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Variable Domain Eigenvalue Problems for the Place Laplace Operator with Density

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Abstract. We consider variable domain eigenvalue problems for the placeLaplace operator with density function. The first variation of the eigenvalues is calculated with respect to domain, their properties are investigated when the domain varies.

Key Words and Phrases: Eigenvalue problems, domain variation, support function

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1. Introduction

Eigenvalue problems are one of the intensively investigated fields of the spectral theory. These problems have significant applications, since some mechanical characteristics of certain systems indeed are described by the eigenvalues of the corresponding operators ([1]) . But in some cases the considered problems, as well as, the problems of the stability of vibrating bodies, the propagation of waves in composite media, and the thermic insulation of conductors lead to the eigenvalue problems with variable domain ([2]). Investigation of such kind of problems meets some difficulties, because of in this case one have to deal with domain funtionals instead of usual ones. In its mathematical formulation the problem consists of taking an operator and considering its eiganvalues as a functionals of the domain ([3]). In spite of actuality these investigation meet some difficulties related mainly with the definition of the domain variation. In [4-7] some properties of the eigenvalues are investigated using the different definitions of the domain variation.

Here we use the new definition of the domain variation based on the single valued correspondence between bounded convex domains and continuous positively homogeneous functions. This function defined as $P_A(x) = \sup_{l \in A}(l, x)$, $x \in R^m$ is called support function of the domain. Using this correspondence the variation of the domain is expressed by the variation of its support function. This technique allows us to avoid some of the difficulties during the investigation of the variable domain eigenvalue problems. In present work the formula is obtained for the first variation relatively domain and some properties are proved for the eigenvalues of placeLaplace operator with density function when the domain varies.

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Consider the following eigenvalue problem

$$-\Delta\, u = \lambda\rho(x)u, \quad x \in D, \tag{1}$$

$$u(x) = 0, \quad x \in S_D, \tag{2}$$

Here Δ is placeLaplace operator, $\rho(x)$is positive differentiable function in R^m,$D \subset R^m$,$S_D = \partial D$- its boundary. It is known ([8]) that the first eigenvalue of the problem (1), (2) may be calculated by the formula

$$\lambda_1(D) = \inf \frac{J_1(u,D)}{J_2(u,D)}, \tag{3}$$

where $\nabla\, u = (u_{x_1}, u_{x_2} ... u_{x_m})$,

$$J_1(u,D) = \int_D |\nabla u_x(x)|^2\, dx\ , \quad J_2(u,D) = \int_D \rho(x)u^2(x)dx,$$

and *inf* is taken over all functions $u \in C^2(D)$,being equal to zero at S_D. Let's denote by M the set of all convex bounded domains$D \subset R^m$. Define

$$K = \left\{D \in M, S_D \in C^2\right\} \tag{4}$$

It is known [3] that under given conditions eigenfunctions of the problem (1), (2) belong to the class$u \in C^2(D) \bigcap C^1(\overline{D})$ and eigenvalues are positive and may be numbered as $0 \le \lambda_1 \le \lambda_2 \le ...$ considering their multiplicity [1,8].

Using (3) we can consider the eigenvalue $\lambda_1$of the problem (1), (2) as a functional of D. We'll calculate the first variation of this functional. Using the obtained formula we investigate various properties of the eigenvalue relatively domain.

2. The Space of The Pairs of Convex Sets

Let's define the operations of addition and multiplication by the non-negative number in $M$by the following relations

$$A + B = \{c = a + b;\ a \in A,\ b \in B\},$$
$$\lambda A = \{\lambda a : \quad a \in A,\ \},\ \lambda \ge 0.$$

M is not a linear space (the operation of subtraction is not defined in M). Let's consider the pairs $(A, B) \in M \times M$and define the operations:

$$\begin{aligned} &(A_1,\, B_1) + (A_2,\, B_2) = (A_1 +\, A_2,\ B_1 + B_2)\,, \\ &\lambda(A,B) = (\,\lambda A,\ \lambda B)\,,\ \ if\ \lambda \ge 0, \\ &(-1)\cdot(A,B) = (B,A), \end{aligned} \tag{5}$$

$(A, B) \approx (C, D)$ if $A + D = B + C$

As a zero element of this space is taken the pair (0, 0), i.e. the set of elements $(A,\ A)$, $A \in M$.

The set of all such pairs forms a structure of a linear space. We'll introduce a scalar product in this space below.

For any $A \in M$ the function

$$P_A(x) = \sup_{l \in A}(l, x)\,,\ \ x \in R^m \tag{6}$$

is called a support function of the set A. This function is continues convex and positive homogeneous i.e.$P_A(\lambda x) = \lambda P_A(x),\ \ \lambda \geq 0$. Also it is known [9], that for each continues convex positive-homogeneous function $P(x)$ there exists a convex bounded set $A \in M$, such that

$$P(x) = \ P_A\,(x)\,.$$

The set $A \in M$ is reconstructed as a subdifferential of the function $P_D(x)$ at the origin [9,10], i.e.

$$A = \partial P_A(0) = \{l \in R^m : P_A(x) \geq (l, x), x \in R^m\}. \tag{7}$$

Let's take any $a = (A_1, A_2)\ ,\ \ \ b = (B_1, B_2)\ ,\ \ A_i,\ B_i \in M,\ i = 1, 2$and define the scalar product as

$$(a, b) = \int_{S_B} p(x)q(x)ds. \tag{8}$$

Here $p(x) = P_{A_1}(x) - P_{A_2}(x)\ ,\ \ \ \ q(x) = P_{B_1}(x) - P_{B_2}(x)\,;\ \ P_{A_i}(x), P_{B_i}(x)$ are support functions of the sets $A_i,\ B_i$ respectively; S_B is a surface of the unit sphere B. Norm in this space is defined as

$$\|\,a\,\|_{ML_1} = \sqrt{(a, a)} = \left(\int_{S_B} \left(P_{A_1}(x) - P_{A_2}(x)\right)^2 ds\right)^{1/2}. \tag{9}$$

For one dimensional case $m = 1$, the formula (8) turns to

$$(a, b) = p(-1)q(-1) + p(1)q(1). \tag{10}$$

It may be shown that this definition satisfies all requirements of the scalar product. We define this space by $ML_2(B)$orML_2. If in (8) to take any $D \in M$instead of B, then corresponding space with a scalar product

$$(a, b) = \int_{S_D} p(n(x))\ q(n(x))ds$$

is defined by $ML_2(D)$. Here $n(x)$is an outward normal to S_D in the point x.

3. First Variation of the Eigenvalue

To investigate the behaviour of the eigenvalues of the problem (1), (2) relatively domain, let's calculate the first variation of the functional $\lambda(D)$.

We define the differentiability of the functional $\lambda_1(D)$ similarly to [6].

Definition. The functional$\lambda_1(D)$ is called differentiable in D_0 in the direction$D$, if there exists the limit

$$\delta\lambda_1(D_0, D) = \lim_{\varepsilon\to+0} \frac{\lambda_1((1-\varepsilon)D_0+\varepsilon D) - \lambda(D_0)}{\varepsilon}. \qquad (11)$$

This definition may be written in the following equivalent form

$$\delta\lambda_1(d_0, d) = \lim_{\varepsilon\to+0} \frac{\lambda_1(d_0+\varepsilon d) - \lambda(d_0)}{\varepsilon},$$

It follows from the relation

$$\begin{aligned} & d_0+\varepsilon d = (D_0, 0) + \varepsilon(D, D_0) = \\ & = (D_0+\varepsilon D, \varepsilon D_0) = ((1-\varepsilon)D_0+\varepsilon D, 0)\,. \end{aligned}$$

For the simplicity here and later on we define $\delta\lambda_1(D_0, D)$ by $\delta\lambda_1(D_0)$.

Theorem 1. *The functional $\lambda_1(D)$is differentiable on K in the direction $\overline{D} \in K$ and*

$$\delta\lambda_1(D, \bar{D}) = -\frac{\int_{S_D} |\nabla u(x)|^2 \left[P_{\bar{D}}(n(x)) - P_D(n(x))\right] ds}{\int_D \rho(x)u^2(x)dx}. \qquad (12)$$

Remark 1. *If $S_D = S_1 \bigcup S_2$, where $S_1$is fixed and S_2 is unknown, then the formula (12) instead of S_D involves S_2.*

Corollary 1. *If domain D depends of on the parameter $t \in R$and support function of the domain $D = D(t)$ is differentiable then*
$\lambda_1'(t) = -\frac{\int_{S_D} |\nabla u(x)|^2 P'_{D(t)}(n(x))ds}{\int_{D(t)} \rho(x)\, u^2(x)dx}$, *(13) where* $S = \partial D(t)$, $\lambda_1 = \lambda_1(D(t))$, $P'_{D(t)} = \frac{\partial P_{D(t)}}{\partial t}$.

Proof. It is easy to obtain following relation from the formula (12)

$$\begin{aligned} & \lambda\left(t+\Delta t\right) - \lambda\left(t\right) = \lambda\left(D\left(t+\Delta t\right)\right) - \lambda\left(D\left(t\right)\right) = \\ & = -\frac{\int_{S_{D(t)}} |\nabla u(x)|^2 \left[P_{D(t+\Delta t)}(n(x)) - P_{D(t)}(n(x))\right] ds}{\int_{D(t)} \rho(x)u^2(x)dx} + o(\Delta t). \end{aligned}$$

From this considering the differentiability of the function$P_{D(t)}\left(x\right)$ relatively t, dividing by Δt, one may get (13). The theorem is proved.

In the case of multidimensional parameter $t=(t_1,t_2,...,t_k)\in \mathrm{T}\subset R^k$, where $S(t)\in C^2$for each $t\in \mathrm{T}$, the formula (13) takes a form

$$\frac{\partial\lambda_1(t)}{\partial t_i}=-\frac{\int_{S(t)}|\nabla u(x)|^2\,\frac{\partial P_{D(t)}(n(x))}{\partial t_i}dx}{\int_{D(t)}\rho(x)\,u^2(x)dx},\tag{13}$$

Corollary 2. *Let* $m=1,\ \ D(t)=(t_1,t_2),\ t_1,t_2\in R.$ *Then*

$$\frac{\partial\lambda_1}{\partial t_1}=\frac{u_x^2(t_1)}{\int_{t_1}^{t_2}\rho(x)u^2(x)dx},\frac{\partial\lambda_1}{\partial t_2}=-\frac{u_x^2(t_2)}{\int_{t_1}^{t_2}\rho(x)u^2(x)dx},\tag{14}$$

Proof. Since the support function of the interval has a form

$$P_{(t_1,t_2)}(x)=\begin{cases} t_2x\,, & x\geq 0,\\ t_1x\,, & x<0\end{cases}\quad(16)$$

from (14) we get (15). The theorem is proved.

4. Some Properties of the Eigenvalues

Now we investigate some properties of the first eigenvalue of the problem (1), (2) relatively domain using the formula (12).

From (13) one may immediately get the following

Corollary 3. *If*

$$P'_{D(t)}(x)\geq 0\quad\left(P'_{D(t)}(x)\leq 0\right),\qquad \forall x\in R^m\ ,$$

then $\lambda(t)$ *decreases (increases) relatively* t*.*

Example 1. **Let** $D(t)=D+tD_0,\ \ t>0,\ \ D,D_0\in K$**.**

Then following to the property of the support function [9]

$$P_{D(t)}(x)=P_D(x)+tP_{D_0}(x)\ ,\ \ P'_{D(t)x}(x)=P_{D_0(x)}(x)\,.$$

If $0\in D_0$, then

$$P_{D_0}(x)=\sup_{l\in D_0}(l,x)\geq(l,0)=0.$$

From this considering the corollary (13) $\lambda(t)$ decreases.

Example 2. Let's take $D(t)=(a(t)\,,\ b(t))$, where $a(t)\leq b(t)\ ,\ \ a(t)\ ,\ b(t)$ are differentiable functions. Then it is clear that for this case

$$P_{D(t)}(x)=\begin{cases} b(t)x\,, & x\geq 0,\\ a(t)x\,, & x<0\end{cases}$$

And

$$P'_{D(t)}(x) = \begin{cases} b'(t)x\,, & x \geq 0, \\ a'(t)x\,, & x < 0 \end{cases}$$

It shows that $\lambda(t)$ increases if $b'(t) \leq 0,\ a'(t) \geq 0$ and decreases if $b'(t) \geq 0,\ a'(t) \leq 0$.

Example 3. Let $m = 1$ and $D(t) = D(t_1, t_2) = (t_1, t_2)$. Then as follows from the corollary 3 $\lambda(t)$ increases with respect to t_1 and decreases with respect to t_2.

Theorem 2. *Let $\rho(tx) = \rho(x) \cdot t^{\alpha},\ \alpha \neq -2$. Then for the egenvalues of the problem* (1), (2) *in the domain D the following formula is true*

$$\lambda_1(D) = \frac{\int_{S_D} |\nabla u|^2 P_D(n(x))ds}{(\alpha+2)\int_D \rho(x)u^2(x)dx} \tag{15}$$

Proof. Take $D_0 \in K,\ D(t) = t \cdot D_0,\ t > 0$. The first eigenfunction of the problem (1), (2), corresponding to the domain D_0, define by $u(x)$. Then

$$-\Delta u(x) = \lambda_1(D_0)\rho(x)u(x)\,,\ x \in D_0.$$

This equation one can write in the following equivalent form

$$-\frac{1}{t^2}\Delta_{\left(\frac{x}{t}\right)}u\left(\frac{x}{t}\right) = \frac{\lambda_1(D_0)\rho\left(\frac{x}{t}\right)u\left(\frac{x}{t}\right)}{t^2},\ \ x \in D(t)\,. \tag{16}$$

Denote $\tilde{u}(x) = u\left(\frac{x}{t}\right)$. Since $\Delta\tilde{u}(x) = \Delta u\left(\frac{x}{t}\right)\,,\ x \in D(t)$ satisfies to the relation

$$\Delta\tilde{u}(x) = \frac{1}{t^2}\Delta u_{\left(\frac{x}{t}\right)}\left(\frac{x}{t}\right),$$

and $\rho\left(\frac{x}{t}\right) = \rho(x) \cdot t^{-\alpha},\ \alpha \neq -2$following the condition, form (18) we obtain

$$-\Delta\tilde{u}(x) = \frac{\lambda_1(D_0)}{t^{2+\alpha}}\rho(x)\tilde{u}(x)\,.$$

It shows that $\Delta\tilde{u}(x)$ is an eigenfunction and $\frac{\lambda_1(D_0)}{t^{2+\alpha}}$ - eigenvalue of the problem (1), (2) in the domain $D(t)$. Considering this in (13), we get

$$-(2+\alpha)\frac{\lambda_1(D_0)}{t^{\alpha+3}} = -\frac{\frac{1}{t^2}\int_{S_{D(t)}} \left|\nabla u\left(\frac{x}{t}\right)\right|^2 P_{D_0}(n(x))\,ds}{\int_{D(t)} \rho\left(\frac{x}{t}\right)u^2\left(\frac{x}{t}\right)dx}.$$

Taking $t = 1$ one may get

$$(2+\alpha)\lambda_1(D_0) = \frac{\int_{S_{D_0}} |\nabla u(x)|^2 P_{D_0}(n(x))\,ds}{\int_{D_0} \rho(x)\,u^2(x)\,dx}. \tag{17}$$

From last relation we obtain (15).

The theorem is proved.

If $u = u(x)$ is normalized eigenfunction for the problem (1), (2), e.i.

$$\int_D \rho(x)u^2(x)dx = 1,$$

then

$$\lambda_1(D) = \frac{1}{\alpha+2}\int_{S_D} |\nabla u(x)|\,.^2 P_D(n(x))ds. \tag{18}$$

Formula (20) shows that the boundary value of the function $|\nabla u(x)|$unequivocally defines λ_1.

From the formula (17) we obtain following result for the case $\alpha = -2$.

Corollary 4. Let $t^2\rho(tx) = \rho(x)$. Then

$$\int_{S_{D_0}} |\nabla u(x)|^2 P_{D_0}(n(x))\, ds = 0.$$

Corollary 5. Let $m = 1,\ \ D(t) = (t_1, t_2),\ t_1, t_2 \in R$and $\alpha \neq -2$. Then

$$\lambda_1 = \frac{u_x^2(t_2)t_2 - u_x^2(t_1)t_1}{(\alpha+2)\int_{t_1}^{t_2}\rho(x)u^2(x)dx}, \tag{19}$$

The first eigenvalue for the normalized eigenfunction we obtain following formula

$$\lambda_1(t_1,t_2) = \frac{1}{\alpha+2}\left[u^2(t_2)\cdot t_2 - u^2(t_1)t_1\right] \tag{20}$$

For the case $\alpha = -2$ satisfy

$$u^2(t_2)\cdot t_2 - u^2(t_1)t_1 = 0. \tag{21}$$

Example 3. Let $D(t) = B_t$ be a sphere of radius t, $\rho(x)$ satisfies to the condition $\rho(tx) = \rho(x)\cdot t^\alpha,\ \ \alpha \neq -2$. Considering $P_{B_t}(x) = t\,\|n(x)\| = t$, from (18) we get

$$\lambda_1(t) = \frac{1}{2+\alpha}\,\|\nabla u\|^2_{L_2(S(t))}\,.$$

Theorem 3. Let λ be a simple eigenvalue of the problem (1), (2) and the following relation is satisfied for the support function $P(t,x)$ of the domain $D(t)$

$$\sum_{i=1}^{k} a_i(t)\frac{\partial P(t,x)}{\partial t_i} = b(t)P(t,x), \quad x \in S_B,\ t \in T. \tag{22}$$

where S_B is a surface of the sphere B, $a_i(t),\ b(t)$ are given functions.

Then the eigenvalues $\lambda(t)$ of the problem (1), (2) in the domain$D(t)$ satisfy the equation

$$\sum_{i=1}^{k} a_i(t)\frac{\partial \lambda(t)}{\partial t_i} = -(\alpha+2)b(t)\lambda(t). \tag{23}$$

Proof. Multiplying both sides of (14) by $a_i(t)$ and taking a sum we get

$$\sum_{i=1}^{k} a_i(t)\cdot\frac{\partial\lambda}{\partial t_i} = -\int_{S_{D(t)}} \left|\nabla u\,(t,x)\right|^2 b(t)P_{D(t)}(n(x))dx.$$

Using (15) from last relation one may get (24).

The theorem is proved.

Example 4. Let's take$D(t)=[t_1\ t_2]$. Then as follows from (**??**)

$$t_1\frac{\partial P(t,x)}{\partial t_1}+t_2\frac{\partial P(t,x)}{\partial t_2}=P(t,x).$$

Thus, we get the following equation for the eigenvalues

$$t_1\frac{\partial \lambda(t)}{\partial t_1}+t_2\frac{\partial \lambda(t)}{\partial t_2}=-(\alpha+2)\lambda(t).$$

For the unequivocal calculation of the eigenvalues from (25) the boundary condition has to be given no on its characteristics.

Theorem 3.

$$-P'_{D(t)}(x)\le \mu P_{D_(t)}(x),\quad t\in[t_0,t_1]\,,\ \mu>0. \tag{24}$$

Then the estimate

$$\lambda(t)\le\lambda(t_0)e^{\mu p(t-t_0)}, t\in[t_0,t_1]. \tag{25}$$

is valid.

Proof. From (13), (15) considering (24) we have

$$\lambda'(t)\le\mu(\alpha+2)\lambda(t),\quad t\in[t_0,t_1].$$

Multiplying this by$\lambda(t)$, considering that $\lambda(t)\ge 0$ and integrating we get

$$\int_{t_0}^{t}\lambda'(\tau)\lambda(\tau)\,d\tau\le(\alpha+2)\mu\int_{t_0}^{t}\lambda^2(\tau)d\tau. \tag{26}$$

Integration by parts of the left hand side of (26) gives

$$\int_{t_0}^{t}\lambda'(\tau)\lambda(\tau)d\tau=\frac{1}{2}\lambda^2(t)-\frac{1}{2}\lambda^2(t_0).$$

Putting this in (21) we obtain

$$\lambda^2(t)\le 2(\alpha+2)m\int_{t_0}^{t}\lambda^2(\tau)d\tau+\lambda^2(t_0).$$

From last relation using Granule's lemma (see [10], p.450) we get

$$\lambda^2(t) \le \lambda^2(t_0) \cdot e^{2(\alpha+2)m(t-t_0)},$$

or the same (25).

The theorem is proved.

Example 5. Let $D(t) = tD_1 + D_0$, $t \ge 0$ and $0 \in (1+t)D_1 + D_0$ for all $t \in [0, T]$. Then

$$\lambda(t) \le \lambda(0)e^{(\alpha+2)t}, \tag{27}$$

where $\lambda(0) = \lambda(D_0)$.

Really, in this case

$$(1+t)P_{D_1}(x) + P_{D_0}(x) \ge 0,$$

or

$$-P_{D_1}(x) \le tP_{D_1}(x) + P_{D_0}(x).$$

Since $P'_{D(t)}(x) = \frac{\partial P_{D(t)}(x)}{\partial t}$ and $t \cdot P_{D_1}(x) + P_{D_0}(x) = P_{D(t)}(x)$ ([9]), (**??**) is satisfied by $\mu = 1$. From this we obtain (27).

5. Proof of Theorem 1.

Denote

$$\Lambda_1(u, D) = \int_D |\nabla u(x)|^2 \, dx,$$

$$\Lambda_2(u, D) = \int_D \rho(x) \, |u(x)|^2 \, dx.$$

$$\Lambda(u, D) = \Lambda_1(u, D)/\Lambda_2(u, D).$$

Then

$$\lambda_1(D) = \inf \Lambda(u, D), \tag{28}$$

Where inf is taken over all functions $u \in C^2$, being equal to zero on S_D. This class of the function we define by $\dot{C}^2(D)$.

First we calculate the increment of the functional $\Lambda\,(u, D)$. To do this we take any $D, D_0 \in K$ and calculate the increments of the functionals I_1 I_2.

Define

$$D_\varepsilon = (1-\varepsilon)D_0 + \varepsilon D, \quad D(\varepsilon) = D_\varepsilon \bigcap D_0,$$

$$S_\varepsilon = \partial D_\varepsilon, \quad S(\varepsilon) = \partial D(\varepsilon), \quad 0 \le \varepsilon \le 1. \tag{29}$$

The pair (u, D) is called to be admissible if $u \in \dot{C}^2(D),\ D \in K$.

Let $(u^\varepsilon, D_\varepsilon)$ be an admissible pair. Then

$$\Delta\Lambda_1 \equiv \Lambda_1(u^\varepsilon, D_\varepsilon) - \Lambda_1(u^0, D_0) = \int_{D_\varepsilon} |\nabla u^\varepsilon(x)|^2\, dx - \int_{D_\circ} |\nabla u^\circ(x)|^2\, dx =$$

$$= \left[\int_{D(\varepsilon)} |\nabla u^\varepsilon(x)|^2\, dx \ - \int_{D(\varepsilon)} \left|\nabla u^0(x)\right|^2 dx\right] + \left[\int_{D_\varepsilon} |\nabla u^\varepsilon(x)|^2\, dx \ - \int_{D(\varepsilon)} |\nabla u^\varepsilon(x)|^2\, dx\right] +$$

$$+ \left[\textstyle\int_{D(\varepsilon)} \left|\nabla u^0(x)\right|^2 dx \ - \int_{D_0} \left|\nabla u^0(x)\right|^2 dx\right] == \Lambda_4 + \Lambda_5 + \Lambda_6. \tag{30}$$

Here by $\Lambda_4\,, \Lambda_5, \Lambda_6$ are defined the first, second and third brackets correspondingly.

Now let's calculate each $\Lambda_i,\ i = 4,\ 5,\ 6$ separately.

In [6] the functional

$$J(D) = \int_D f(x)\, dx,$$

have been considered and the following formulae was obtained

$$\delta J(D, \overline{D}) = \int_{S_B} f(x)\ \left[p_{\overline{D}}(n(x)) - p_D(n(x))\right]\ ds \tag{31}$$

for its first variation.

Using this formula for $\Lambda_5\ \ \Lambda_6$, we get

$$.\Lambda_5 = \int_{S(\varepsilon)} |\nabla u^\varepsilon(x)|^2\ \left[P_{D_\varepsilon}(n(x)) - P_{D(\varepsilon)}(n(x))\right] ds + o(\varepsilon)\, .34)$$

Similarly

$$\Lambda_6 = -\int_{S(\varepsilon)} \left|\nabla u^0(x)\right|^2\ \left[P_{D_0}(n(x)) - P_{D(\varepsilon)}(n(x))\right]\ dx + o(\varepsilon)\,. \tag{32}$$

Adding obtained expressions we obtain

$$\Lambda_5 + \Lambda_6 = \int_{S(\xi)} \left|\nabla u^0(x)\right|^2\ \left[P_{D_\varepsilon}(n(x)) - P_{D_0}(n(x))\right] dx + o(\varepsilon) + o\left(\|\delta u\|_{C^1(D(\varepsilon))}\right), \tag{33}$$

where $\delta u = u^\varepsilon(x) - u^0(x)$,

Now let's calculate Λ_4

$$\Lambda_4 = \int_{D(\varepsilon)} 2\left(\nabla u^0(x), \nabla \delta u(x)\right) dx = \int_{D(\varepsilon)} 2div(\nabla u^0(x)\delta u(x)dx - \\ -2\int_{D(\varepsilon)} \Delta u^0(x)\delta(u)dx + o\left(\|\delta u\|_{C^1(D(\varepsilon))}\right). \tag{34}$$

Define the first integral by Λ_7 and transform it taking into account that $u^\varepsilon(x) = 0$ on S_ε and $u^0(x) = 0$ on S_{D_0}

$$\Lambda_7 = \int_{D(\varepsilon)} 2div(\nabla u^0(x)\delta u(x)dx - 2\int_{S_\varepsilon} (\nabla u^0(x), n(x))u^\varepsilon(x)ds + \\ +2\int_{S_{D_0}} (\nabla u^0(x), n(x))u^0(x)ds. \tag{35}$$

Considering $\delta u = u^\varepsilon(x) - u^0(x)$,

$$\Lambda_7 = 2\int_{D(\varepsilon)} div(\nabla u^0(x)u^\varepsilon(x)dx - 2\int_{D_\varepsilon} div(\nabla u^0(x)u^\varepsilon(x))dx + \\ +2\int_{D_0} div(\nabla u^0(x)u^0(x))dx - 2\int_{D(s)} div(\nabla u^0(x)u^\varepsilon(x))dx.$$

Using the formulae (33), we obtain

$$\Lambda_7 = -2\int_{S(\varepsilon)} div(\nabla u^0(x)u(x))\left[P_{D_\varepsilon}(n(x)) - P_{D(\varepsilon)}(n(x))\right] ds + o(\varepsilon) + o\left(\|\delta u\|_{!^1(D(\varepsilon))}\right).$$

Putting the obtained expression for Λ_7 into (37) we get the following formula for Λ_4

$$\Lambda_4 = -2\int_{S(\varepsilon)} div(\nabla u^0(x)u^0(x))\left[P_{D_\varepsilon}(n(x)) - P_{D(\varepsilon)}(n(x))\right] ds - \\ -2\int_{D(\varepsilon)} \Delta u^0(x)\delta u(x)dx + o(\varepsilon) + o\left(\|\delta u\|_{!^1(D(\varepsilon))}\right). \tag{36}$$

Substituting (33) and (39) into (32) we obtain

$$\Delta\Lambda_1 = \int_{S(\varepsilon)} \left[\left|\nabla u^0\right|^2 - 2div(\nabla u^0 u^0)\right] \left[P_{D_\varepsilon}(n(x)) - P_{D_0}(n(x))\right] ds - \\ -2\int_{D(\varepsilon)} \Delta u^0(x)\delta u(x)dx + o(\varepsilon) + o(\varepsilon) + o\left(\|\delta u\|_{C^1(D(\varepsilon))}\right). \tag{37}$$

Considering $D_\varepsilon = (1-\varepsilon)D_0 + \varepsilon D$ or $P_{D_\varepsilon}(x) = (1-\varepsilon)P_{D_0}(x) + \varepsilon P_D(x)$,
we have

$$P_{D_\varepsilon}(x) - P_{D_0}(x) = \varepsilon(P_D(x) - P_{D_0}(x)). \tag{38}$$

Taking into account (38) in (40) after some transformations one may get

$$\Delta\Lambda_1 = \int_{S_{D_0}} \left[\left|\nabla u^0\right|^2 - 2div(\nabla u^0 u^0)\right] \left[P_{D_\varepsilon}(n(x)) - P_{D_0}(n(x))\right] ds - \\ -2\int_{D_0} \Delta u^0(x)\delta u(x)dx + o(\varepsilon) + o\left(\|\delta u\|_{C^1(D(\varepsilon))}\right). \tag{39}$$

The similar considerations lead us to the following formula for the increment of the functional Λ_2

$$\Delta\Lambda_2 = 2\int_{D_0} \rho(x)u^0\delta u(x)dx + \varepsilon\int_{S_{D_0}} \rho(x)\left|u^0(x)\right|^2\ \left[P_D\left(n(x)\right) - P_{D_0}\left(n(x)\right)\right]ds+ \\ +o(\varepsilon) + o\left(\left\|\delta u\right\|_{C^1(D(\varepsilon))}\right)\ . \tag{40}$$

Now let $u^0(x)$ and u^ε be normalized eigenfunctions of the problem (1), (2) in the domains $D_0$and $D_\varepsilon$correspondingly. Then as it is clear from (30)

$$\delta\,\Lambda = \delta\,\Lambda_1 - \Lambda_1(u^0, D_0)\delta\,\Lambda_2.$$

Putting here the obtained formulas for $\Delta\Lambda_1,\ \Delta\Lambda_2$, considering $\lambda_1(D_0) = \Lambda_1(u^0, D_0)/\Lambda_2(u^0, D_0)$ and boundary condition $u^0(x) = 0,\ x \in S_{D_0}$ one may get

$$\Delta\Lambda = \frac{1}{\Lambda_2(u^0,D_0)}\int_{S_{D_0}}\left[\left|\nabla u^0\right|^2 - 2div(\nabla u^0u^0)\right]\left[P_{D_\varepsilon}(n(x)) - P_{D_0}(n(x))\right]ds+ \\ +2\int_{D_0}\left[-\Delta u^0(x) - \lambda_1\rho(x)u^0(x)\right]\delta u(x)dx\ + o(\varepsilon) + o\left(\left\|\delta u\right\|_{C^1(D(\varepsilon))}\right)$$

In this expression the second integral equals to zero since, u^0 is a solution of the problem (1), (2) when $D = D_0$. Thus, from last equality under boundary condition $u^0(x) = 0, x \in S_{D_0}$ we get

$$\delta\lambda_1 = -\frac{\int_{S_{D_0}}\left|\nabla u^0(x)\right|^2\left[P_D(n(x)) - P_{D_0}(n(x))\right]ds}{\int_D \rho(x)\left|u^0(x)\right|^2dx}\quad .$$

Theorem is proved.

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