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On Stability of Bases Consisting of Perturbed Exponential Systems in Grand Lebesgue Spaces

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Abstract. In this work perturbed exponential system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ is considered in grand-Lebesgue spaces and here $\{\lambda_n\}$ is a sequence of real numbers. Since grand Lebesgue space is non-separable, the subspace $G^{p)}(-\pi,\pi)$ of $L^{P)}(-\pi,\pi)$ generated by the shift operator is defined, and thus the system which is mentioned above is not complete in them. A condition on the sequence $\{\lambda_n\}$, which is sufficient for the above system to form the basis for the subspace $G^{p)}(-\pi,\pi)$is found. The results which have been acquired are the analogs of those obtained earlier for the Lebesgue spaces L^p. The analog of the classical Levinson's theorem on the completeness of system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$in the spaces L^p, $1<p<+\infty$ is established.

KeyWords and Phrases: basicity, grand Lebesgue space, Levinson's theorem, exponential system, q-Hilbert system.

1. Introduction

Let's consider an exponential system

$$\left\{e^{i\lambda_n x}\right\}_{n\in Z}, \tag{1.1}$$

where Z is a set of integers and $\{\lambda_n\}\in R$ is some sequence of real numbers.

This system is a natural perturbation of the classical system of exponent and it is also an eigenfunction of a second-order ordinary differential operator with integral boundary condition. This is the reason why there is pervasive enthusiasm in studying the basis properties of these systems in different kinds of function spaces.

The first results in this field belong probably to Paley Wiener [1] and N. Levinson [2].

The well-known Kadets 1/4 theorem also belongs to this field (see [3]).

When $\lambda_n$has a constant shift $\lambda_n = n+\alpha sign n$,$(\alpha\in R)$ then sinus and cosinus analogs of (1.1) system arise in the solution of mixed or elliptic type differential equations by the Fourier method (see e.g., [4-6]).

In view of this, many authors have studied the basis properties of systems (1.1) (see, e.g., [4; 7-15]). All above-mentioned works treat basis properties in the Lebesgue spaces.

Recently there has been increasing interest in non-standard function spaces in the context of applications to the problems of mechanics and mathematical physics. Among those spaces, we can mention Lebesgue spaces with variable summability index, Morrey spaces, grand-Lebesgue spaces, etc. Each of these spaces has its differences compared to classical Lebesgue-type spaces. For example, Morrey and grand-Lebesgue spaces are not separable, and therefore, a countable system of elements of course cannot be complete in these spaces. So one has to choose reasonable subspaces of these spaces to treat some approximation problems. In the context of classical trigonometric systems and their linear perturbations (concerning a phase), approximation problems in these spaces have been studied in varying degrees (for more details see, e.g., [15;19-23]).

In this work, we consider a perturbed exponential system $\left\{e^{i\lambda_k}\right\}_{k\in Z}$in grand Lebesgue space $L^{p)}(-\pi;\pi)$ based on the continuity of the shift operator, we define a subspace $G^{p)}(0,\pi) \subset L^{p)}(0,\pi)$of this space. We establish an analog of the classical Levinson's theorem on the replacement of a finite number of elements of this system by other elements. We find a condition on the sequence $\{\lambda_k\}$ which is sufficient for this system to form a basis for $L^{p)}(0,\pi)$ Our results are the analogs of the corresponding results obtained for Lebesgue spaces (see, e.g., [16]).

2. Necessary information

We need some information on the theory of bases and from a theory of grand Lebesgue spaces to be used to obtain our main results.

Let $L^{p)}(-\pi;\pi)$, $1 < p < +\infty$, be a grand-Lebesgue space [25-28] of measurable (a Lebesgue-measurable) functions f on $[-\pi;\pi]$, which satisfy the condition

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t)|^{p-\varepsilon}\, dt \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

Space $L^{p)}(-\pi;\pi)$ is a complete normed space with the norm $\|f\|_{p)}$. The embeddings

$$L^{p}(-\pi;\pi) \subset L^{p)}(-\pi;\pi) \subset L^{p-\varepsilon}(-\pi;\pi),$$

and the relations

$$\left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} \leq \|f(\cdot)\|_{p)} \leq 2\pi(p-1) \|f(\cdot)\|_{p}, \forall f \in L^{p}(-\pi;\pi), 0 < \varepsilon < p-1, \quad (2.1)$$

hold. For $\forall\delta > 0$, consider the shift operator

$$T_\delta f(x) = \begin{cases} f(x+\delta), x+\delta \in [-\pi;\pi], \\ 0, x+\delta \in R\backslash[-\pi;\pi], \end{cases} \forall f \in L^{p)}(-\pi;\pi).$$

Denote by $\tilde{G}^{p)}(-\pi;\pi)$ the linear manifold of functions $f \in L^{p)}(-\pi;\pi)$ satisfying the condition

$$\|T_\delta f - f\|_{p)} \to 0, \delta \to 0.$$

Let $G^{p)}(-\pi;\pi)$ be a closure of $\tilde{G}^{p)}(-\pi;\pi)$ in $L^{p)}(-\pi;\pi)$. As $\forall f \in L^{p}(-\pi;\pi)$ we have $\|T_\delta f - f\|_p \to 0$, $\delta \to 0$, from (2.1) it follows that $L^{p}(-\pi;\pi) \subset G^{p)}(-\pi;\pi)$. The space $C_0^\infty[-\pi;\pi]$ of infinitely differentiable finite functions on $[-\pi;\pi]$ is dense in $G^{p)}(-\pi;\pi)$.

The space $L^{p)}(-\pi,\pi)$ is nonseparable complete space with the norm $\|f\|_{p)}$. Consider the family of functions $\{f_a\}$(see, e.g., [17-18]) :

$$f_a(t) = \begin{cases} 0, \ 0 \le t \le a, \\ (t-a)^{-\frac{1}{p}}, \ a < t \le 1 \, , \end{cases}$$

for $\forall a \in (0,1)$we have $\{f_\alpha\} \subset L^{p)}(0,1)$. In fact

$$\|f_a\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_a^1 (t-a)^{-1+\frac{\varepsilon}{p}}\, dt\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} \left(p(t-a)^{\frac{\varepsilon}{p}} \,|_a^1\right)^{\frac{1}{p-\varepsilon}} =$$
$$= \sup_{0<\varepsilon<p-1} \left(p(1-a)^{\frac{\varepsilon}{p}}\right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

So $\{f_a\} \subset L^{p)}(0,1)$. Now let for $a, b \in [0,1)$so that $b > a$, then we have

$$\|f_b - f_a\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f_b(t) - f_a(t)|^{p-\varepsilon}\, dt\right)^{\frac{1}{p-\varepsilon}} \ge \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_a^b (t-a)^{-1+\frac{\varepsilon}{p}}\, dt\right)^{\frac{1}{p-\varepsilon}} =$$
$$= \sup_{0<\varepsilon<p-1} \left(p(b-a)^{\frac{\varepsilon}{p}}\right)^{\frac{1}{p-\varepsilon}} \ge \lim_{\varepsilon\to+0} \left(p(b-a)^{\frac{\varepsilon}{p}}\right)^{\frac{1}{p-\varepsilon}} = p^{\frac{1}{p}}.$$

It directly follows that the space $L^{p)}(0,1)$ is non-separable.

We will need the following concepts and facts to get the main results.

Definition 2.1. *A system $\{f_n\}_{n\in N} \subset L^{p)}(a,b)$ is called q-Hilbert $(q > 0)$ if there exists $C > 0$ such that for every finite set of complex numbers $\{c_n\}$ the inequality*

$$\|\{c_n\}\|_{l_q} \le C \left\| \sum_n c_n f_n \right\|_{L^{p)}(a,b)},$$

holds.

Definition 2.2. *A sequence $\{\lambda_n\}$ is called separated if $\inf\limits_{i\ne j} |\lambda_i - \lambda_j| > 0$ holds.*

We will also need the following

Statement 2.3. *Suppose a finite number of elements in the basis of some Banach space are replaced by the other elements of this space. Then the following properties are equivalent for the newly obtained system:i) it forms a basis;*
ii) it is complete;
iii) it is minimal.

We will also use some concepts and facts from the theory of Banach function spaces. Let's state needful facts from this theory.

Let $(M;\mathcal{M},\mu)$ be a measurable space with a measure μ. Denote a set of all measurable functions $f : M \to \mathrm{C}$ (where C is a complex plane) by $\mathcal{F}$. Denote a subspace of functions from $\mathcal{F}$, which take on values μ -a.e., by $\mathcal{F}_0$. Let $\mathcal{F}^+ = \{f \in \mathcal{F} : f \geq 0\}$.

Definition 2.4. *A mapping $\rho : \mathcal{F}^+ \to [0, +\infty]$ is called a Banach function norm (or simply a b.f.n.) if, for all $f, g, f_n \in \mathcal{F}^+ (n = 1, 2, 3...)$ for all constants $k \geq 0$, and for all μ-measurable subsets $E$of $\mathcal{M}$, the following properties hold:*

(P1) $\rho(f) = 0 \Leftrightarrow f = 0 \;\; \mu - a.e.; \;\; \rho(kf) = k\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;
(P2) $0 \leq g \leq f \;\; \mu - a.e. \Rightarrow \rho(g) \leq \rho(f)$;
(P3) $0 \leq f_n \uparrow f \;\; \mu - a.e. \Rightarrow \rho(f_n) \uparrow \rho(f)$;
(P4) $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;
(P5) $\mu(E) < +\infty \Rightarrow \exists C_E > 0 : \int_E f d\mu \leq C_E \rho(f), \; \forall f \in \mathcal{F}^+$.

Definition 2.5. *Let ρ be a function norm. The collection $X = X(\rho)$of all functions $f \in \mathcal{F}$, for which $\rho(|f|) < +\infty$, is called a Banach function space (or simply a b.f.s.). For $f \in X$ define, $\|f\|_X = \rho(|f|)$.*

Definition 2.6. . *Suppose $f \in \mathcal{F}_0$. The decreasing rearrangement of f is the function f^*defined on $[0, +\infty)$ by*

$$f^* = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \; t \geq 0.$$

where $\mu_f(\lambda) = \mu\{t : |f(t)| > \lambda\}, \lambda \geq 0$, is a distribution function of f.

Definition 2.7. *Let X be a Banach function space. The closure of the set ofsimple function M_s in X is denoted by X_b .Let X be a Banach function space over $(M;\mu)$. Let $\rho'(g) = \sup\left\{\int_M fgd\mu : f \in \mathcal{F}^+; \rho(f) \leq 1\right\}, \forall \mathrm{g} \in \mathcal{F}^+$.*

Space
$X' = \{g \in \mathcal{F} : \rho'(|g|) < +\infty\}$,is called an associate space of X.

The functions $g \in \mathcal{F}_0$ f and g from $\mathcal{F}_0$ are called equimeasurable if$\mu_f(\lambda) = \mu_g(\lambda), \forall \lambda \geq 0$. Banach function $\rho : \mathcal{F}^+ \to [0, +\infty)$ is called rearrangement invariant if for arbitrary equimeasurable functions $f, g \in \mathcal{F}^+$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Lorentz,Orlicz, Lorentz-Orlicz spaces are r.i.s..

Definition 2.8. *Let X be a r.i.s. over a resonant space (about resonant space see e.g., [23, p.45]) $(M;\mu)$. For each finite value of t belonging to the range of μ , let $E \in \mathcal{M}$:$\mu(E) = t$ and $\varphi_X(t) = \|\chi_E\|_X$. The function φ_X is called the fundamental function of X.*

Definition 2.9. *A function f in a b.f.s. X is said to have absolutely continuous norm in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\} \subset \mathcal{M}$:$E_n \to \emptyset \; \mu - a.e.$. The set of all functions in X of absolutely continuous norm is denoted by X_a.*

We also need the following theorems from the monograph [29].

Theorem 2.10. *[29]. The subspaces X_a and X_b coincide if and only if the characteristic function χ_E has absolutely continuous norm for every set $E \in \mathcal{M}$ of finite measure.*

Theorem 2.11. *The Banach space dual X^* of a b.f.s. X is canonically isometrically isomorphic to the associate space X' if and only if X has absolutely continuous norm.*

We will also use the following statement from [29, p.14].

Statement 2.12. *Let X be a b.f.s. over $(M;\mu)$with norm $\|\cdot\|_X$. A function $f \in X$ has absolutely continuous norm if and only if $\|f\chi_{E_n}\|_X \downarrow 0$ for every sequence $\{E_n\}_{n\in N}$ satisfying $E_n \downarrow \emptyset$ μ -a.e..*

For more details about these facts see, e.g., [29].

3. Sufficient condition for separation of $\{\lambda_n\}$

The following lemma is true.

Lemma 3.1. *Let $\{\lambda_n\}_{n\in Z} \in R$ be some sequence of real numbers. If the system $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ is q-Hilbert in $L^{p)}(-\pi;\pi)$,$p \in (1,+\infty)$ then $\{\lambda_n\}_{n\in Z}$ is separated.*

Proof. According to the definition of q-Hilbertness we have

$$\left(\sum_{k=1}^{\infty}|c_k|^q\right)^{\frac{1}{q}} \leq C \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=1}^{\infty}c_k f_k\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}$$

where if we take $c_n = 1, c_m = -1, c_k = 0, k \neq n \neq m$ and $f_k = e^{i\lambda_k t}$ then we have

$$(1+|-1|+0+...)^{\frac{1}{q}} \leq C \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left|e^{i\lambda_n t}-e^{i\lambda_m t}\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \Rightarrow$$

$$\Rightarrow \quad 2^{\frac{1}{q}} \leq C\left\|e^{i\lambda_n t}-e^{i\lambda_m t}\right\|_{L^{p)}(-\pi,\pi)}. \tag{3.1}$$

Taking into account the inequality

$$\left|e^{i\lambda_n t}-e^{i\lambda_k t}\right| = |\cos(\lambda_n t)+i\sin(\lambda_n t)-\cos(\lambda_k t)-i\sin(\lambda_k t)| =$$

$$= |\cos(\lambda_n t)-\cos(\lambda_k t)+i(\sin(\lambda_n t)-\sin(\lambda_k t))| =$$
$$= \sqrt{(\cos(\lambda_n t)-\cos(\lambda_k t))^2+(\sin(\lambda_n t)-\sin(\lambda_k t))^2}$$
$$= \sqrt{2-2\cos(\lambda_n-\lambda_k)t} = 2\left|\sin\left(\frac{\lambda_n-\lambda_k}{2}t\right)\right| \leq$$
$$\leq |\lambda_n-\lambda_k|\,|t| \leq \pi\,|\lambda_n-\lambda_k|\,.$$

We have

$$\left\| e^{i\lambda_n t} - e^{i\lambda_k t} \right\|_{L_{p)}(-\pi,\pi)} \le \pi \left| \lambda_n - \lambda_k \right| \|1\|_{L_{p)}(-\pi,\pi)} =$$

$$= \pi \sup_{0<\varepsilon<p-1} (2\pi)^{\frac{1}{p-\varepsilon}} \left| \lambda_n - \lambda_k \right| = 2\pi^2 \left| \lambda_n - \lambda_k \right|.$$

The remaining part follows directly from (3.1).

$$2^{\frac{1}{q}} \le 2\pi^2 C \left| \lambda_n - \lambda_k \right| \quad \Rightarrow \quad \left| \lambda_n - \lambda_k \right| \ge \frac{2^{\frac{1}{q}}}{2\pi^2 C} \quad \Rightarrow \quad \inf_{n \ne k} \left| \lambda_n - \lambda_k \right| > 0.$$

The lemma is proved. □

The lemma below can be proved in the same way.

Lemma 3.2. *Let $\{\lambda_n\}_{n\in Z} \in R$ be some sequence. If the system (1.1) is q -Hilbert $L^{p)}(-\pi,\pi)$,$p \in (1,+\infty)$, then $\{\lambda_n\}_{n\in Z}$ is separated.*

4. The $G^{p)}$ analog of Levinson's Theorem

In this section, we set an analog of Levinson's theorem in $G^{p)}(-\pi;\pi)$. Moreover, denote by $(G^{p)}(a,b))'$ the associate space of $G^{p)}(-\pi;\pi)$ i.e.

$$(G^{p)}(a,b))' = \left\{ g \in \mathrm{T}(a,b) : \rho'(|g|) < +\infty \right\},$$

where

$$\rho_p'(g) = \sup \left\{ \int_a^b fg dt : f \in \mathrm{T}^+(a,b) : \|f\|_{L_{p)(a,b)}} \le 1 \right\},$$

$\mathrm{T}(a,b)$ are Lebesgue-measurable functions on (a,b) and

$$\mathrm{T}^+(a,b) = \{ f \in \mathrm{T}(a,b) : f \ge 0 \}.$$

The following analog of Levinson's theorem is true.

Theorem 4.1. *Let $\{\lambda_k\}_{k\in N} \subset C$ be some sequence. For the exponential system $\left\{e^{i\lambda_k t}\right\}_{k\in N}$ not to be complete in $G^{p)}(-\pi;\pi)$ it is necessary and sufficient that there exists an entire function $F(\lambda)$ vanishing at all points λ_k, $k \in N$, and admitting representation*

$$F(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda t} \overline{\psi(t)} dt,$$

where $\psi \in (G^{p)}(-\pi,\pi))'$ is some function.

Proof. Let the system $\left\{e^{i\lambda_k t}\right\}_{k\in N}$ be not complete in $G^{p)}(-\pi,\pi), 1<p<+\infty$.
Then there exists a non-zero functional $\nu \in \left(G^{p)}(-\pi,\pi)\right)^*$ such that

$$v(e^{i\lambda_k t}) = 0, \forall k \in N.$$

Let's show that the spaces $(G^{p)}(-\pi,\pi))'$ and $(G^{p)}(-\pi,\pi))^*$ are isometrically isomorphic, i.e. they can be equated with each other.
According to Theorem 2.11 to show this it suffices to prove that $G^{p)}(-\pi,\pi)$ has absolutely continuous norm. Let $f \in G^{p)}(-\pi,\pi)$ be an arbitrary function. As $C\left[-\pi,\pi\right]$ (a space of continuous functions on $[-\pi,\pi]$) is dense in $G^{p)}(-\pi,\pi)$, for $\forall\varepsilon>0$, $\exists f_0 \in C\left[-\pi,\pi\right]$ we have

$$\|f-f_0\|_{L^{p)}(-\pi,\pi)} < \varepsilon.$$

Let $\{E_n\}_{n\in \mathrm{N}} \subset (-\pi,\pi)$be an arbitrary sequence of (Lebesgue) measurable sets such that $E_n \downarrow \emptyset$ m-a.e. (m-is a Lebesgue measure).

Recall that $E_n \downarrow \emptyset$ m-a.e. means that $\chi_{E_n} \downarrow 0$ m-a.e. Let's show that $\|f\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} \downarrow 0$.
So, let $\varepsilon>0$ be an arbitrary number. We have

$$\|f\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} \le \|(f-f_0)\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} + \|f_0\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} < \varepsilon + \|f_0\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} . \tag{4.1}$$

Let $C = \|f_0\|_{L_\infty(-\pi,\pi)}$. We have

$$\|f_0\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} \le C\left\|\chi_{E_n}\right\|_{L^{p)}(-\pi,\pi)} = C \sup_{0<\varepsilon<p-1} \left(\varepsilon\left|E_n\right|\right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le C\left|E_n\right|^{\frac{1}{p}} \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} = C\left|E_n\right|^{\frac{1}{p}} p^{\frac{1}{p}}.$$

where $|\cdot|$ is a Lebesgue measure. It is clear that $\lim\limits_n E_n = \bigcap\limits_n E_n = \emptyset$ m -a.e.. Consequently

$$\lim_n |E_n| = \left|\lim_n E_n\right| = 0.$$

Then from (4.1), it follows that $\|f\chi_{E_n}\|_{L^{p)}(-\pi,\pi)} \to 0, n\to\infty$.
Thus, by Statement 2.12, space $G^{p)}\left(-\pi,\pi\right)$has absolutely continuous norm. Then from Theorem 2.11 it follows that $(G^{p)}(-\pi,\pi))^* = (G^{p)}(-\pi,\pi))'$ and hence it is clear that $\exists\psi \in (G^{p)}(-\pi,\pi))' : v(f) = \int_{-\pi}^{\pi} f(t)\overline{\psi(t)}dt,\ \forall f \in G^{p)}(-\pi,\pi)$. If we write here $f\left(t\right) = e^{i\lambda t}$ we have

$$F(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda t}\overline{\psi(t)}dt,\ \lambda \in C. \tag{4.2}$$

Obviously, $F(\cdot)$is an entire function and $F\left(\lambda_k\right) = 0, \forall k \in N$.
The theorem is proved. $\square$

Theorem 4.2. *If the entire function $F(\cdot)$ is represented in the form* (4.1)*,* $\psi \in \left(G^{p)}(-\pi,\pi)\right)'$, $1 < p < +\infty$, $F(\lambda_0) = 0$ *and* $\mu \in C$ *is an arbitrary number, then the function*

$$F_1(\lambda) = \frac{\lambda - \mu}{\lambda - \lambda_0} F(\lambda),$$

is also represented in the form (4.1).

Proof. Similar to the proof of Levinson's theorem, let define $\varphi(x)$ it as follows

$$\overline{\varphi(x)} = \overline{\psi(x)} + i(\mu - \lambda_0)e^{-i\lambda_0 x} \int_{-\pi}^{x} e^{i\lambda_0 x}\overline{\psi(y)}dy. \tag{4.3}$$

By multiplying both sides by $e^{i\lambda x}$ and integrating from $-\pi$ to π we obtain

$$\int_{-\pi}^{\pi} e^{i\lambda x}\overline{\varphi(x)}dx = \int_{-\pi}^{\pi} e^{i\lambda x}\overline{\psi(x)}dx + i(\mu-\lambda_0)\int_{-\pi}^{\pi} e^{i(\lambda-\lambda_0)x}\left(\int_{-\pi}^{x} e^{i\lambda_0 y}\overline{\psi(y)}dy\right)dx,$$
$$\int_{-\pi}^{\pi} e^{i\lambda x}\overline{\varphi(x)}dx = F(\lambda) + i(\mu-\lambda_0)\int_{-\pi}^{\pi} e^{i(\lambda-\lambda_0)x}\left(\int_{-\pi}^{x} e^{i\lambda_0 y}\overline{\psi(y)}dy\right)dx.$$

Changing the order of integration (Fubini's theorem), we have

$$\int_{-\pi}^{\pi} e^{i\lambda x}\overline{\varphi(x)}dx = F(\lambda) + i(\mu-\lambda_0)\int_{-\pi}^{\pi}\int_{y}^{\pi} e^{i(\lambda-\lambda_0)x}e^{i\lambda_0 y}\overline{\psi(y)}dxdy =$$

$$= F(\lambda) + \frac{\mu-\lambda_0}{\lambda-\lambda_0}\int_{-\pi}^{\pi}\left(e^{i(\lambda-\lambda_0)\pi} - e^{i(\lambda-\lambda_0)y}\right)e^{i\lambda_0 y}\overline{\psi(y)}dy =$$

$$= F(\lambda) - \frac{\mu-\lambda_0}{\lambda-\lambda_0}\int_{-\pi}^{\pi} e^{i\lambda y}\overline{\psi(y)}dy = F(\lambda) - \frac{\mu-\lambda_0}{\lambda-\lambda_0}F(\lambda) = \frac{\lambda-\mu}{\lambda-\lambda_0}F(\lambda) = F_1(\lambda),$$

i.e.

$$F_1(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda x}\overline{\varphi(x)}dx.$$

In order to represent the function $\varphi(\cdot)$ by a functional on $G^{p)}(-\pi,\pi)$, it is sufficient to show that $\varphi \in \left(G^{p)}(-\pi,\pi)\right)'$. It is clear that $\left|e^{i\lambda_0 x}\right| \le const < +\infty, \forall x \in [-\pi,\pi]$. Therefore, from the expression (4.3) for $\varphi(\cdot)$, it follows that it now suffices to prove that $\int_{-\pi}^{x}|\psi(y)|\,dy \in \left(G^{p)}(-\pi,\pi)\right)'$ but this is obvious because $\int_{-\pi}^{x}|\psi(y)|\,dy \in C[-\pi,\pi]$

The theorem is proved. □

This theorem has the following direct corollary.

Corollary 4.3. *Let the system* $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ *be complete in* $G^{p)}(-\pi;\pi)$, $1 < \varepsilon < p-1$. *If* n *arbitrary functions are removed from this system and* n *other functions* $e^{i\mu_j x}$, $j = \overline{1,n}$ *(where* $\mu_1, \mu_2, ..., \mu_n$ *are arbitrary complex numbers different from any of* λ_k*) are added instead of them, then the newly obtained system will be complete in* $G^{p)}(-\pi;\pi)$.

5. On Stability of Exponential Basis in $G^{p)}(-\pi;\pi)$

Now, let's show the stability of the bases property of an exponential system with regard to $\{\lambda_n\}_{n\in Z}$ sequence in $G^{p)}(-\pi;\pi)$

Theorem 5.1. *Let $r\in(1,\ \min(p,q))$, $\frac{1}{p}+\frac{1}{q}=1$ and $\{\lambda_n\}_{n\in Z}$, $\{\mu_n\}_{n\in Z}$ such sequences that the relation*

$$\sum_{n=-\infty}^{\infty}|\lambda_n-\mu_n|^r<+\infty,$$

holds. If the system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ forms a basis for $G^{p)}(-\pi,\pi)$, equivalent to the basis $\left\{e^{inx}\right\}_{n\in Z}$, then the system $\left\{e^{i\mu_n x}\right\}_{n\in Z}$ also forms a basis for $G^{p)}(-\pi,\pi)$, equivalent to $\left\{e^{inx}\right\}_{n\in Z}$.

Proof. Denote $\varphi_n(x)=e^{i\lambda_n x},\psi_n(x)=e^{i\mu_n x}$. We have

$$\|\varphi_n-\psi_n\|^r_{L^{p)}(-\pi,\pi)}\le C\,|\lambda_n-\mu_n|^r\,,$$

where $C>0$ is a constant independent of n. Consequently

$$\sum_{n=-\infty}^{+\infty}\|\varphi_n-\psi_n\|^r_{L^{p)}(-\pi,\pi)}<+\infty.$$

Let $\{c_n\}\in C$ be an arbitrary finite set of complex numbers. Then from the Hausdorff-Young theorem, we obtain

$$\|\{c_n\}\|_{l_{r'}}\le C\left\|\sum_n c_n e^{inx}\right\|_{L^r(-\pi,\pi)}, \tag{5.1}$$

where $C>0$ is a constant independent of $c_n$and $\frac{1}{r}+\frac{1}{r'}=1$ holds. By $C>0$ we will denote absolute constants (which can be different in different places) As the bases $\{\varphi_n\}_{n\in Z}$ and $\left\{e^{inx}\right\}_{n\in Z}$ are equivalent, from (5.1) it follows

$$\|\{c_n\}\|_{l_{r'}}\le C\left\|\sum_n c_n\varphi_n(x)\right\|_{L^r(-\pi,\pi)}.$$

For some m, n natural numbers consider

$$f_n=\begin{cases}\varphi_n,\ n=0,1,...,m-1\\ \psi_n,\ n=m+1,m+2,...\end{cases}$$

Taking into account (5.1) and the continuous embedding
$L^p\subset L^{p)}\subset L^{p-\varepsilon}\Rightarrow\|f\|_{p-\varepsilon}\le c\,\|f\|_{p)}\,,r=p-\varepsilon$, relationship we have

$$\left\|\sum_n c_n(f_n-\varphi_n)\right\|_{L^{p)}(-\pi,\pi)}\le\sum_n|c_n|\,\|f_n-\varphi_n\|_{L^{p)}(-\pi,\pi)}\le$$

$$\leq \|\{c_n\}\|_{l_{r'}} \left(\sum_n \|f_n - \varphi_n\|^r_{L^{p)}(-\pi,\pi)} \right)^{\frac{1}{r}} \leq$$

$$\leq C \left\| \sum_n c_n \varphi_n \right\|_{L^r(-\pi,\pi)} \left(\sum_{|n|\geq m} \|\psi_n - \varphi_n\|^r_{L^{p)}(-\pi,\pi)} \right)^{\frac{1}{r}}$$

$$\leq C \left\| \sum_n c_n \varphi_n \right\|_{L^{p)}(-\pi,\pi)} \left(\sum_{|n|\geq m} \|\psi_n - \varphi_n\|^r_{L^{p)}(-\pi,\pi)} \right)^{\frac{1}{r}} < +\infty. \qquad (5.2)$$

We can choose m such that it holds

$$C \left(\sum_{|n|\geq m} \|\psi_n - \varphi_n\|^r_{L^{p)}(-\pi,\pi)} \right)^{\frac{1}{r}} < 1.$$

Then from the Paley-Wiener theorem (for Banach case; see, e.g., [24, p.187]) and the relation (5.2) it follows that the system $\{f_n\}_{n\in Z}$ forms a basis for $G^{p)}(-\pi,\pi)$, equivalent to $\{\varphi_n\}_{n\in Z}$. From the completeness of the system $\{f_n\}_{n\in Z}$ in $G^{p)}(-\pi,\pi)$ and Corollary (4.3) it follows that the system $\{\psi_n\}_{n\in Z}$ is also complete in $G^{p)}(-\pi,\pi)$. Then by Statement 2.3 the system $\{\psi_n\}_{n\in Z}$ also forms a basis for $G^{p)}(-\pi,\pi)$, equivalent to $\{\varphi_n\}_{n\in Z}$.

The theorem is proved. □

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