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A New Method for Investigation and Recognizing of Self-Adjoint Boundary Value Problems

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Abstract. Boundary value problems (BVPs) are one of the most important fields in engineering and mathematical physics. In self-adjoint case of these problems, there are some facilities to solve them, such as eigenvalues of adjoint equations are real numbers and associated eigenfunctions make an orthogonal basis system.
In this paper a new method for investigation of self-adjoint B.V.Ps including ordinary differential equations (O.D.Es) is introduced. Based on this method, at first, some necessary conditions are obtained by making use of fundamental solutions of adjoint equations. Then an algebraic system is made by this necessary conditions and boundary conditions of given boundary value problem. Finally, by making use of Lagrangian identity and boundary values of unknown function, sufficient conditions for having a self-adjoint problem are presented.

Key Words and Phrases: Boundary value problem, Self-adjoint equation, Necessary conditions, Fundamental solution

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1. Introduction

As we know, the mathematical models of many physical phenomena and engineering problems are appeared in the form of boundary value problems (B.V.Ps). For this, these problems have been considered by many of mathematicians and scientists [1], [2]. In recent researches self-adjoint problems have been studied by Naimark [3]. He introduced a method for investigation and recognizing these problems. In Naimark's method, in the number of boundary conditions of given problem, additional relations with arbitrary constants are added to boundary conditions to make an algebraic system. Adding these conditions caused that the adjoint problem for the given problem is not unique, because there are arbitrary constants in the additional relations. In this paper, we present a method such that the arbitrary constants appear naturally in the necessary conditions and this fact yields that the corresponding adjoint problem be unique. These necessary conditions are used instead of above mentioned additional relations with arbitrary constants and are obtained by fundamental solution of adjoint equation. In final step, by making use of Lagrangian identity, sufficient conditions will be presented for being self-adjoint

problem.
For the first time, necessary conditions were used by Russian mathematician A. V. Bitsadze [5] and especially Azerbaijanian mathematician N. Aliev. During 1995-2010, Aliev with Iranian mathematicians M. Jahanshahi and S. M. Hosseini used these conditions for investigation and reduction of B.V.Ps including partial differential equations (P.D.Es) to the second kind of Fredholm integral equations [7-11].
In this paper we use these necessary conditions for investigating and recognizing self-adjoint B.V.Ps including ordinary differential equations (O.D.Es). This method also can be used for B.V.Ps including P.D.Es, especially for elliptic P.D.Es with constant coefficients:

$$P(D)u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + au$$

Because we need only to have generalized solution of given P.D.E. Therefore this method can be used for the wide class of B.V.Ps which they have generalized solutions [4]. It is necessary to say that $U(x-\xi)$ is a generalized solution when we have: $P(D)U(x-\xi) = \delta(x-\xi)$ where $\delta(x-\xi)$ is the Delta-Dirac function. On the other hand, this method can be done for initial-boundary value problems including hyperbolic and parabolic P.D.Es such as

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = h(x,t), \quad x \in (a,b), \quad t > 0 \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x) \\ &u(a,t) = p(t), \quad u(b,t) = q(t) \end{aligned}$$

$$\begin{aligned} &u_t = k\, u_{xx}, \quad x \in (a,b), \quad t > 0 \\ &u(x,0) = f(x) \\ &u(a,t) = p(t), \quad u(b,t) = q(t) \end{aligned}$$

After applying integral transforms and Fourier methods for these problems, we obtain usually a spectral problem with respect to parameter λ. The resulted spectral problem can be a self-adjoint problem or not.
At the end of paper, some unsloved problems will be presented.

2. Mathematical Statement of Problem

We consider the following boundary value problem with non-local boundary conditions:

$$ly \equiv y''(x) + a_1\, y'(x) + a_2\, y(x) = \phi(x)\,; \; x \in (0,1) \tag{1}$$

$$\ell_i y \equiv \alpha_{i1}\, y'(0) + \alpha_{i2}\, y(0) + \beta_{i1}\, y'(1) + \beta_{i2}\, y(1) = 0\,; \; i = 1,2 \tag{2}$$

Where $\phi(x)$ is a known continuous real-valued function and the coefficients $\alpha_{ij},\ \beta_{ij}\ (i,j = 1,2)$ are real constants.

3. Adjoint problem of main problem

We consider the adjoint equation of (1) in the following form:

$$l^*z \equiv z''(x) - a_1 z'(x) + a_2 z(x), \tag{3}$$

such that

$$(ly, z) = B(y, z) + (y, l^*z),$$

where

$$B(y, z) = [y'(x)z(x) - y(x)z'(x) + a_1 z(x)y(x)]|_0^1. \tag{4}$$

3.1. Naimark method

According to the Naimark method [3], for obtaining self-adjoint problem, the two following relations are added to the boundary conditions (2.2):

$$\ell_i y \equiv \alpha_{i1}\, y'(0) + \alpha_{i2}\, y(0) + \beta_{i1}\, y'(1) + \beta_{i2}\, y(1), \quad i = 3, 4 \tag{5}$$

Where α_{ij}, β_{ij} $(i = 3, 4,\ j = 1, 2)$ are arbitrary constants.
If we consider the two boundary conditions (2) and added relations (5) as an algebraic system, then its determinant is

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \alpha_{31} & \alpha_{32} & \beta_{31} & \beta_{32} \\ \alpha_{41} & \alpha_{42} & \beta_{41} & \beta_{42} \end{vmatrix}.$$

The constants α_{ij}, β_{ij} $(i = 3, 4,\ \ j = 1, 2)$ are chosen such that this determinant is not vanish. By letting A_{ij} $(i, j = 1, \ldots, 4)$ as the minor of $ij\,th$ element in Δ, then for the unknowns of this system we have the following relations:

$$y'(0) = 1/\Delta \sum_{i=1}^{4} l_i y\, A_{i1} \quad , \quad y(0) = 1/\Delta \sum_{i=1}^{4} l_i y\, A_{i2}$$

$$y'(1) = 1/\Delta \sum_{i=1}^{4} l_i y\, A_{i3} \quad , \quad y(1) = 1/\Delta \sum_{i=1}^{4} l_i y\, A_{i4}$$

And for $B(y, z)$ we have

$$B(y, z) = z(1)/\Delta \sum_{i=1}^{4} l_i y\, A_{i3} - z'(1)/\Delta \sum_{i=1}^{4} l_i y\, A_{i4} + a_1 z(1)/\Delta \sum_{i=1}^{4} l_i y\, A_{i4} -$$

$$z(0)/\Delta \sum_{i=1}^{4} l_i y\, A_{i1} + z'(0)/\Delta \sum_{i=1}^{4} l_i y\, A_{i2} - a_1 z(0)/\Delta \sum_{i=1}^{4} l_i y\, A_{i2}.$$

Therefore the boundary conditions of adjoint problem are

$$A_{i3}\, z(1) - A_{i4}\, z'(1) + a_1 A_{i4}\, z(1) - A_{i1}\, z(0) + A_{i2}\, z'(0) - a_1 A_{i2}\, z(0) = 0, \quad i = 3, 4.$$

Regarding this fact that the coefficients $\alpha_{ij},\ \beta_{ij}$ $(i = 3, 4,\ j = 1, 2)$ are arbitrary constants, therefore the resulted adjoint problem is not unique.

3.2. New method

In this section, we shall construct the related adjoint problem which it will be unique. Because of the coefficients in added relations are determined uniquely by using the necessary conditions.
For this, at first we consider the fundamental of equation (3.1);

$$Z(x-\xi) = \frac{e(x-\xi)}{r_1 - r_2}\,[e^{r_1(x-\xi)} - e^{r_2(x-\xi)}], \tag{6}$$

Where e is symmetric Heaviside function and r_1, r_2 are the distinct real roots of characteristic equation of (3).
On the other hand we can write

$$Z''(x-\xi) - a_1\, Z'(x-\xi) + a_2\, Z(x-\xi) = \delta(x-\xi),$$

Where δ is the Delta Dirac function.[4]
Now, we are going to calculate the necessary conditions over differential equation (1). For this, the fundamental solution (6) is multiplied to both sides of equation (1), then it is integrated over [0,1] as came in [7], [8];

$$\begin{aligned}\int_0^1 \phi(x) Z(x-\xi) dx &= \int_0^1 [y''(x) + a_1 y'(x) + a_2 y(x)] Z(x-\xi)\, dx \\ &= [y'(x)Z(x-\xi) - y(x)Z'(x-\xi) + a_1 y(x) Z(x-\xi)]|_0^1 + \\ &\int_0^1 y(x)[Z''(x-\xi) - a_1 Z'(x-\xi) + a_2 Z(x-\xi)]\, dx.\end{aligned}$$

The last integration is

$$\begin{aligned}\int_0^1 y(x)[Z''(x-\xi) - a_1 Z'(x-\xi) + a_2 Z(x-\xi)]\, dx &= \int_0^1 y(x)\delta(x-\xi)\, dx \\ &= \begin{cases} y(\xi) & \xi \in (0,1), \\ \frac{1}{2} y(\xi) & \xi = 0 \text{ or } \xi = 1, \\ 0 & \xi \notin [0,1]. \end{cases}\end{aligned}$$

By considering the boundary value of unknown function, we have

$$\int_0^1 \phi(x)Z(x-\xi)\,dx = [y'(x)Z(x-\xi)-y(x)Z'(x-\xi)+a_1y(x)Z(x-\xi)]|_0^1+\frac{1}{2}y(\xi);\ \xi = 0 \ or\ 1.$$

For $\xi = 0$, we have

$$\begin{aligned}\int_0^1 \phi(x)Z(x)\,dx = y'(1)Z(1) - y(1)Z'(1) + a_1y(1)Z(1)-\\ y'(0)Z(0) + y(0)Z'(0) - a_1y(0)Z(0) + \tfrac{1}{2}y(0).\end{aligned} \tag{7}$$

For $\xi = 1$, we have

$$\begin{aligned}\int_0^1 \phi(x)Z(x-1)\,dx = y'(1)Z(0) - y(1)Z'(0) + a_1y(1)Z(0)-\\ y'(0)Z(-1) + y(0)Z'(-1) - a_1y(0)Z(-1) + \tfrac{1}{2}y(1).\end{aligned} \tag{8}$$

We note that

$$Z(x) = e(x)\,z(x) \quad , \quad Z'(x) = e(x)\,z'(x)$$

Where $z(x)$ is the solution of associated homogeneous equation of (3). [4]
If we let

$$\lambda_1 = \frac{e^{r_1} - e^{r_2}}{r_1 - r_2} \quad , \quad \lambda_2 = \frac{e^{-r_1} - e^{-r_2}}{r_1 - r_2} \quad , \quad \lambda_3 = \frac{r_1e^{r_1} - r_2e^{r_2}}{r_1 - r_2} \quad , \quad \lambda_4 = \frac{r_1e^{-r_1} - r_2e^{-r_2}}{r_1 - r_2}$$

Then we have

$$\begin{aligned} Z(1) = 1/2\,\lambda_1 \quad , \quad Z'(1) = 1/2\,\lambda_3 \quad , \quad Z(0) = 0\\ Z(-1) = -1/2\,\lambda_2 \quad , \quad Z'(-1) = -1/2\,\lambda_4 \quad , \quad Z'(0) = 0\end{aligned} \tag{9}$$

We will obtain the following relations:

$$\begin{cases} (a_1\lambda_1 - \lambda_3)y(1) + \lambda_1\, y'(1) + y(0) = \displaystyle\int_0^1 \phi(x)z(x)dx, \\ (a_1\lambda_2 - \lambda_4)y(0) + \lambda_2\, y'(0) + y(1) = -\displaystyle\int_0^1 \phi(x)z(x-1)dx. \end{cases} \tag{10}$$

Which they are necessary conditions over equation (1).

4. Construction of Algebraic System

In this step, we make the following algebraic system by using the necessary conditions (10) and boundary conditions (2):

$$\begin{cases} (a_1\lambda_1-\lambda_3)y(1)+\lambda_1 y'(1)+y(0)=\displaystyle\int_0^1 \phi(x)z(x)dx, \\ (a_1\lambda_2-\lambda_4)y(0)+\lambda_2 y'(0)+y(1)=-\displaystyle\int_0^1 \phi(x)z(x-1)dx, \\ \alpha_{11}\,y^{'}(0)+\alpha_{12}\,y(0)+\beta_{11}\,y'(1)+\beta_{12}\,y(1)=0, \\ \alpha_{21}\,y'(0)+\alpha_{22}\,y(0)+\beta_{21}\,y'(1)+\beta_{22}\,y(1)=0. \end{cases} \tag{11}$$

We consider the following amounts for boundary values of $y'(1),\ y(1), y'(0), y(0)$ of unknown function by Cramer's rule

$$y'(0)=\frac{1}{\Delta}\begin{vmatrix} \displaystyle\int_0^1 \phi(x)z(x)dx & 1 & \lambda_1 & a_1\lambda_1-\lambda_3 \\ -\displaystyle\int_0^1 \phi(x)z(x-1)dx & a_1\lambda_2-\lambda_4 & 0 & 1 \\ 0 & \alpha_{12} & \beta_{11} & \beta_{12} \\ 0 & \alpha_{22} & \beta_{21} & \beta_{22} \end{vmatrix},$$

$$y(0)=\frac{1}{\Delta}\begin{vmatrix} 0 & \displaystyle\int_0^1 \phi(x)z(x)dx & \lambda_1 & a_1\lambda_1-\lambda_3 \\ \lambda_2 & -\displaystyle\int_0^1 \phi(x)z(x-1)dx & 0 & 1 \\ \alpha_{11} & 0 & \beta_{11} & \beta_{12} \\ \alpha_{21} & 0 & \beta_{21} & \beta_{22} \end{vmatrix},$$

$$y'(1)=\frac{1}{\Delta}\begin{vmatrix} 0 & 1 & \displaystyle\int_0^1 \phi(x)z(x)dx & a_1\lambda_1-\lambda_3 \\ \lambda_2 & a_1\lambda_2-\lambda_4 & -\displaystyle\int_0^1 \phi(x)z(x-1)dx & 1 \\ \alpha_{11} & \alpha_{12} & 0 & \beta_{12} \\ \alpha_{21} & \alpha_{22} & 0 & \beta_{22} \end{vmatrix},$$

$$y(1) = \frac{1}{\Delta} \begin{vmatrix} 0 & 1 & \lambda_1 & \int_0^1 \phi(x)z(x)dx \\ \lambda_2 & a_1\lambda_2 - \lambda_4 & 0 & -\int_0^1 \phi(x)z(x-1)dx \\ \alpha_{11} & \alpha_{12} & \beta_{11} & 0 \\ \alpha_{21} & \alpha_{22} & \beta_{21} & 0 \end{vmatrix},$$

Where

$$\Delta = \begin{vmatrix} 0 & 1 & \lambda_1 & a_1\lambda_1 - \lambda_3 \\ \lambda_2 & a_1\lambda_2 - \lambda_4 & 0 & 1 \\ \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \end{vmatrix} \neq 0. \tag{12}$$

We suppose this determinate is not vanish.

If we let

$$A_{12} = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \quad A_{13} = \begin{vmatrix} \alpha_{11} & \beta_{11} \\ \alpha_{21} & \beta_{21} \end{vmatrix}, \quad A_{14} = \begin{vmatrix} \alpha_{11} & \beta_{12} \\ \alpha_{21} & \beta_{22} \end{vmatrix}, \quad A_{23} = \begin{vmatrix} \alpha_{12} & \beta_{11} \\ \alpha_{22} & \beta_{21} \end{vmatrix}$$

$$A_{34} = \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}, \quad A_{24} = \begin{vmatrix} \alpha_{12} & \beta_{12} \\ \alpha_{22} & \beta_{22} \end{vmatrix}, \quad \lambda_1^* = a_1\lambda_1 - \lambda_3, \quad \lambda_2^* = a_1\lambda_2 - \lambda_4$$

$$B_1 = \int_0^1 \phi(x)z(x)dx, \quad B_2 = -\int_0^1 \phi(x)z(x-1)dx$$

We have the following relations:

$$\begin{cases} y'(0) = 1/\Delta\,[\,B_1(\lambda_2^*\,A_{34} + A_{23}) - B_2(A_{34} - \lambda_1\,A_{24} + \lambda_1^*\,A_{23})\,], \\ y(0) = 1/\Delta\,[\,-B_1\lambda_2\,A_{34} + B_2(-\lambda_1\,A_{14} + \lambda_1^*\,A_{13})\,], \\ y'(1) = 1/\Delta\,[\,B_1(\lambda_2\,A_{24} - \lambda_2^*\,A_{14} + A_{12}) + B_2(A_{14} - \lambda_1^*\,A_{12})\,], \\ y(1) = 1/\Delta\,[\,-B_1(\lambda_2\,A_{23} - \lambda_2^*\,A_{13}) + B_2(-A_{13} + \lambda_1\,A_{12})\,]. \end{cases} \tag{13}$$

5. Boundary conditions for adjoint equation

Now, for obtaining of boundary conditions of adjoint equation (3), we put the boundary values (13) in bilinear form $B(y,z)$.

$B(y,z) =$

$1/\Delta\,[\,B_1(\lambda_2\,A_{24} - \lambda_2^*\,A_{14} + A_{12} - a_1\lambda_2 A_{23} + a_1\lambda_2^* A_{13}) + B_2(A_{14} - \lambda_1^*\,A_{12} - A_{13} + \lambda_1 A_{12})\,]\,z(1)+$

$1/\Delta\,[\,B_1(\lambda_2\,A_{23} - \lambda_2^*\,A_{13}) + B_2(A_{13} - \lambda_1\,A_{12})\,]\,z'(1)+$

$1/\Delta\,[\,-B_1(\lambda_2^*\,A_{34} + A_{23} - a_1\lambda_2 A_{34}) + B_2(A_{34} - \lambda_1\,A_{24} + \lambda_1^*\,A_{23} + a_1\lambda_1 A_{14} - a_1\lambda_1^* A_{13})\,]\,z(0)+$

$1/\Delta\,[\,-B_1\lambda_2\,A_{34} + B_2(-\lambda_1\,A_{14} + \lambda_1^*\,A_{13})\,]\,z'(0).$

Now for being $B(y,z) = 0$, we should have the following relations:

$$\begin{cases} (\lambda_2\,A_{24} - \lambda_2^*\,A_{14} + A_{12} - a_1\lambda_2 A_{23} + a_1\lambda_2^* A_{13})\,z(1) + (\lambda_2\,A_{23} - \lambda_2^*\,A_{13})\,z'(1) - \\ \qquad (\lambda_2^*\,A_{34} + A_{23} - a_1\lambda_2 A_{34})\,z(0) - \lambda_2\,A_{34}\,z'(0) = 0. \\ \\ (A_{14} - \lambda_1^*\,A_{12} - A_{13} + \lambda_1 A_{12})\,z(1) + (A_{13} - \lambda_1\,A_{12})\,z'(1) + \\ \qquad (A_{34} - \lambda_1\,A_{24} + \lambda_1^*\,A_{23} + a_1\lambda_1 A_{14} - a_1\lambda_1^* A_{13})\,z(0) + (-\lambda_1\,A_{14} + \lambda_1^*\,A_{13})\,z'(0) = 0. \end{cases} \tag{14}$$

6. Main Results

To sum up we can conclude the following theorem and corollary.

Theorem: By considering the adjoint equation (3) and adjoint boundary conditions (14), we can write the adjoint problem of main problem $(1) - (2)$ by (3) and (14).

Corollary: Let the main problem $(1) - (2)$ and its related adjoint problem (3) and (14) under condition (12), then the main problem $(1) - (2)$ is a self-adjoint problem, if $a_1 = 0$ and the following conditions are hold:

$$\alpha_{11} = -\lambda_2\,A_{34} \quad , \quad \alpha_{12} = -\lambda_2^*\,A_{34} - A_{23} \quad , \quad \beta_{11} = \lambda_2\,A_{23} - \lambda_2^*\,A_{13}$$

$$\beta_{12} = \lambda_2\,A_{24} - \lambda_2^*\,A_{14} + A_{12} \quad , \quad \alpha_{21} = -\lambda_1\,A_{14} + \lambda_1^*\,A_{13} \quad , \quad \alpha_{22} = A_{34} - \lambda_1\,A_{24} + \lambda_1^*\,A_{23}$$

$$\beta_{21} = A_{13} - \lambda_1 A_{12} \quad , \quad \beta_{22} = A_{14} - \lambda_1^* A_{12} - A_{13} + \lambda_1 A_{12}$$

Some unsloved probelms:

1. Consider the following B.V.P with general non-local boundary conditions;

$$\Delta u + a u_{x_1} + b u_{x_2} + cu = 0, \quad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$$

$$\alpha_j(x_1)\,u(x_1, \gamma_1(x_1)) + \beta_j(x_1)\,u(x_1, \gamma_2(x_1)) = \phi_j(x_1), \quad j = 1, 2, \; x_1 \in (a, b)$$

Where $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 |\; x_1 \in (a,b),\; x_2 \in (\gamma_1(x_1), \gamma_2(x_1))\}$ and $\phi_j, \alpha_j, \beta_j$ are real continuous functions and $u(x_1, x_2)$ is the unknown function. Give some sufficient conditions which the above mentioned B.V.P is a self-adjoint problem.

2. Consider the following initial-boundary value problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \quad x \in (0,1), \quad t > 0$$
$$u(0,t) = u(1,t)$$
$$\frac{\partial u(x,t)}{\partial x}|_{x=0} + \frac{\partial u(x,t)}{\partial x}|_{x=1} = 0, \quad t \geq 0$$
$$\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} = \phi_k(x), \quad k = 0, 1, \quad x \in [0,1]$$

i) Determine the related spectral problem is a self-adjoint problem or not.
ii) Determine sufficient conditions which the resulted spectral problem be a self-adjoint problem.

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