Journal of Contemporary Applied Mathematics
V. 11, No 2, 2021, December
ISSN 2222-5498

A Unified Treatment of The Generalised Poisson Distribution with Implications for Characterization and Estimation of Properties

T.O. Obilade, S.D. Francis, S.O. Ezeah, A.R. Babalola

Abstract. Noting the wide variety in the consideration and parameter estimation of the Generalized Poison Distribution (GPD), this paper presents a unified treatment of the GPD suitable for any configuration of mean and variance, and making allowance for the q-truncated normalization factor $F_q(\lambda, \theta)$ such that $F_q(\lambda, \theta) \leq 1 \leq F_{q+1}(\lambda, \theta)$.
It is established that the GPD model satisfies the second-order linear partial differential equation (PDE):

$$\Phi^2 \left(\frac{L(\lambda, \theta)}{N} + F_q^*(\lambda, \theta) \right) = -m_1,$$

where $L(\lambda, \theta)$ and $F_q^*(\lambda, \theta)$ are the log of the likelihood and the normalization factor respectively, λ, θ are the parameters, N is the sample size and Φ^2 is the curvature operator:

$$\frac{\lambda^2 \partial^2}{\partial \lambda^2} + 2\lambda\theta \frac{\partial^2}{\partial \lambda \partial \theta} + \theta^2 \frac{\partial^2}{\partial \theta^2}$$

This suggests a PDE for which the case $\lambda > 0,\ 0 \leq \theta < 1$ manifests as a credible boundary condition. We provide an illustration for maximum likelihood estimation purposes with a copiously used data from the literature.

KeyWords and Phrases: Generalized Poisson Distribution, Normalization Factor, Moment Z Transform, Partial Differential Equation.

2010 Mathematics Subject Classifications: 6215, 62F10, 62F12

1. Introduction

Ever since [1] introduced the generalized Poisson Distribution (GPD) as a generalization of the discrete Poisson distribution, a variety of authors have made many valuable contributions on its properties and estimation procedures. The list includes [3], [4], [8], [10], as well as [12]. As pointed out by [2], the variance of the GPD can be greater (over dispersed), equal (equi dispersed), or less (under dispersed) than the mean as one of the parameters (θ) is positive, zero or negative. The model thus provides a very good fit

to a wide range of data, from biological or queuing applications for which we may have contagious, crowding or bursty data as well as actuarial or even fertility studies for which there can be some under dispersion as found with multinomial situations. The applied estimation procedures have included maximum likelihood, Bayes procedures, minimum chi-squared, weighted discrepancies, empirical rates of change as well as moment estimators and minimum variance unbiased estimators.
In this paper, we specifically allow for the q-truncated normalization factor $F_q(\lambda,\theta)$ such that $F_q(\lambda,\theta) \leq 1 \leq F_{q+1}(\lambda,\theta)$ and established that GPD can be characterized by a pair of curvature partial differential equations (PDE)

$$\frac{\lambda\partial H}{\partial\lambda} + \theta\frac{\partial H}{\partial\theta} = -(\lambda - (1-\theta)\ m_1)$$

and

$$\frac{\lambda^2\partial^2 H}{\partial\lambda^2} + 2\lambda\theta\frac{\partial^2 H}{\partial\lambda\partial\theta} + \theta^2\frac{\partial^2 H}{\partial\theta^2} = -m_1,$$

where

$$H = \frac{L(\lambda,\theta)}{N} + F_q^*(\lambda,\theta),$$

$L(\lambda,\theta)$ is the log-likelihood, $F_q^*(\lambda,\theta)$ is the log-normalization factor, $\lambda,\ \theta$ are the parameters, N is the sample size and m_1 is the first raw sample moment. This suggests a PDE for which the less problematic case $\lambda > 0$ and $0 \leq \theta < 1$ manifests as a credible boundary condition for which H is replaced by $L(\lambda,\theta)/N$.
Let $X_\infty,\ X_q$ be random variables defined by

$$\Pr\{X_\infty = x\} = f(\lambda,\theta,x) = \frac{\lambda}{x!}(\lambda + x\theta)^{x-1}e^{-(\lambda+\theta x)} \tag{1.1}$$

and (See [6]),

$$\Pr\{X_q = x\} = g(\lambda,\theta,x) = \begin{cases} \frac{f(\lambda,\theta,x)}{F_q(\lambda,\theta)} & \text{for} \quad x = 0,\ 1,\ \ldots,\ q \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

with

$$F_q(\lambda,\theta) = \Pr\{X_\infty \leq q\} = \sum_{x=0}^{q} f(\lambda,\theta,x), \qquad q = 0,\ 1,\ 2,\ \ldots \tag{1.3}$$

As noted in [9], the random variables $X_\infty,\ X_q$ thus have the generalized Poisson distribution (GPD) with $f(\lambda,\theta,x)$, appropriate only when $\lambda > 0;\ \theta \geq 0$, while $g(\lambda,\theta,x)$ is appropriate, amongst the other cases, in the proven case of under dispersion when $\lambda > 0;\ \theta < 0$, and $\max\left(-1, \frac{-\lambda}{q}\right) \leq \theta < 1$ and $q(> 4)$ is the largest positive integer for which $\lambda + \theta q \geq 0$ when $\theta < 0$. It is considered apt to adopt $g(\lambda,\theta,x)$ as a unified case with $q = \infty$ when $\theta \geq 0$ such that $F_\infty(\lambda,\theta) = 1$. It should be noted in the general case that

$$\begin{aligned} F_q(\lambda,\theta) \equiv F_q &= \sum_{x=0}^{q} \frac{\lambda(\lambda+\theta x)^{x-1}}{x!} e^{-(\lambda+\theta x)}, \\ \text{with} \qquad F_q(\lambda,\theta) &\leq 1 < F_{q+1}(\lambda,\theta) \end{aligned} \tag{1.4}$$

Suppose we introduce the following notations:

$$\mu_j = E[X^j] = \sum_{x=0}^{q} g(\lambda,\theta,x)\ x^j = j^{\text{th}} \text{ population raw moment.}$$

$m_j = \sum_{x=0}^{q} n_x\, x^j/N = j^{\text{th}}$ sample raw moment for counts, n_x for values of x and sample size N.

$$G(\mu,z) = \sum_{j=0}^{\infty} \mu_j z^j = z \text{ transform for population raw moments of, } X_q.$$

$$M(z) = \sum_{j=0}^{\infty} m_j z^j = z \text{ transform for population raw moments of } X_q.$$

Denote $G = G(\mu, -\theta/\lambda)$ and $M = M(-\theta/\lambda)$, we note as follows:

$$\sum_{x=0}^{q} \frac{g(\lambda,\theta,x)(x-1)}{\lambda+\theta x} = -\frac{1}{\theta}\left(\frac{\lambda+\theta}{\lambda}G - 1\right) \tag{1.5}$$

$$\sum_{x=0}^{q} \frac{g(\lambda,\theta,x)x(x-1)}{\lambda+\theta x} = \frac{1}{\theta^2}\left((\lambda+\theta)(G-1) + \theta\mu_1\right) \tag{1.6}$$

$$\sum_{x=0}^{q} \frac{n_x(x-1)}{\lambda+\theta x} = -\frac{1}{\theta}\left(\frac{\lambda+\theta}{\lambda}M - 1\right) N \tag{1.7}$$

$$\sum_{x=0}^{q} \frac{n_x x(x-1)}{\lambda+\theta x} = \frac{1}{\theta^2}\left((\lambda+\theta)(M-1) + \theta m_1\right) N \tag{1.8}$$

It follows that

$$\frac{\partial F^*}{\partial \lambda} = \frac{1}{F_q(\lambda,\theta)} \frac{\partial F_q(\lambda,\theta)}{\partial \lambda} = \frac{1}{\lambda\theta}\left((\lambda+\theta)(1-G) - \lambda\theta\right) \tag{1.9}$$

$$\frac{\partial F^*}{\partial \theta} = \frac{1}{F_q(\lambda,\theta)} \frac{\partial F_q(\lambda,\theta)}{\partial \theta} = \frac{1}{\theta^2}\left((\lambda+\theta)(G-1) + \theta(1-\theta)\mu_1\right) \tag{1.10}$$

$$\lambda^2 \frac{\partial^2 F_q^*(\lambda,\theta)}{\partial \lambda^2} = -(1-G) - \frac{\lambda}{\theta}(\lambda+\theta)G'_\lambda \tag{1.11}$$

$$\theta^2 \ \frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\theta^2} = -\left(\frac{2\lambda+\theta}{\theta}\right)(G-1) - \mu_1 + \theta(1-\theta)\frac{\partial\mu_1}{\partial\theta} + (\lambda+\theta)G'_\theta \tag{1.12}$$

$$\lambda\theta \ \frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\lambda\partial\theta} = \frac{\lambda}{\theta}\left(G - 1 + (\lambda+\theta)G'_\lambda + \theta(1-\theta)\frac{\partial\mu_1}{\partial\lambda}\right) \tag{1.13}$$

$$\lambda\theta \ \frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\theta\partial\lambda} = \frac{\lambda}{\theta}(G-1) - (\lambda+\theta)G'_\theta \tag{1.14}$$

2. Log-likelihood Function

The log-likelihood function $L(\lambda,\ \theta,\ n_1,\ n_2,\ \ldots,\ n_q) \equiv L(\lambda,\theta)$ for GPD is given by

$$L(\lambda,\theta) = Ln\left[\prod_{x=0}^{q}(g(\lambda,\theta,x))^{n_x}\right] \tag{2.1}$$

Differentiating 2.1 w.r.t. $\lambda,\ \theta$ we have

$$\frac{\partial L}{\partial\lambda} = N\left(\frac{1-\lambda}{\lambda}\right) - \frac{N}{F_q(\lambda,\theta)}\frac{\partial F_q(\lambda,\theta)}{\partial\lambda} + \sum_{x=0}^{q} n_x\left(\frac{x-1}{\lambda+\theta x}\right) \tag{2.2}$$

$$\frac{\partial L}{\partial\theta} = -\frac{N}{F_q(\lambda,\theta)}\frac{\partial F_q(\lambda,\theta)}{\partial\theta} + \sum_{x=0}^{q} n_x\left(\frac{x(x-1)}{\lambda+\theta x}\right) - m_1 N \tag{2.3}$$

So that, using 1.7-1.10, we have

$$\frac{\partial L}{\partial\lambda} = \left(\frac{\lambda+\theta}{\lambda\theta}\right)(G-M)\,N \tag{2.4}$$

$$\frac{\partial L}{\partial\theta} = \left(\left(\frac{\lambda+\theta}{\theta^2}\right)(M-G)\right)N + \left(\frac{1-\theta}{\theta}\right)(m_1-\mu_1)N \tag{2.5}$$

and

$$\frac{\lambda^2}{N}\ \frac{\partial^2 L}{\partial\lambda^2} = (M-G) - \frac{\lambda}{\theta}(\lambda+\theta)\left(M'_\lambda - G'_\lambda\right) \tag{2.6}$$

$$\frac{\theta^2}{N}\frac{\partial^2 L}{\partial\theta^2} = -\left(\frac{2\lambda+\theta}{\theta}\right)(M-G) + (\lambda+\theta)\left(M'_\theta - G'_\theta\right) + (\mu_1 - m_1) - \theta(1-\theta)\frac{\partial\mu_1}{\partial\theta} \tag{2.7}$$

$$\frac{\lambda\ \theta}{N}\frac{\partial^2 L}{\partial\theta\partial\lambda} = \left(\frac{\lambda}{\theta}\right)(M-G) - (\lambda+\theta)\left(M'_\theta - G'_\theta\right) \tag{2.8}$$

$$\frac{\lambda\theta}{N}\frac{\partial^2 L}{\partial\lambda\partial\theta} = \left(\frac{\lambda}{\theta}\right)(M-G) + \left(\frac{\lambda(\lambda+\theta)}{\theta}\right)\left(M'_\lambda - G'_\lambda\right) - \lambda(1-\theta)\frac{\partial\mu_1}{\partial\lambda} \tag{2.9}$$

Under the assumptions of existence of partial derivatives continuity for $F_q(\lambda,\theta)$ and $L(\lambda,\theta)$, we have

$$\frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\theta\partial\lambda} \equiv \frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\lambda\partial\theta}$$
$$\frac{\partial^2 L(\lambda,\theta)}{\partial\theta\partial\lambda} \equiv \frac{\partial^2 L(\lambda,\theta)}{\partial\lambda\partial\theta}$$

So that from Equations (1.13), (1.14), (2.8) and (2.9),

$$(\lambda+\theta)\left(\lambda G'_\lambda + \theta G'_\theta\right) = -\lambda\theta(1-\theta)\frac{\partial\mu_1}{\partial\lambda} \tag{2.10}$$

$$\lambda M'_\lambda + \theta M'_\theta = 0 \tag{2.11}$$

3. Preliminary Remarks

Remark 3.1. *The normalizing constant* $F_q(\lambda,\theta)$ *for the GPD* (λ,θ), *satisfies the partial differential equation*

$$\lambda\frac{\partial F_q(\lambda,\theta)}{\partial\lambda} + \theta\frac{\partial F_q(\lambda,\theta)}{\partial\theta} = -(\lambda-(1-\theta)\mu_1)F_q(\lambda,\theta) \tag{3.1}$$

as well as the parabolic equation

$$\lambda^2\frac{\partial^2 F_q(\lambda,\theta)}{\partial\lambda^2} + 2\lambda\theta\frac{\partial^2 F_q(\lambda,\theta)}{\partial\theta\partial\lambda} + \theta^2\frac{\partial^2 F_q(\lambda,\theta)}{\partial\theta^2} = (\lambda-(1-\theta)\mu_1)^2F_q(\lambda,\theta) - \mu_1 F_q(\lambda,\theta) + (1-\theta)\left(\lambda\frac{\partial\mu_1}{\partial\lambda} + \theta\frac{\partial\mu_1}{\partial\theta}\right)F_q(\lambda,\theta) \tag{3.2}$$

Proof. Equation 3.1 follows from Equations 1.9 and 1.10. Differentiating Equations 1.9 and 1.10 as appropriate and applying Equation 2.10, we obtain Equation 3.2 □

Remark 3.2. *The logarithm of the normalization factor* $F_q^*(\lambda,\theta)$ *for the GPD* (λ,θ), *satisfies the partial differential equation*

$$\lambda\frac{\partial F_q^*(\lambda,\theta)}{\partial\lambda} + \theta\frac{\partial F_q^*(\lambda,\theta)}{\partial\theta} = -(\lambda-(1-\theta)\mu_1) \tag{3.3}$$

as well as the parabolic equation

$$\lambda^2\frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\lambda^2} + 2\lambda\theta\frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\theta\partial\lambda} + \theta^2\frac{\partial^2 F_q^*(\lambda,\theta)}{\partial\theta^2} = -\mu_1 + (1-\theta)\left(\lambda\frac{\partial\mu_1}{\partial\lambda} + \theta\frac{\partial\mu_1}{\partial\theta}\right) \tag{3.4}$$

where

$$F_q^*(\lambda,\theta) = \log F_q(\lambda,\theta) \tag{3.5}$$

Proof. Results follow easily from Equations 1.9 to 1.13.

Furthermore, by the method of characteristics [7], Equation 3.1 can be replaced by the ordinary differential equation called characteristics form

$$\frac{dF}{d\theta} = -(\lambda - (1-\theta)E[X])\theta^{-1}F_q(\lambda,\theta) \tag{3.6}$$

along the characteristic curve obtained by solving

$$\frac{d\lambda}{d\theta} = \frac{\lambda}{\theta} \tag{3.7}$$

It follows therefore, and without prejudice to equation 1.3, that $F_q(\lambda,\theta)$ can be given in general by

$$F_q(\lambda,\theta) = f_q\left(\frac{\lambda}{\theta}\right) e^{-\int(\lambda-(1-\theta)E[X])\theta^{-1}d\theta} \tag{3.8}$$

where $f_q\left(\frac{\lambda}{\theta}\right)$ is an arbitrary function. In view of the fact that the dynamics of the generalized Poisson process may involve some natural process of mutation selection, growth, division synchronization and the like which may translate in some populations into some affine transformation of parameters, we are motivated to consider relations between processes

$$N_0 \approx \text{ GPD}(\lambda,\theta),\ N_\pm \approx \text{ GPD}(\lambda(1-\pm\alpha),\theta(1\pm\alpha))$$

and some others. We also generally denote that corresponding to $\text{GPD}(\lambda,\theta)$ are mgf $M_0(t)$ and empirical moment generating function (emgf) ψ_0, etc.
We observe that

(i.)

$$\sum_{n=0}^{\infty}\frac{\lambda}{n!}(\lambda+n\theta)^{n-1}z^n e^{-z(\lambda+n\theta)} = 1 \tag{3.9}$$

(ii.)

$$\sum_{n=0}^{\infty}\frac{\lambda}{n!}(\lambda+n\theta)^{n-1}(ze^{\theta z})^n = e^{(-\frac{\lambda}{\theta}W(-\theta ze^{-\theta z}))} \tag{3.10}$$

where $W(z)$ is the product, log or Lambert function [5] defined by $W(z)e^{W(z)} = z$ as well as $W(\theta e^{\theta}) = \theta$.

(iii.)

$$(\lambda+r\theta)(n)_r f(\lambda,\theta,n) = \lambda(\lambda+n\theta)^r f(\lambda+r\theta,\theta,n-r) \tag{3.11}$$

Of course, n_r is the r^{th} falling factorial. We state the following preliminary remarks: The moment generating function (mgf) $M_{X_\infty}(t)$ and probability generating function (pgf) $P_{X_\infty}(t)$ for the random variable X_∞ are given by [.e.g [11]]

$$M_{X_\infty}(t) = e^{-\frac{\lambda}{\theta}[W(-\theta e^{(-\theta+t)})+\theta]} \tag{3.12}$$

$$P_{X_\infty}(t) = e^{-\frac{\lambda}{\theta}[W(-\theta t e^{(-\theta+t)})+\theta]} \tag{3.13}$$

Of course,

$$M_{X_\infty}(t) = P_{X_\infty}(t)$$

It is also the case that all moments of are obtainable from $M_{X_\infty}(t)$.

(iv.) Taking summations over n as appropriate in the recurrence relations Equation 3.11, we remark that

$$E[(N_0)_r] = \frac{\lambda}{\lambda + r\theta} \frac{F_{q-r}(\lambda + r\theta, \theta)}{F_q(\lambda, \theta)} E[(\theta N_r + \lambda + r\theta)^r] \tag{3.14}$$

where $N_0 \approx$ GPD(λ, θ) and $N_r \approx$ GPD$(\lambda + r\theta, \theta)$. Of course, $(N_0)_r$ denotes $N_0(N_0 - 1), \ldots, (N_0 - r + 1)$.

(v.) Similarly, with $N_{-r} \approx$ GPD$(\lambda - r\theta, \theta)$, we have

$$E[(N_{-r})_r] = \frac{\lambda - r\theta}{\lambda} \frac{F_q(\lambda, \theta)}{F_{q-r}(\lambda - r\theta, \theta)} E[(\theta N_0 + \lambda)^r] \tag{3.15}$$

$$E[(N_{-r_1})_{r_1+r_2}] = \frac{\lambda - r_1\theta}{\lambda + r_2\theta} \frac{F_{q-r_2}(\lambda + r_2\theta, \theta)}{F_{q+r_1}(\lambda + r_1\theta, \theta)} E[(\theta N_{r_2} + \lambda + r_2\theta)^{r_1+r_2}] \tag{3.16}$$

(vi.) Further, putting $r \equiv q$ in Equation 3.14 and $r_2 \equiv q$ in Equation 3.16, we obtain

$$F_{q+r}(\lambda - r\theta, \theta) = e^{\lambda + q\theta} \frac{\lambda - r\theta}{\lambda + q\theta} \frac{E[(\theta N_q + \lambda + q\theta)^{r+q}]}{E[(N_{-r})_{r+q}]} \tag{3.17}$$

as well as

$$F_q(\lambda, \theta) = e^{\lambda + q\theta} \frac{\lambda}{\lambda + q\theta} \frac{E[(\theta N_q + \lambda + q\theta)^q]}{E[(N_0)_q]} \tag{3.18}$$

so that

$$F_{q+r}(\lambda - r\theta, \theta) = \frac{\lambda - r\theta}{\lambda} \frac{E[(\theta N_q + \lambda + q\theta)^{r+q}] E[(N_0)_q]}{E[(\theta N_q + \lambda + q\theta)^q] E[(N_{-r})_{r+q}]} F_q(\lambda, \theta)$$

(vii.) Taking summations over n as appropriate in the recurrence relations Equation 3.12, we remark that for

$$N_0 \approx \text{GPD}(\lambda, \theta),\ N_{-\alpha} \approx \text{GPD}(\lambda(1-\alpha), \theta(1-\alpha))$$
$$\text{and } N_\beta \approx \text{GPD}(\lambda(1+\beta), \theta(1+\beta))$$

respectively with mgf $M_0(t)$, $M_{-\alpha}(t)$ and $M_\beta(t)$, we have

$$F_q(\lambda, \theta) M_0(In(1-\alpha)) = e^{-\lambda\alpha} M_{-\alpha}(-\theta\alpha) F_q(\lambda(1-\alpha), \theta(1-\alpha)) \tag{3.19}$$

$$F_q(\lambda, \theta) M_0(-\theta\beta) = e^{\lambda\beta} M_\beta(-In(1+\beta)) F_q(\lambda(1+\beta), \theta(1+\beta)) \tag{3.20}$$

and Equations 3.14 - 3.20 gives the recurrence relation for the normalization factors involving parameters λ and θ and their affine transformations.

□

4. Main Results

Theorem 4.1. *The log-likelihood* $L(\lambda,\theta)$ *satisfies the partial differential equation*

$$\frac{1}{N}\left(\lambda\frac{\partial L}{\partial\lambda}+\theta\frac{\partial L}{\partial\theta}\right)=(1-\theta)(m_1-\mu_1) \tag{4.1}$$

as well as the parabolic equation

$$\frac{1}{N}\left(\lambda^2\frac{\partial^2 L}{\partial\lambda^2}+2\lambda\theta\frac{\partial^2 L}{\partial\theta\partial\lambda}+\theta^2\frac{\partial^2 L}{\partial\theta^2}\right)=(\mu_1-m_1)-(1-\theta)\left(\lambda\frac{\partial\mu}{\partial\lambda}+\theta\frac{\partial\mu_1}{\partial\theta}\right) \tag{4.2}$$

Proof. Equation 4.1 follows from Equations 2.4 and 2.5. Equation 4.2 follows from Equations 2.6 - 2.8 and applying Equations 2.10 and 2.11. □

Theorem 4.2. *The logarithm of the normalized constant* $F_q^*(\lambda,\theta)$ *and the log-likelihood* $L(\lambda,\theta)$ *are connected by the partial differential equation*

$$\lambda\frac{\partial}{\partial\lambda}\left(\frac{L(\lambda,\theta)}{N}+F_q^*(\lambda,\theta)\right)+\theta\frac{\partial}{\partial\theta}\left(\frac{L(\lambda,\theta)}{N}+F_q^*(\lambda,\theta)\right)=-(\lambda-(1-\theta)m_1) \tag{4.3}$$

as well as the second order linear partial differential equation

$$\Phi^2\left(\frac{L(\lambda,\theta)}{N}+F_q^*(\lambda,\theta)\right)=-m_1 \tag{4.4}$$

where $\Phi^2(.)$ *is the operator* $\lambda^2\frac{\partial^2}{\partial\lambda^2}+2\lambda\theta\frac{\partial^2}{\partial\theta\partial\lambda}+\theta^2\frac{\partial^2}{\partial\theta^2}$.

Proof. Equation 4.3 follows from adding Equations 3.3 and 4.1. Equation 4.4 follows from adding Equations 3.4 and 4.2. □

Remark 4.3. *Equations 4.3 and 4.4 give a characteristics pair of curvature equation for the unified* GPD *model defined in Equation 1.1. For the* GPD, X_∞ *for which* $F_{\infty(\lambda,\theta)}=1$ *and* $F^*_{\infty(\lambda,\theta)}=0$, *equation 4.3 and 4.4 reduce to*

$$\lambda\frac{\partial L}{\partial\lambda}+\theta\frac{\partial L}{\partial\theta}=-(\lambda-(1-\theta)m_1)$$

and

$$\Phi^2\left(\frac{L(\lambda,\theta)}{N}\right)=-m_1 \tag{4.5}$$

Equation 4.5 will be valid for all cases where $\lambda>0$ *and* $0\le\theta<1$. *This suggests that one only needs to replace* $\frac{L(\lambda,\theta)}{N}$ *with* $\frac{L(\lambda,\theta)}{N}+F_q^*(\lambda,\theta)$ *estimation purposes in the general case. We remark that a solution to Equation 4.4 must involve certain boundary conditions. It follows from Equations 3.1 and 3.3 that the idea that the first population mean is given by*

$$\mu_1=\lambda/(1-\theta)$$

can be taken as a mere necessary boundary condition for the solution to the second order linear partial differential equation 4.4 on the curve $\theta = 0$ *for which* $F_q^*(\lambda, \theta) = 0$. *However, for Equation 4.3,a maximum likelihood estimator for the population mean is given by*

$$m_1 = \lambda/(1-\theta).$$

5. MLE Numerical Illustration

In clear terms, the appropriate MLE equation for the unified system are:

$$\begin{aligned}\frac{\partial H}{\partial \lambda} &= \left(\frac{\lambda+\theta}{\lambda\theta}\right)\left(1 - M\left(\frac{-\theta}{\lambda}\right)\right) - 1 \\ &= \left(\frac{1-\lambda}{\lambda}\right) + \frac{1}{N}\sum_{x=2}^{q}\frac{(x-1)n_x}{\lambda+\theta x},\end{aligned}$$

$$\begin{aligned}\frac{\partial H}{\partial \theta} &= \left(\frac{\lambda+\theta}{\theta^2}\right)\left(M\left(\frac{-\theta}{\lambda}\right) - 1\right) + \left(\frac{1-\theta}{\theta}\right)m_1 \\ &= -\left(\frac{1}{\theta}\right)(1-(1-\theta)m_1) - \frac{\lambda}{\theta N}\sum_{x=2}^{q}\frac{(x-1)n_x}{\lambda+\theta x},\end{aligned}$$

with

$$\hat{\lambda} = (1-\hat{\theta})m_1.$$

Table 1 gives a goodness of fit analysis of the data on number of Dicentrics per cell as reproduced by [4] for 600 doses of rad. Our model gave $\hat{\lambda} = 1.7568, \hat{\theta} = -0.1261$ as did [4].

6. Conclusion

In this note we have, by a proper weighing with a $q-$ truncated normalization constant $F_q(\lambda, \theta)$, considered a unified model of the Generalized Poisson Distribution as applicable for all configurations of parameters λ and θ. Some basic recurrence relations on the factor $F_q(\lambda, \theta)$ are presented. We have expressed all relevant derivatives for the maximum likelihood estimation of the parameters in terms of the raw moments z transform $M(z)$. We have also derived a modified characterization of the model. As these relations have implications for hypothesis testing and estimation procedures, these contributions promote a better and less confusing utilization of the Generalized Poisson Distribution for testing and fitting purposes.

Table 1: Fitted Values and Observed Number of Dicentrics ([4])

	600 Rad	
No of Dicentrics	Observed	Fitted
0	27	26.1152
1	50	52.0451
2	45	44.4139
3	18	21.2089
4	10	6.2169
5	-	-
χ^2		2.5056

Acknowledgement: The authors hereby acknowledge the editors for their helpful comments on an initial draft of this paper.

References

[1] P. C. Consul and G. C. Jain. *A Generalization of Poisson Distribution.* Technometrics, **15** (4), 791-799, 1973a.

[2] P. C. Consul and G. C. Jain. *Some Interesting Properties of the Generalized Poisson Distribution.* Biometrics, **15**, 495-500, 1973b.

[3] P. C. Consul and M. M. Shoukri. *The Generalized Poisson Distribution when the sample size is larger than the variance.* Commu. statist.simulation and computation, **14** (3), 667-681, 1985.

[4] P. C. Consul and F. Famoye. *Maximum Likelihood Estimate for the Generalized Poisson Distribution when the Sample Mean is Larger than variane.* Commu. statist.simulation and computation, **14** (3), 667-681, 1988.

[5] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffery and D. E. Knuth. *On the Lambert W Function.* Advances in Computational Mathematics, **3**, 329-359, 1996.

[6] F. Famoye and P. C. Cosul. *Interval Estimation and Hypothesis Testing for the Generalized Poisson Distribution.* Advances in Computational Mathematics, **10**, 127-158, 1990.

[7] A. Jeffrey. *Partial Differential Equations and Special Functions.* Handbook of Mathematical Formulas and Integrals, academic Press, London, 231-234, 1995.

[8] K. A. Mir and M. Ahmad. *On Size-Based Generalized Poisson Distribution.* Pak. J. Statist, **29** (3), 261-270, 2013.

[9] T. O. Obilade, S. D. Francis and A. O. Adelodun. *A Note of the Goodness-of-fit Tests for the Generalized Poisson Distribution Based on the Empirical Moment Generating Function.* European Journal of Scientific Research, **11** (1), 37-44, 2005.

[10] D. P. M. Scollink. *On the Analysis of the Truncated Generalized Poisson Distribution Using a Bayesian Method.* Astin Bulletin, **28** (1), 135-152, 1998.

[11] R. Vernic. *A Multivariate Generalization of the Generalized Poisson Distribution.* Astin Bulletin, **30** (1), 57-67, 2000.

[12] Y. S. Wagh, K. K. Kamalja. *Simulation and Computation.* Communications in Statistics, **46** (5), 4098-4112, 2017.

T.O. Obilade
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria
E-mail: tobilade@yahoo.com

S.D. Francis
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria
E-mail: tobilade@yahoo.com

S.O. Ezeah
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria
E-mail: sandra_adioha@yahoo.co.uk

A.R. Babalola
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria
E-mail: boyebabalola@gmail.com

Received 01 June 2021
Accepted 07 August 2021