{j+8,7}\left(x_1\right)+\alpha_{j+8,8}\left(x_1\right)\right)U_1\left(x_1,-\sqrt{r^2-x_1^2}\right)=\varphi_{j+8}\left(x_1\right) \qquad (4')$$

Thus, we get a system of linear algebraic equations. Let satisfies the following condition:

$$\begin{vmatrix} \alpha_{9,1}+\alpha_{9,6} & \alpha_{9,2}+\alpha_{9,5} & \alpha_{9,3}+\alpha_{9,4} & \alpha_{9,7}+\alpha_{9,8} \\ \alpha_{10,1}+\alpha_{10,6} & \alpha_{10,2}+\alpha_{10,5} & \alpha_{10,3}+\alpha_{10,4} & \alpha_{10,7}+\alpha_{10,8} \\ \alpha_{11,1}+\alpha_{11,6} & \alpha_{11,2}+\alpha_{11,5} & \alpha_{11,3}+\alpha_{11,4} & \alpha_{11,7}+\alpha_{11,8} \\ \alpha_{12,1}+\alpha_{12,6} & \alpha_{12,2}+\alpha_{12,5} & \alpha_{12,3}+\alpha_{12,4} & \alpha_{12,7}+\alpha_{12,8} \end{vmatrix}\neq 0 \qquad (8)$$

Then we have

**Theorem 2.** *Suppose that the functions* $\alpha_{i,j}\left(x_1\right),\varphi_i\left(x_1\right)\left(i=\overline{9.12},j=\overline{1,8}\right)$ *are continuous in the interval* $\left(r/\sqrt{2},r\right)$ *and satisfies the condition (8). Then the problem (4),* $\left(4^{'}\right)$ *allow, and its solution has an explicit form.*

**Note.** The solution of (4), $\left(4^{'}\right)$ in an explicit form as follows:We solve the system of linear algebraic equations $\left(4^{'}\right)$ and the result is substituted into (6). This solution can be written in the right-hand side of (3) and prove the Fredholm property of problems $(3)\,,\left(3^{'}\right)$. Here we proceed as well as for the solution of (4), $\left(4^{'}\right)$ that is, first build the fundamental solution of the equation and then calculate.

Necessary condition: The fundamental solution of equation (3) has the following form:

$$\varepsilon(x-\xi) = -\frac{1}{2\pi}\ln\sqrt{\frac{4}{3}\left[(x_1-\xi_1)^2+(x_1-\xi_1)(x_2-\xi_2)+(x_2-\xi_2)^2\right]}, \tag{9}$$

From which the derivatives have:

$$\frac{\partial\varepsilon(x-\xi)}{\partial x_1} = -\frac{1}{4\pi}\frac{2(x_1-\xi_1)+(x_2-\xi_2)}{(x_1-\xi_1)^2+(x_1-\xi_1)(x_2-\xi_2)+(x_2-\xi_2)^2}, \tag{10}$$

We have the following

**Theorem 3.** *Every solution of equation (3) defined in D, its derivatives and their boundary values satisfy the following relations:*

$$\int_\Gamma U(x)\left[\frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_1) - \frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_2) + \frac{\partial\varepsilon(x-\xi)}{\partial x_2}\cos(n\hat{}x_2)\right]dl-$$
$$-\int_\Gamma \frac{\partial U(x)}{\partial x_1}\varepsilon(x-\xi)\cos(n\hat{}x_1)\,dl+$$
$$+\int_\Gamma \frac{\partial U(x)}{\partial x_1}\varepsilon(x-\xi)\left(\cos(n\hat{}x_1)-\cos(n\hat{}x_2)\right)dl+$$
$$+\int_D U_1(x)\,\varepsilon(x-\xi)\,dx = \begin{cases} U(\xi), & \xi\in D;\\ \frac{1}{2}U(\xi), & \xi\in\Gamma.\end{cases} \tag{12}$$

$$\int_\Gamma \frac{\partial U(x)}{\partial x_1}\left[\frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_1) - \frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_2) + \frac{\partial\varepsilon(x-\xi)}{\partial x_2}\cos(n\hat{}x_2)\right]dl+$$
$$+\int_\Gamma \frac{\partial U(x)}{\partial x_2}\left[\frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_2) - \frac{\partial\varepsilon(x-\xi)}{\partial x_2}\cos(n\hat{}x_1)\right]dl-$$
$$-\int_D U_1(x)\frac{\partial\varepsilon(x-\xi)}{\partial x_1}dx = \begin{cases} \frac{\partial U(\xi)}{\partial\xi_1}, & \xi\in D;\\ \frac{1}{2}\frac{\partial U(\xi)}{\partial\xi_1}, & \xi\in\Gamma.\end{cases} \tag{13}$$

$$\int_\Gamma \frac{\partial U(x)}{\partial x_1}\left[\frac{\partial\varepsilon(x-\xi)}{\partial x_2}\cos(n\hat{}x_1) - \frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_2)\right]dl+$$

$$+\int_\Gamma \frac{\partial U(x)}{\partial x_2}\left[\frac{\partial\varepsilon(x-\xi)}{\partial x_1}\cos(n\hat{}x_1) - \frac{\partial\varepsilon(x-\xi)}{\partial x_2}\cos(n\hat{}x_1) + \frac{\partial\varepsilon(x-\xi)}{\partial x_2}\cos(n\hat{}x_2)\right]dl-$$
$$-\int_D U_1(x)\frac{\partial\varepsilon(x-\xi)}{\partial x_2}dx = \begin{cases} \frac{\partial U(\xi)}{\partial\xi_2}, & \xi\in D;\\ \frac{1}{2}\frac{\partial U(\xi)}{\partial\xi_2}, & \xi\in\Gamma.\end{cases} \tag{14}$$

From the expressions (10) and (11) we see that in (12) -(14), which are derivatives of the fundamental solutions, when $\xi$ and $x$ be in one part of the $Gamma_k$, boundary having a singularity. Since we are interested only in the singular integrals (regular integrals will mark the dots):

$$U\left((-1)^m \xi_1, (-1)^n \sqrt{r^2-\xi_1^2}\right) =$$

$$= (-1)^n (-1)^m \frac{1}{2\pi} \int_{r/\sqrt{2}}^{r} \frac{U\left((-1)^m x_1, (-1)^n \sqrt{r^2-x_1^2}\right)}{x_1-\xi_1} dx_1 + \ldots m, n = 0, 1. \quad (15)$$

$$U\left((-1)^m \sqrt{r^2-\xi_1^2}, (-1)^n \xi_1\right) =$$

$$= (-1)^{m+n} \frac{1}{2\pi} \int_{r/\sqrt{2}}^{r} \frac{U\left((-1)^m \sqrt{r^2-x_1^2}, (-1)^n x_1\right)}{x_1-\xi_1} dx_1 + \ldots m, n = 0, 1 \quad (16)$$

$$\frac{\partial U(\tau)}{\partial \tau_1}\Big|_{\tau_1=(-1)^m \xi_1; \tau_2=(-1)^n \sqrt{r^2-\xi_1^2};} =$$

$$= (-1)^{m+n} \int_{r/\sqrt{2}}^{r} \frac{dx_1}{x_1-\xi_1} \left[\frac{\partial U(y)}{\partial y_1} - 2\frac{\partial U(y)}{\partial y_2}\right]\Big|_{\frac{dx_1}{x_1-\xi_1}} + \cdots$$

$$y_2 = (-1)^n \sqrt{r^2-x_1^2}, y_1 = (-1)^m x_1; \quad m, n = 0, 1. \quad (17)$$

$$\frac{\partial U(\tau)}{\partial \tau_1}\Big|_{\tau_1=(-1)^n \sqrt{r^2-\xi_1^2}; \tau_2=(-1)^m \xi_1} =$$

$$= \frac{(-1)^{m+n}}{2\pi} \int_{r/\sqrt{2}}^{r} \frac{dx_1}{x_1-\xi_1} \left[\frac{\partial U(y)}{\partial y_1} - 2\frac{\partial U(y)}{\partial y_2}\right]\Big|_{\frac{dx_1}{x_1-\xi_1}} + \ldots$$

.

$$y_1 = (-1)^n \sqrt{r^2-x_1^2}, y_2 = (-1)^m x_1; \quad m, n = 0, 1. \quad (18)$$

$$\frac{\partial U(\tau)}{\partial \tau_1}\Big|_{\tau_1=(-1)^m \xi_1; \tau_2=(-1)^n \sqrt{r^2-\xi_1^2};} =$$

$$= \frac{(-1)^{m+n}}{2\pi} \int_{r/\sqrt{2}}^{r} \frac{dx_1}{x_1-\xi_1} \left[\frac{\partial U(y)}{\partial y_1} - 2\frac{\partial U(y)}{\partial y_2}\right]\Big|_{y_2=(-1)^m x_1, y_1=(-1)^n \sqrt{r^2-x_1^2};} + \ldots$$

.

$$m, n = 0, 1. \quad (19)$$

$$\frac{\partial U(\tau)}{\partial \tau_1}\Big|_{\tau_1=(-1)^n \sqrt{r^2-\xi_1^2}; \tau_2=(-1)^m \xi_1} =$$

$$= \frac{(-1)^{m+n}}{2\pi} \int_{r/\sqrt{2}}^{r} \frac{dx_1}{x_1 - \xi_1} \left[ \frac{\partial U(y)}{\partial y_1} - 2 \frac{\partial U(y)}{\partial y_2} \right] \Big|_{y_1 = (-1)^n \sqrt{r^2 - x_1^2}, y_2 = (-1)^m x_1;} + \cdot\cdot$$

.

$$m, n = 0, 1. \tag{20}$$

With this set

**Theorem 4.** *Under the conditions of Theorem 2, every solution of equation (3) (after substitution $U_1(x)$ defined in $D$ satisfy singular necessary conditions (15) - (20).*

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