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The Steady-State Solution Of Serial Queuing Processes With Feedback And Reneging

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Abstract. O'Brien [6], Jackson [3] and Hunt [4] studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer [1] obtained the steady-state solution of a single channel queuing model having Poisson input, exponential holding time, random selection introducing reneging. Finch [2] studied simple queues with customers at random for service at a number of service stations in series with feedback. Singh [8] studied the problem of serial queues introducing the concept of reneging. Punam, Singh and Ashok [7] found the steady-state solution of serial queuing processes where feedback is not permitted.
In our present work, the steady-state solutions are obtained for serial queuing processes with feedback and reneging in which

1. The number of serial service channel is M
2. A customer may join any channel from outside and leave the system at any stage after getting service.
3. Feedback is permitted from each channel to its previous channel.
4. The impatient customer leaves the service facility after wait of certain time.
5. Poisson arrivals and exponential service times are followed.
6. The queue discipline is random selection for service
7. Waiting space is infinite

Key Words and Phrases: Steady-State, random selection, feedback and reneging.

1. Formulation of Model

The system consists of the queues $Q_j\,(j = 1, 2, 3, ..., M)$with respective servers S_j $(j = 1, 2, 3, ..., M)$. Customers demanding different types of service arrive from outside the system with Poisson stream with respective parameters $\lambda_j\,(j = 1, 2, ..., M)$at $Q_j\,(j = 1, 2, 3, ..., M)$respectively. Further the impatient customers after joining any queue may leave the queue without service after a wait of certain time. After the completion of service at S_j, the customer either leaves the system with probability p_j or joins the

next channel with probability q_j or joins back the previous channel with probability r_j such that $p_j + q_j + r_j = 1$,$(j = 1, 2, 3, ...M)$. It is being mentioned here that $r_j = 0$ when $j = 1$ as there is no previous channel of the first channel and $q_j = 0$ when $j = M$ since there is no next channel after Mth channel. The service time distribution for severs S_j are mutually independent negative distribution with parameters $\mu_j\,(j = 1, 2, 3, ..., M)$.
The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block development officer, Tehsildar, Sub-divisional magistrate, District magistrate etc. Here, the officers of the district correspond to the servers of the model. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The impatient customers after joining the queue may leave the queue without getting service at any stage. The senior officer may send any customer to his junior if some information regarding the customer's problem is lacking.

2. Formulation of Equations

Define: $P\,(n_1, n_2, n_3, ...n_{M-1}, n_M; t)$= the probability that at time 't' there are n_j customers (which may leave the system after service or join the next phase or join back the previous channel or renege) waiting before $S_j\,(j = 1, 2, 3, ..., M-1, M)$

We define the operators $T_{i\cdot}$, $T_{\cdot i}$, $T_{\cdot i,_i} + 1.$, $T_{i-1\cdot,\,\cdot i}$ to act upon the vector $\tilde{n} = (n_1, n_2, n_3, ...n_M)$ as fallows

$$T_{i\cdot}\,(\tilde{n}) = (n_1, n_2, n_3, ..., n_i - 1, ...n_M)$$

$$T_{\cdot i}\,(\tilde{n}) = (n_1, n_2, n_3, ..., n_i + 1, ..., n_M)$$

$$T_{\cdot i, i+1\,\cdot}\,(\tilde{n}) = (n_1, n_2, n_3, ..., n_i + 1, n_{i+1} - 1, ..., n_M)$$

$$T_{i-1\cdot,\,\cdot i}\,(\tilde{n}) = (n_1, n_2, n_3, ..., n_{i-1} - 1, n_i + 1...., n_M)$$

Following the procedure given by Kelly [5], we write the difference – differential equations as under

$$\frac{dP\,(\tilde{n}; t)}{dt} = -\left[\sum_{i=1}^{M} \lambda_i + \sum_{i=1}^{M} \delta\,(n_i)\,(\mu_i + C_{in_i})\right] P\,(\tilde{n}; t)$$
$$+ \sum_{i=1}^{M} \lambda_i P\,(T_{i\cdot}\,(\tilde{n})\,; t) + \sum_{i=1}^{M} (\mu_i\, p_i + C_{i\,n_i+1})\; P\,(T_{\cdot i}\,(\tilde{n})\,; t) +$$
$$+ \sum_{i=1}^{M-1} \mu_i q_i\, P\,(T_{\cdot i, i+1\,\cdot}\,(\tilde{n})\,; t) + \sum_{i=1}^{M} \mu_i r_i\, P\,(T_{i-1\cdot,\,\cdot i}\,(\tilde{n})\,; t)\,. \qquad (1)$$

for $n_i \geq 0 \qquad (i = 1, 2, 3, ..., M)$

where$\delta\,(x) = \begin{cases} 1 & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$

and $P\,(\tilde{n}; t) = \tilde{0}$ if any of the arguments in negative.

3. Steady – State Equations

We write the following Steady–state equations of the queuing model by equating the time - derivates to zero in the equation (1)

$$\left[\sum_{i=1}^{M}\lambda_i+\sum_{i=1}^{M}\delta\left(n_i\right)\left(\mu_i+C_{in_i}\right)\right]P\left(\tilde{n}\right)=\sum_{i=1}^{M}\lambda_i P\left(T_i.\left(\tilde{n}\right)\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{i\,n_i+1}\right)P\left(T._i\left(\tilde{n}\right)\right)$$

$$+\sum_{i=1}^{M-1}\mu_i q_i\,P\left(T._{i,i+1}\,.\left(\tilde{n}\right)\right)+\sum_{i=1}^{M}\mu_i r_i\,P\left(T_{i-1}.,._i\left(\tilde{n}\right)\right) \qquad (2)$$

for $n_i\geq 0 \qquad (i=1,2,3,....,M)$

4. Steady-State Solutions

The solutions of the Steady-State equations (2) can be verified to be

$$P\left(\tilde{n}\right)=P\left(\tilde{0}\right)\left(\frac{\left(\lambda_1+\frac{\mu_2 r_2\rho_2}{\mu_2+C_{2n_2+1}}\right)^{n_1}}{\prod_{i=1}^{n_1}\left(\mu_1+C_{1i}\right)}\right)\cdot\left(\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{\mu_1+C_{1n_1+1}}+\frac{\mu_3 r_3\rho_3}{\mu_3+C_{3n_3+1}}\right)^{n_2}}{\prod_{i=1}^{n_2}\left(\mu_2+C_{2i}\right)}\right)\times$$

$$\times\left(\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{\mu_2+C_{2n_2+1}}+\frac{\mu_4 r_4\rho_4}{\mu_4+C_{4n_4+1}}\right)^{n_3}}{\prod_{i=1}^{n_3}\left(\mu_3+C_{3i}\right)}\right)$$

$$.....\left(\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}+\frac{\mu_M r_M\rho_M}{\mu_M+C_{Mn_M+1}}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}\left(\mu_{M-1}+C_{M-1i}\right)}\right)\times$$

$$\times\left(\frac{\left(\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\right)^{n_M}}{\prod_{i=1}^{n_M}\left(\mu_M+C_{Mi}\right)}\right) \qquad (3)$$

Where

$$\rho_1=\lambda_1+\frac{\mu_2 r_2\rho_2}{\mu_2+C_{2n_2+1}}$$

$$\rho_2=\lambda_2+\frac{\mu_1 q_1\rho_1}{\mu_1+C_{1n_1+1}}+\frac{\mu_3 r_3\rho_3}{\mu_3+C_{3n_3+1}}$$

$$\rho_3=\lambda_3+\frac{\mu_2 q_2\rho_2}{\mu_2+C_{2n_2+1}}+\frac{\mu_4 r_4\rho_4}{\mu_4+C_{4n_4+1}} \qquad (4)$$

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$$\rho_{M-1}=\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}+\frac{\mu_M r_M\rho_M}{\mu_M+C_{Mn_M+1}}$$

$$\rho_M=\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}$$

Solving these (4) M-equations for ρ_M with the help of determinants, we get

$$\rho_M=\frac{\left(\begin{array}{l}\lambda_M\Delta_{M-1}+\frac{q_{M-1}\mu_{M-1}\lambda_{M-1}\Delta_{M-2}}{\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}+\frac{q_{M-1}\mu_{M-1}q_{M-2}\mu_{M-2}\lambda_{M-2}\Delta_{M-3}}{\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\ldots\\+\frac{q_{M-1}\mu_{M-1}}{\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}.\frac{q_{M-2}\mu_{M-2}}{\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}.....\frac{q_3\mu_3}{\left(\mu_3+C_{3n_3+1}\right)}\lambda_3\Delta_2+\\\frac{q_{M-1}\mu_{M-1}}{\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}.\frac{q_{M-2}\mu_{M-2}}{\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}.....\frac{q_3\mu_3}{\left(\mu_3+C_{3n_3+1}\right)}\frac{q_2\mu_2}{\left(\mu_2+C_{2n_2+1}\right)}\lambda_2\Delta_1+\\\frac{q_{M-1}\mu_{M-1}}{\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}.\frac{q_{M-2}\mu_{M-2}}{\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}.....\frac{q_3\mu_3}{\left(\mu_3+C_{3n_3+1}\right)}\frac{q_2\mu_2}{\left(\mu_2+C_{2n_2+1}\right)}\frac{q_1\mu_1}{\left(\mu_1+C_{1n_1+1}\right)}\lambda_1\end{array}\right)}{\Delta_M} \tag{5}$$

Where $\Delta_M=\Delta_{M-1}-\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}.\frac{r_M\mu_M}{\mu_M+C_{Mn_M+1}}\Delta_{M-2}$

$$\Delta_{M-1}=\Delta_{M-2}-\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}.\frac{r_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{Mn_{M-1}+1}}\Delta_{M-3} \tag{6}$$

Continuing in this way

$$\Delta_3=\Delta_2-\frac{q_2\mu_2}{\mu_2+C_{2n_2+1}}.\frac{r_3\mu_3}{\mu_3+C_{3n_3+1}}$$

Where $\Delta_M=$

$$=\left|\begin{array}{cccccccc}1 & -\frac{r_2\mu_2}{\mu_2+C_{2n_2+1}} & 0 & 0 & \cdots & 0 & 0 & \\ -\frac{q_1\mu_1}{\mu_1+C_{1n_1+1}} & 1 & -\frac{r_3\mu_3}{\mu_3+C_{3n_3+1}} & 0 & \cdots & 0 & 0 & \\ 0 & -\frac{q_2\mu_2}{\mu_2+C_{2n_2+1}} & 1 & -\frac{r_4\mu_4}{\mu_4+C_{4n_4+1}} & \cdots & 0 & 0 & 0\\ \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}} & 1 & -\frac{r_M}{\mu_M+C} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}} & \end{array}\right.$$

$$\Delta_2=\left|\begin{array}{cc}1 & -\frac{r_2\mu_2}{\mu_2+C_{2n_2+1}}\\ -\frac{\mu_1q_1}{\mu_1+C_{1n_1+1}} & 1\end{array}\right|$$

$$\Delta_1=|1|=1$$

Since $\rho_M$is obtained,so we can get $\rho_{M-1}$by putting the value of ρ_M in the last equation of (4.2) , $\rho_{M-2}$by putting the values of $\rho_{M-1}$and $\rho_M$in the last but one equation of $(4,2)$ .Continuing in this way, we shall obtain $\rho_{M-3},\rho_{M-4},---,\rho_3,\rho_2$,and $\rho_1,.$

Thus, we write $(4,1)$as under

$$p(\tilde{n}) = P(\tilde{0}) \left(\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1} (\mu_1 + C_{1i})} \right) \left(\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2} (\mu_2 + C_{2i})} \right) \left(\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3} (\mu_3 + C_{3i})} \right) \ldots. \left(\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}} (\mu_{M-1} + C_{M-1i})} \right) \cdot \left(\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M} (\mu_M + C_{Mi})} \right) \tag{4.5}$$

for$n_i \geq 0 \qquad (i = 1, 2, 3, ..., M)$
We obtain $P(\tilde{0})$ from the normalizing conditions.

$$\sum_{\tilde{n}=\tilde{0}}^{\infty} P(\tilde{n}) = 1 \tag{7}$$

and with the restriction that traffic intensity of each service channel of the system is less than unity.
C_{in_i} is the reneging rate at which customer renege after a wait of time T_{0i} whenever there are n_i customer in the service channel Q_i.

$$C_{in_i} = \frac{\mu_{1i} e^{-\frac{\mu_{1i} T_{0i}}{n_i}}}{1 - e^{-\frac{\mu_{1i} T_{0i}}{n_i}}} \qquad (i = 1, 2, 3, ...M)$$

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive. Putting $C_{in_i} = C_i \qquad (i = 1, 2, 3, ...M)$. In the steady-state solution (3), the steady- state solution reduces to

$$P(\tilde{n}) = P(\tilde{0}) \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} \left(\frac{\rho_3}{\mu_3 + C_3} \right)^{n_3} \ldots. \left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right)^{n_{M-1}} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M} \tag{4.7}$$

We obtain $P(\tilde{0})$ from (7) and (**??**) as

$$P(\tilde{0}) = \left(1 - \frac{\rho_1}{\mu_1 + C_1} \right) \left(1 - \frac{\rho_2}{\mu_2 + C_2} \right) \left(1 - \frac{\rho_3}{\mu_3 + C_3} \right) \ldots.. \left(1 - \frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right) \left(1 - \frac{\rho_M}{\mu_M + C_M} \right)$$

Thus $P(\tilde{n})$is completely determined.

5. Steady-State Marginal Probabilities

Let $P(n_1)$be the steady-state marginal probability that there are n_1 units in the queue before the first serverS_i. This is determined as

$$P(n_1) = \sum_{n_2, n_3, \ldots n_M = 0}^{\infty} P(\tilde{n})$$
$$= \sum_{n_2, n_3, \ldots n_M = 0}^{\infty} P(\tilde{0}) \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} \left(\frac{\rho_3}{\mu_3 + C_3} \right)^{n_3} \ldots. \left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right)^{n_{M-1}} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M}$$
$$P(n_1) = \left(1 - \frac{\rho_1}{\mu_1 + C_1} \right) \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \qquad n_1 > 0$$

Similarly

$$P\left(n_2\right)=\left(1-\frac{\rho_2}{\mu_2+C_2}\right)\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2} \qquad n_2>0$$

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$$P\left(n_M\right)=\left(1-\frac{\rho_M}{\mu_M+C_M}\right)\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M} \qquad n_M>0$$

6. Mean Queue Length

Marginal queue length before the server S_1 is determined by

$$L_1=\sum_{n_1=0}^{\infty} n_1 P\left(n_1\right)$$
$$=\sum_{n_1=0}^{\infty} n_1\left(1-\frac{\rho_1}{\mu_1+C_1}\right)\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}$$

$$L_1=\frac{\frac{\rho_1}{\mu_1+C_1}}{\left(1-\frac{\rho_1}{\mu_1+C_1}\right)}=\frac{\rho_1}{\left(\mu_1+C_1-\rho_1\right)}$$

Similarly

$$L_2=\frac{\rho_2}{\left(\mu_2+C_2-\rho_2\right)}$$

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$$L_{M-1}=\frac{\rho_{M-1}}{\left(\mu_{M-1}+C_{M-1}-\rho_{M-1}\right)}$$

$$L_M=\frac{\rho_M}{\left(\mu_M+C_M-\rho_M\right)}$$

Thus mean queue length $L=\sum_{i=1}^{M} L_i$

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