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On Tensor Fields Generated by Differentiable Functions

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Abstract. In the paper it is considered the question on differentiable properties of maps generated by real smooth functions. In one dimensional case the set of sequential derivatives well defines the structure of a set of singular points. In this paper we attempt to generalize this theory for multivariate functions.

Key words: differentiable function, tensor field, functional matrix, tensor product.

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1. Introduction

We shall use the notations from [1]. Let $\mathbb{R}^n$ be n-dimensional real space, $U \subset \mathbb{R}^n$ some open set, and let $f(\bar{x}) \in C^{(m)}(U)$ be a function having all continuous partial derivatives of orders up to m. We sketch remind the basic notions of the theory. Let $\bar{x}, \bar{h} \in U, \bar{x} + \bar{h} \in U$ be some points in the set $U \subset \mathbb{R}^n$. Denote by $\mathbf{f}(\bar{x})$ a vector -valued map, that is, some map $\mathbf{f} : U \to V \subset \mathbb{R}^k$, where k is a natural number. If there is a linear map $A \in L(\mathbb{R}^n, \mathbb{R}^k)$ such that the limit

$$\lim_{\bar{h}\to 0} \frac{\left|\mathbf{f}(\bar{x} + \bar{h}) - \mathbf{f}(\bar{x}) - A\bar{h}\right|}{\left|\bar{h}\right|} \tag{1}$$

exists then the function $\mathbf{f}(\bar{x})$ is called a differentiable function at the point $\bar{x} \in U$. Here $\left|\bar{h}\right|$ denotes the usual Euclidean norm of the vector $\bar{h}$. When $\mathbf{f}(\bar{x})$ is differentiable at the point $\bar{x}$, then we shall use the notation for the linear map A: $\mathbf{f}'(\bar{x}) = A$. If $\mathbf{f}(\bar{x})$ is differentiable at any point of the set U, then we say that the map $\mathbf{f}$ is differentiable in U. When k=1 we get the definition of the notion of derivative of the function, in which the linear map A is substituted by the number defining the linear map. Taking $\bar{h} = h\bar{e}_j$ where h is a real number, $\bar{e}_j$ is a basis vector, we get partial derivative with respect to x_j.

Considering gradient vector $\nabla\mathbf{f} = (D_1 f, ..., D_n f)$ we get map $\nabla\mathbf{f} : U \to V \subset \mathbb{R}^n$. If to take second derivative, in accordance with the definition above, we get the linear operator $A \in L(\mathbb{R}^n, \mathbb{R}^n)$. The matrix of this operator is a Jacoby matrix of coordinate functions of the vector $\nabla \mathbf{f}$ at the point $\bar{x}$(acting on the tangential subspace, that is, subspace with the origin at $\bar{x}$):

$$F(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}; \ f_j = \partial f / \partial x_j.$$

In metric problems connected with this matrix, for example in the question of estimates of trigonometric integrals, or estimates for areas of surfaces and so on, it stands necessary to study normal forms of this operator. Let $r_1(\bar{x}), ..., r_n(\bar{x})$ be singular numbers of this matrix at the point $\bar{x}$. Then singular value decomposition of this matrix by using of singular bases can be written as follows:

$$Q^T F(\bar{x}) T = \Lambda,$$

where Q and T are orthogonal functional matrices of order n, and Λ is a diagonal matrix containing singular number of the matrix $F(\bar{x})$ (see [6])on the diagonal. The columns of the matrices Q and T set up, so called, singular bases of the matrix$F(\bar{x})$. Following relations are best known (see [3-4]):

$$r_j(\bar{x}) = (F\bar{t}_j, \bar{q}_j), \ F\bar{t}_j = r_j \bar{q}_j, F^T \bar{q}_j = \bar{t}_j. \qquad (2)$$

As it was shown in [2, 6, 7, 9], the functions $r_j(\bar{x})$ are differentiable functions. These functions are defined by the matrix $F(\bar{x})$ univaludely, if do not count their order of placement. Therefore, if we define the map $\bar{r} : U \to \mathbb{R}^n$, then its Jacoby matrix will be defined by the matrix $F(\bar{x})$ in some neighborhood of the point $\bar{x}$. In this article we shall investigate this question, studying the structure of the tensor fields connected with the function $F(\bar{x})$. It should be noted that we consider here simplest case of map in which the full space is mapped into itself. The method of investigation used here can be applied to maps of manifolds. In both cases the relations (2) plays an essential role.

2. Singular value decomposition of derivative matrix

Let us differentiate the first of relations with respect to any varying direction $\bar{\xi} = \bar{\xi}(\bar{x})$. It means that we take in the relation (1) $\bar{h} = h\bar{\xi}$, and shall pass to the limit as $h \to 0$. Using relation (2) we get:

$$D_{\bar{\xi}} r_j(\bar{x}) = (D_{\bar{\xi}} F \bar{t}_j, \bar{q}_j) + (F D_{\bar{\xi}} \bar{t}_j, \bar{q}_j) + (F\bar{t}_j, D_{\bar{\xi}} \bar{q}_j) =$$

$$= (D_{\bar{\xi}} F \bar{t}_j, \bar{q}_j) + (D_{\bar{\xi}} \bar{t}_j, F^T \bar{q}_j) + (F\bar{t}_j, D_{\bar{\xi}} \bar{q}_j) =$$
$$= (D_{\bar{\xi}} F \bar{t}_j, \bar{q}_j) + r_j (D_{\bar{\xi}} \bar{t}_j, \bar{t}_j) + r_j (\bar{q}_j, D_{\bar{\xi}} \bar{q}_j).$$

Since columns of the matrices Q and T are the unite vectors, then $(\bar{t}_j, \bar{t}_j) = 1$. So, for any vector $\bar{\xi}$

$$D_{\bar{\xi}} (\bar{t}_j, \bar{t}_j) = 2 (D_{\bar{\xi}} \bar{t}_j, \bar{t}_j) = 0.$$

Therefore,

$$D_{\bar{\xi}} r_j(\bar{x}) = D_{\bar{\xi}}(F\bar{t}_j, \bar{q}_j) = ((D_{\bar{\xi}}F)\bar{t}_j, \bar{q}_j). \tag{3}$$

The last relation shows that the vectors of singular bases remain unchanged when we differentiate the singular numbers of the matrix. So, despite that these vectors are the functions of independent variables, they behave themselves as a constant vectors when differentiation of singular numbers is taken with respect arbitrary direction. From the relation (3) it is seen that the Jacoby matrix of the map $\bar{s} : \bar{x} \mapsto \bar{r}(\bar{x})$, where $\bar{r}^T(\bar{x}) = (r_1(\bar{x}), ..., r_n(\bar{x}))$ is possible write as below

$$\frac{\partial(r_1, ..., r_n)}{\partial(x_1, ..., x_n)} = (((D_{\bar{x}_i}F)\bar{t}_j, \bar{q}_j)) = \begin{pmatrix} ((D_{\bar{x}_1}F)\bar{t}_1, \bar{q}_1) & \cdots & ((D_{\bar{x}_1}F)\bar{t}_n, \bar{q}_n) \\ ((D_{\bar{x}_2}F)\bar{t}_1, \bar{q}_1) & \cdots & ((D_{\bar{x}_2}F)\bar{t}_n, \bar{q}_{1n}) \\ \vdots & \ddots & \vdots \\ ((D_{\bar{x}_n}F)\bar{t}_1, \bar{q}_1) & \cdots & ((D_{\bar{x}_n}F)\bar{t}_n, \bar{q}_n) \end{pmatrix}.$$

İt is not difficult to see that

$$((D_{\bar{x}_i}F)\bar{t}_j, \bar{q}_j) = \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial F}{\partial x_i} t_{mj} q_{kj} =$$

$$= \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial^3 f}{\partial x_m \partial x_k \partial x_i} t_{mj} q_{kj}. \tag{4}$$

Consider now the linear map $\mathbf{f}'' : \mathbb{R}^n \to \mathbb{R}^{n^2}$ as a vector function the components of which are elements of the matrix F. Let's arrange the elements of each column of the matrix F consequently in a line, and take the transposed Jacoby matrix of the system of functions

$$B = \left(\frac{\partial^2 f_1}{\partial x_1 \partial x_1}, ..., \frac{\partial^2 f_1}{\partial x_1 \partial x_n}, \frac{\partial^2 f_2}{\partial x_1 \partial x_1}, ..., \frac{\partial^2 f_2}{\partial x_1 \partial x_n}, ..., \frac{\partial^2 f_n}{\partial x_1 \partial x_1}, ..., \frac{\partial^2 f_n}{\partial x_1 \partial x_n} \right).$$

It is easy to observe that this matrix will be transposed to the matrix of the linear map conditionally denoted as above $\mathbf{f}''' : \mathbb{R}^n \to \mathbb{R}^{n^3}$. Then the relations can be written more briefly as follows:

$$((D_{\bar{x}_i}F)\bar{t}_j, \bar{q}_j) = ((D_{x_i}B)(\bar{t}_i \otimes \bar{q}_i)),$$

where the expression $\bar{t}_i \otimes \bar{q}_i$ means the vector-column

$$\bar{t}_i \otimes \bar{q}_i = \begin{pmatrix} t_{i1}q_{i1} \\ \vdots \\ t_{i1}q_{in} \\ \vdots \\ t_{in}q_{i1} \\ \vdots \\ t_{in}q_{in} \end{pmatrix}.$$

So, the Jacoby matrix $S(\bar{x})$ of the map $\bar{s} : \bar{x} \mapsto \bar{r}(\bar{x})$ is possible express as follows:

$$S(\bar{x}) = \begin{pmatrix} D_{x_1}B \\ \vdots \\ D_{x_n}B \end{pmatrix} (\bar{t}_1 \otimes \bar{q}_1 \cdots \bar{t}_n \otimes \bar{q}_n) = B'(T * Q), \tag{5}$$

where the symbol B' used for "derivative" of B, and the symbol $T * Q$ is used for the matrix constructed from columns $\bar{t}_1 \otimes \bar{q}_1, ..., \bar{t}_n \otimes \bar{q}_n$.

Following two theorems is proved for matrices with polynomials entries ([6]). Below we generalize it for smooth functions and clarify the structure of the tensor fields generated by real smooth functions, giving the singular value decomposition for functional matrices entered above.

Theorem 2.1. *Let $f(\bar{x})$ be a real smooth function, and the matrices $S(\bar{x})$ and B' defined as above. If $\det(B' \cdot B'^T) \neq 0$ for all $\bar{x} \in U$ then the relation below holds:*

$$(\det S(\bar{x}))^2 = \det(B' \cdot B'^T).$$

Proof. Now express the number $(\det S(\bar{x}))^{-1}$ as follows (see [5, p. 131, the problem 35]):

$$(\det S(\bar{x}))^{-1} = c^{-1} \int_{\|U(\bar{x})\bar{w}\| \leq 1} dw_1 \cdots dw_n.$$

Substituting the expression (5) for the matrix $S(\bar{x})$, we find:

$$(\det S(\bar{x}))^{-1} = c^{-1} \int_{\|B'(\bar{x})(T*Q)\bar{w}\| \leq 1} dw_1 \cdots dw_n.$$

where c expresses the volume of the unite ball $c = \pi^{-n/2}\Gamma(n/2)$. Let's change variables under the last integral by the formula $\bar{v} = (T * Q)\bar{w}$. Since the columns of the matrix $T * Q$ are pare wisely orthogonal, we can represent the last integral in the form of a surface integral

$$(\det S(\bar{x}))^{-1} = c^{-1} \int_{\|B'(\bar{x})\bar{v}\| \leq 1} ds$$

(indeed, the surface element equals $\sqrt{\det((T * Q)^T (T * Q)} dw_1 \cdots dw_s = dw_1 \cdots dw_s)$. The matrix B' has the size $n \times n^2$ and has a singular value decomposition of the form:

$$B' = L \Lambda M$$

where the matrices L and M are orthogonal matrices of order n and n^2, correspondingly. Let's make orthogonal change of variables under the surface integral above by using formula

$$\bar{y} = M\bar{x}.$$

Since this is an orthogonal transformation, the surface integral does not change its value. We shall use arguments of the works [6, 10] to complete the proof. Assume that singular numbers are placed at the first n columns of the matrix Λ (they can placed in any order). Noting that all of these singular numbers are distinct from zero, we take the columns at which they are placed. If some of them are equal to zero, then we must consider all possible their situation (as it was made in [8], we can assume that all of singular numbers are non-zero, because if not, then we can instead of it take small perturbation of the matrix $F(\bar{x})$ of a view

$$F(\bar{x}) + \begin{pmatrix} \varepsilon x_1 & 0 & \cdots & 0 \\ 0 & \varepsilon x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon x_n \end{pmatrix}$$

with next passing to the limit). We get

$$\int_{\|B'(\bar{x})\bar{v}\|\leq 1} ds = \int_{\lambda_1^2 u_1^2+\cdots+\lambda_n^2 u_n^2 \leq 1} du_1 \cdots du_n = c(\lambda_1 \cdots \lambda_n)^{-1} = (\det(B' \cdot B'^T))^{-1/2}.$$

This is a needed result. Theorem 2.1 is proven. □

Note that the got result does not mean that singular numbers of the matrices $S(\bar{x})$ and B' are identical. From the proof of the theorem 2.1 it is visible that these matrices have different transformations to get singular value decomposition. For the matrix $S(\bar{x})$ such reducing demands transformation in the subspace generated by the vectors $\bar{t}_1 \otimes \bar{q}_1, ..., \bar{t}_n \otimes \bar{q}_n$ in $\mathbb{R}^n \otimes \mathbb{R}^n$, but singular value decomposition of the matrix B' is possible find by orthogonal transformation in the full space $\mathbb{R}^n \otimes \mathbb{R}^n$.

Corollary 2.2. *Jacoby matrix of the map $\bar{s} : \bar{x} \mapsto \bar{r}(\bar{x})$ is non-singular at every inner point in some neighborhood of which the matrix B' has maximal rank.*

This corollary is an easy consequence of the theorem 2.1. It shows that the Jacobian of the corollary can vanish only in a subset of zero Jordan measure, where the matrix B' degenerates.

Theorem 2.1 expresses important property of tensor fields generated by smooth functions. From the proof of the theorem we conclude that the forms which arise during differentiation can be reduced to normal forms by engaging convenient geometric interpretation. Since the scalar product in the right hand side of (2) is a bilinear form, then it can be written as a linear form in $\mathbb{R}^n \otimes \mathbb{R}^n$, as a scalar product $(\tilde{F}, (\bar{t}_j \otimes \bar{q}_j))$ in which $(\tilde{F})^T$ denote the row got by arranging of the elements of columns of the matrix F consequently in a row. So, $(\tilde{F})^T(T * Q)$ is a matrix of the size $1 \times n$, which consists of singular numbers of the matrix F. Let's write this matrix symbolically as $F \circ (T * Q)$.

After of differentiation we get rows $(D_{x_i}F) \circ (T * Q)$. Placing all partial derivatives in a column, we get a cubic matrix every layer of which acts to the matrix $T * Q$ by operation defined as $\circ$. So, every layer of the cubic matrix define a row of the matrix $S(\bar{x})$ in consent with the formula (5). The operation $T * Q$ defines a subspace in $\mathbb{R}^n \otimes \mathbb{R}^n$, which we shall call as "prime subspace" and denote by $Pm(\mathbb{R}^n \otimes \mathbb{R}^n)$. Note that the prime subspace depends on taken bases in both spaces. If we take singular value decomposition for the matrix $S(\bar{x})$ (or equivalently for the matrix $B'(T * Q)$ from (5)) we get singular bases at which the matrix of diagonal section of the cubic matrix reduces to diagonal form (diagonal of the cubic matrix). We can now continue the same reasoning to go forward by taking higher derivatives.

Consider now the case of derivatives of order 3. Let the matrix $S(\bar{x})$ has following singular value decomposition: $S(\bar{x}) = H\Sigma G$ (equivalently, $Q_1^T S(\bar{x}) T_1 = \Sigma$) in which the H and G are orthogonal matrices, and Σ is a diagonal matrix of singular numbers. Denote by $S_1(\bar{x})$ the Jacoby matrix of the system of functions consisting of singular numbers of $S(\bar{x})$.

Theorem 2.3. *Let $f(\bar{x})$ be real smooth function, and the matrices $S(\bar{x})$ and B' defined as above. If $\det(B'' \cdot B''^T) \neq 0$ for all $\bar{x} \in U$ then the relation below holds:*

$$(\det S_1(\bar{x}))^2 = \det(B'' \cdot B''^T).$$

Proof. Proof of the theorem is carried out by applying the resoning of the proof of Theorem 1. It is easy to observe that

$$S_1(\bar{x}) = B''(Q_1 * ((Q * T)T_1)).$$

The matrix $Q_1 * ((Q * T)T_1)$is a matrix of the size $n^3 \times n$ with orthonormal columns lying in some prime subspace of tensor product $\mathbb{R}^n \otimes (Pm(\mathbb{R}^n \otimes \mathbb{R}^n))$. The remaining part of the proof spends as in the proof of Theorem 2.1. Theorem 2.3 is proved. □

3. General case

Applying the method of mathematical induction we can generalize the results of previous section. To do this we need in some designations. Let $f(\bar{x})$ be some smooth function $f(\bar{x}) \in C^{(m)}(U)$, m is a natural number greater than 2. Consider two sequences of matrices: $S(\bar{x})$, $S_1(\bar{x})$,…,$S_{m-2}(\bar{x})$ and $B(\bar{x})$, $B'(\bar{x})$,…,$B^{(m-2)}(\bar{x})$ which are constructed by following way.

1. Construction of the first sequence. $S(\bar{x})$ is a matrix constructed above, that is, this matrix is a Jacoby matrix of the system of functions consisting of coordinate function of the gradient ∇f. $S_1(\bar{x})$ is a transposed Jacoby matrix of the system of singular numbers of the matrix $S(\bar{x})$. Further, by induction, if the matrix $S_j(\bar{x})$ is defined then by $S_{j+1}(\bar{x})$ we denote the transposed Jacoby matrix for the system of singular number of the matrix $S_j(\bar{x})$.

2. Construction of the second sequence we begin from the matrix $B_0(\bar{x}) = S(\bar{x})$. We take as a matrix $B_1(\bar{x}) = S'(\bar{x})$ the transposed Jacoby matrix of the system of n^2 functions got arranging the elements of columns, consequently, in a line. So, this a matrix is of the size $n \times n^2$. Further, we arrange the elements of columns of the matrix $B_1(\bar{x})$, consequently, in a row, and take the transposed Jacoby matrix of got system of functions, denoting the last one as $B_2(\bar{x})$. Continuing by such way, we get the sequence of matrices $B_1(\bar{x})$,…,$B_{m-2}(\bar{x})$. The matrix $B_{m-2}(\bar{x})$ consists of all partial derivatives of the function $f(\bar{x})$ of order m.

Applying the method used above we can prove the following theorem.

Theorem 3.1. *Let $m \geq 3$, and $0 \leq k \leq m-3$, If $\det(B_{k+1} \cdot B_{k+1}^T) \neq 0$ for all $\bar{x} \in U$ then the relation below holds:*

$$(\det S_k(\bar{x}))^2 = \det(B_k \cdot B_k^T).$$

The following result is a generalization of the corollary 2.2 of the theorem 2.1.

Corollary 3.2. *Let conditions of Theorem 3.1 are satisfied. Then the Jacoby matrix of the map $\bar{s} : \bar{x} \mapsto \bar{r}(\bar{x})$ is non-singular at every inner point in some neighborhood of which the matrix $B_{m-2}(\bar{x})$ has maximal rank.*

Corollary 3.2 shows that in conditions of the theorem 3.1 the set of singular points of the map $\bar{s} : \bar{x} \mapsto \bar{r}(\bar{x})$ cannot have inner points. So, in the conditions of Theorem 3.1 the set of all singular points of the last map has zero Jordan mesure.

Corollary 3.3. *Let conditions of Theorem 3.1 are satisfied. Then the set of singular points of the map $\bar{s} : \bar{x} \mapsto \bar{r}(\bar{x})$ has zero Jordan measure.*

From the corollary 3.3 it follows that in conditions of the theorem 3.1 the set of all of singular points of the function $f(\bar{x})$ and its derivative matrices has zero Jordan measure.

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