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Existence of Solution of Nonlinear Bridge Problem with Time-delay in Aerodynamic Resistance Force

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Abstract. In this paper we consider the mathematical model of the nonlinear bridge problem with a strong delay in linear aerodynamic resistance force. The existence and uniqueness of the solution is investigated by modelling this problem as the Cauchy problem for an operator coefficient equation in a certain space.

Key Words and Phrases: nonlinear bridge problem, aerodynamic resistance force, delay, mixed problem.
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1. Introduction

A suspension bridge is a general construction of civil engineering structure. Many papers have been devoted to the modelling of suspension bridges, for instance, in [1], Lazer and McKenna studied the problem of nonlinear oscillation in a suspension bridge. They introduced a (one-dimensional) mathematical model for the bridge that takes into account of the fact that the coupling provided by the stays connecting the main cable to the deck of the road bed is fundamentally nonlinear, that is, they gave rise to the some system of semi linear hyperbolic equation (see[2-3]), where the first equation describes the vibration of the road bed in the vertical plain and the second equation describes that of the main cable from which the road bed is suspended by the tie cables. In recently, there has been a growing interest in this area.

There are many references to study the existence and asymptotic behavior of solutions for the mathematical model of suspension bridge, (see [4-10] and references there in). In recent studies, the existence of global minimal attractors of dynamic systems created by these problems has been considered (see [3-10]). See also [11] for the existence of global attractors for the coupled system of suspension bridge equations in the case of the tensioning cable has one common point with the roadbed.

In recent years, the control of partial differential equations with time delay effects has become the most requested area of research. Time delays arise in many applications, because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally not only depend on the present state but also on some past occurrences. In many cases it was shown that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used[12- 15].

In this paper, we consider the corresponding mixed problem with a strong delay in linear aerodynamic resistance force. We prove the theorem on the existence and uniqueness of the solution of considered problem.

2. Statement of the problem. Existence and uniqueness of the solution.

We consider the following mathematical model for the oscillations of the bridge with strong delay

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + [u-v]_+ + \lambda_1 u_t(x,t) + \\ \quad + \lambda_2 u_t(x,t-\tau_1) + g_1(u,v) = 0, \\ v_{tt}(x,t) - v_{xx}(x,t) - [u-v]_+ + \mu_1 v_t(x,t) + \\ \quad + \mu_2 v_t(x,t-\tau_2) + g_2(u,v) = 0 \end{cases} \tag{2.1}$$

where $u(x,t)$ is state function of the road bed and $v(x,t)$ is that of the main cable; $\tau_1,\ \tau_2 > 0$ represents the time delay, $\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2$ are real numbers such that $|\lambda_2| < \lambda_1$, $|\mu_2| < \mu_1$, and $g_1(u,v) = |u|^{p-1}|v|^{p+1}u,\ g_2(u,v) = |u|^{p+1}|v|^{p-1}v$, $p \geq 1$. Here $[a]_+ = \max\{a, 0\}$.

Let's define the following initial and boundary conditions for the system (2.1).

$$\begin{cases} u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = v(0,t) = v(l,t) = 0,\ t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,l), \\ u_t(x,t-\tau_1) = f_{01}(x,t-\tau_1), \quad x \in (0,l),\ t \in (0,\tau_1), \\ v(x,0) = v_0(x), v'(x,0) = v_1(x),\ x \in (0,l), \\ v_t(x,t-\tau_2) = f_{02}(x,t-\tau_2), \quad x \in (0,l),\ t \in (0,\tau_2). \end{cases} \tag{2.2}$$

In order to establish the existence of a unique solution to (2.1)-(2.2), we introduce the new variables(see [12-15]):

$$z_1(x,\rho,t) = u_t(x,t-\tau_1\rho), \rho \in (0,1), x \in (0,l), t > 0$$

$$z_2(x,\rho,t) = v_t(x,t-\tau_2\rho), \rho \in (0,1), x \in (0,l), t > 0$$

Obviously, z_1 and z_2 are solutions to the following problems:

$$\begin{cases} \tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0, \quad \rho \in (0,1),\ x \in (0,l),\ t > 0 \\ z_1(x,\rho,0) = f_{01}(x,-\rho\tau_1), \quad x \in (0,l), \rho \in (0,1) \\ \tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, \quad \rho \in (0,1), x \in (0,l), t > 0, \\ z_2(x,\rho,0) = f_{02}(x,-\rho\tau_2), \quad x \in (0,l), \rho \in (0,1) \end{cases} \tag{2.3}$$

So, problem (2.1)-(2.3) takes the form

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + [u-v]_+ + \lambda_1 u_t(x,t) + \lambda_2 z_1(x,1,t) = 0, in\,(0,1)\times(0,\infty), \\ v_{tt}(x,t) - v_{xx}(x,t) - [u-v]_+ + \mu_1 v_t(x,t) + \mu_2 z_2(x,1,t) = 0, \quad in\ \ (0,1)\times(0,\infty), \\ \tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0, in\,(0,l)\times(0,1)\times(0,\infty), \\ \tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, in\,(0,l)\times(0,l)\times(0,\infty), \end{cases} \tag{2.4}$$

$$
\begin{cases}
u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \\
v(0,t) = v(l,t) = 0, \\
z_1(0,\rho,t) = z_1(l,\rho,t) = 0, \\
z_2(0,\rho,t) = z_2(l,\rho,t) = 0,
\end{cases} \tag{2.5}
$$

$$
\begin{cases}
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,l), \\
z_1(x,\rho,0) = f_{01}(x,-\rho\tau_1), \quad x \in (0,l), \quad \rho \in (0,1), \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (0,l), \\
z_2(x,\rho,0) = f_{02}(x,-\rho\tau_2), \quad x \in (0,l), \ \rho \in (0,1)
\end{cases} \tag{2.6}
$$

For investigating the problem (2.4)-(2.6), we introduce the following notations:

$$
\mathrm{H^k}(a,\mathrm{b}) = \left\{ y : \ y, y', ..., y^{(k)} \in L_2(a,b) \right\},
$$

$$
\widehat{\mathrm{H}}^{\mathrm{k}}(a,\mathrm{b}) = \left\{ y : \quad y \in \mathrm{H^k}(a,\mathrm{b}), \quad y^{(2s)}(a) = y^{(2s)}(b) = 0, \ s = 0,1,..., \left[\frac{k}{2}\right] \right\},
$$

where $[r]$ is the integer part of the number r . We will denote the space $\widehat{\mathrm{H}}^{\mathrm{k}}(0,l)$ as $\widehat{\mathrm{H}}^{\mathrm{k}}$. We introduce the following space:

$$
\mathcal{H} = \widehat{\mathrm{H}}^2 \times L_2(0,l) \times \widehat{\mathrm{H}}^1 \times L_2(0,l) \times L_2((0,1)\times(0,l)) \quad \times L_2((0,1)\times(0,l)),
$$

equipped with the scalar product

$$
\langle \omega, \widetilde{\omega} \rangle = \int_0^l u_{1xx}\tilde{u}_{1xx}dx + \int_0^l u_2\tilde{u}_2dx + \int_0^l u_3\tilde{u}_3dx + \int_0^l u_4\tilde{u}_4dx +
$$

$$
+\tau|\lambda_2| \int_0^l \int_0^1 z_1\tilde{z}_1 d\rho dx + \tau|\mu_2| \int_0^l \int_0^1 z_2\tilde{z}_2 d\rho dx,
$$

for all $\omega = (u_1, \ u_2, u_3, u_4, \ z_1, z_2)^T$, $\widetilde{\omega} = (\ \tilde{u}_1, \ \tilde{u}_2, \ \ \tilde{u}_3, \ \ \tilde{u}_4, \ \tilde{z}_1, \widetilde{z_2})^T \in \mathcal{H}$.

Let's define the following operators A_0, $A_1(\cdot)$ and $F(\cdot)$ in the space $\mathcal{H}$. *The linear operator* A_0 is defined by

$$
A_0\omega = \begin{pmatrix}
-u_2 \\
u_{1xxxx} + \lambda_1 u_2 + \lambda_2 z_1(\cdot,1) \\
-u_4 \\
-u_{3xx} + \mu_1 u_4 + \mu_2 z_2(\cdot,1) \\
\frac{1}{\tau_1} z_{1\rho} \\
\frac{1}{\tau_2} z_{2\rho}
\end{pmatrix},
$$

with domain

$$
D(A_0) = \Big\{ \omega : \ \omega = (u_1,u_2,u_3,u_4,z_1,z_2)^T \in \mathcal{H}, \ u_1 \in \widehat{\mathrm{H}}^4, \quad u_2 \in \widehat{\mathrm{H}}^2, \quad u_3 \in \widehat{\mathrm{H}}^2,
$$

$$
u_4 \in \widehat{\mathrm{H}}^1, \quad z_i, z_{i\rho} \in L_2((0,1)\times(0,l)), \ z_i(\cdot,1) \in L_2((0,1)\times(0,l)), \ i = 1,2 \quad \Big\}.
$$

The nonlinear operators $A_1(\cdot)$ and $F(\cdot)$, acting from $\mathcal{H}$ *into the space* $\mathcal{H}$*, are respectively defined as*

$$A_1(\omega) = \begin{pmatrix} 0 \\ [u_1 - u_3]_+ \\ 0 \\ -\ [u_1 - u_3]_+ \\ 0 \\ 0 \end{pmatrix}, F(\omega) = \begin{pmatrix} 0 \\ g_1(u,v) \\ 0 \\ g_2(u,v) \\ 0 \\ 0 \end{pmatrix}.$$

Let $u_1(t) = u(\cdot,t)$, $u_2(t) = u_t(\cdot,t)$, $u_3(t) = v(\cdot,t)$, $u_4(t) = v_t(\cdot,t)$,
$z_1(t) = z_1(\cdot,t), z_2(t) = z_2(\cdot,t)$ and denote by

$$\omega = \omega(t) = (u_1(t),\ u_2(t), u_3(t), u_4(t),\ z_1(t), z_2(t))^T,$$

$$\omega(0) = \omega_0 = (u_{10},\ u_{20},\ u_{30}, u_{40},\ z_1(\cdot - \rho\tau), z_2(\cdot - \rho\tau)).$$

Then the problem (2.4)-(2.6) can be rewritten as an initial- value problem:

$$\begin{cases} \omega' + A_0\omega + A_1(\omega) + F(\omega) = 0 \\ \omega(0) = \omega_0 \end{cases} \tag{2.7}$$

We have the following existence and uniqueness result:

Theorem 2.1. *Assume that*

$$|\lambda_2| \le \lambda_1 \, and \, |\mu_2| \le \mu_1. \tag{2.8}$$

Then for any $\omega_0 \in \mathcal{H}$*, the problem* (2.7) *has a unique solution*

$$\omega(\cdot) \in \mathrm{C}(\,[0,+\infty)\,,\ \mathcal{H}\ ).$$

Moreover, if $\omega_0 \in D(A_0)$, the solution of (2.7) satisfies

$$\omega(\cdot) \in \mathrm{C}^1(\,[0,+\infty)\,,\ \mathcal{H}\ ) \cap \mathrm{C}(\,[0,+\infty)\,,\ D(A_0)\ )\ .$$

Before proving the theorem, we will investigate the auxiliary problem as in [16-18, 21]. For this purpose, we start to assign the following nonlinear operator for each positive K>0:

$$F_K(\omega) = \begin{pmatrix} 0 \\ g_{1K}(u,v) \\ 0 \\ g_{2K}(u,v) \\ 0 \\ 0 \end{pmatrix}$$

where

$$g_{iK}(u,v) = \begin{cases} g_i(u,v), & u^2 + v^2 < K^2, \\ g_i(K\frac{u}{\sqrt{u^2+v^2}}, K\frac{v}{\sqrt{u^2+v^2}}), & u^2 + v^2 \ge K^2. \end{cases}$$

At first, let's take a look at the following problem:

$$\begin{cases} \omega' + A_0\omega + A_1(\omega) + F_K(\omega) = 0 \\ \omega(0) = \omega_0 \end{cases} \tag{2.9}$$

To solve the problem (2.9), we should prove the following Lemmas to use the known results for the operator equations in the monograph [19].

Lemma 2.2. A_0 *is maximal dissipative operator.*

Proof. We start by showing that $(-A_0)$ is dissipative. So, for $\omega = (u_1, u_2, u_3, u_4, z_1, z_2)^T \in D(A_0)$, we have

$$\langle A_0\omega, \omega\rangle = -\int_0^l u_{1xx}u_{2xx}dx + \int_0^l u_2\left(u_{1xxxx} + \lambda_1 u_2 + \lambda_2 z_1(\cdot, 1)\right)dx -$$

$$-\int_0^l u_{3x}u_{4x}dx + \int_0^l u_4\left(-u_{3xx} + \mu_1 u_4 + \mu_2 z_2(\cdot, 1)\right)dx +$$

$$+|\lambda_2|\int_0^l\int_0^1 z_1 z_{1\rho}d\rho dx + |\mu_2|\int_0^l\int_0^1 z_2 z_{2\rho}d\rho dx =$$

$$= \lambda_1\int_0^l |u_2|^2 dx + \lambda_2\int_0^l u_2 z_1(\cdot, 1)dx +$$

$$+\mu_1\int_0^l |u_4|^2 dx + \mu_2\int_0^l u_4 z_2(\cdot, 1)dx + \frac{|\lambda_2|}{2}\int_0^l |z_1(\cdot, 1)|^2 dx -$$

$$-\frac{|\lambda_2|}{2}\int_0^l |u_2|^2 dx + \frac{|\mu_2|}{2}\int_0^l |z_2(\cdot, 1)|^2 dx - \frac{|\mu_2|}{2}\int_0^l |u_4|^2 dx. \tag{2.10}$$

Using Young's inequality for the second and fourth term of (2.10), we arrive at

$$\langle A_0\omega, \omega\rangle \geq (\lambda_1 - |\lambda_2|)\int_0^l |u_2|^2 dx + (\mu_1 - |\mu_2|)\int_0^l |u_4|^2 dx \geq 0$$

by virtue of (2.8).
Next, we show that $(-A_0)$ is maximal dissipative operator. That is, for each $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$ *we have to find* $\omega \in D(A)$ such that

$$\omega + A_0\omega = F, \tag{2.11}$$

i.e.,

$$\begin{cases} u_1 - u_2 = f_1 \\ u_2 + u_{1xxxx} + \lambda_1 u_2 + \lambda_2 z_1(\cdot, 1) = f_2 \\ u_3 - u_4 = f_3 \\ u_4 - u_{3xx} + \mu_1 u_4 + \mu_2 z_2(\cdot, 1) = f_4 \\ \tau_1 z_1 + z_{1\rho} = \tau_1\ f_5 \\ \tau_2 z_2 + z_{2\rho} = \tau_2\ f_6 \end{cases} \tag{2.12}$$

$$u_1(0) = u_1(l) = u_{1xx}(0) = u_{1xx}(l) = 0,$$

$$u_2(0) = u_2(l) = 0,$$

$$u_3(0) = u_3(l) = 0,$$

$$u_4(0) = u_4(l) = 0,$$

$$z_1(\cdot, 0) = u_2 \ = u_1 - f_1$$

$$z_2(\cdot, 0) = u_4 = u_3 - f_3.$$

The first and fifth equations of (2.12) give

$$z_1 (x, \rho, t) = \tau_1 \ \ e^{-\tau_1 \rho} \int_0^\rho f_5 \ (\theta, \cdot) \ e^{\theta \tau_1} d\theta + (u_1 - f_1) \ e^{-\rho \tau_1} \tag{2.13}$$

Analogously, the third and sixth equations of (2.12) give

$$z_2 (x, \rho, t) = \tau_2 \ \ e^{-\tau_2 \rho} \int_0^\rho f_6 \ (\theta, \cdot) \ e^{\theta \tau_2} d\theta + (u_3 - f_3) \ e^{-\rho \tau_2} \tag{2.14}$$

Replacing $u_2 = u_1 - f_1$, $u_4 = u_3 - f_3$,

$$z_1 (\cdot, 1) = \tau_1 \ \ e^{-\tau_1} \int_0^1 f_5 \ (\theta, \cdot) \ e^{\theta \tau_1} d\theta + (u_1 - f_1) \ e^{-\tau_1}$$

and

$$z_2 (\cdot, 1) = \tau_2 \ \ e^{-\tau_2} \int_0^1 f_6 \ (\theta, \cdot) \ e^{\theta \tau_2} d\theta + (u_3 - f_3) \ e^{-\tau_2}$$

from (2.12), we obtain that

$$\begin{cases} \kappa_1 u_1 + \partial^4 u_1 = F_2, & u_1 \in \widehat{\mathrm{H}}^4 \\ \kappa_2 u_3 - \partial^2 u_3 = F_4, & u_3 \in \widehat{\mathrm{H}}^2 \end{cases} \tag{2.15}$$

where

$$\begin{cases} \kappa_1 = 1 + \lambda_1 + \lambda_2 e^{-\tau_1} > 0 \\ \kappa_2 = 1 + \mu_1 + \mu_2 e^{-\tau_2} > 0 \\ F_2 = f_2 + \kappa_1 f_1 - \lambda_2 \tau_1 \ \ e^{-\tau_1} \int_0^1 f_5 \ (\theta, \cdot) \ e^{\theta \tau_1} d\theta \\ F_4 = f_4 + \kappa_2 f_3 - \mu_2 \tau_2 \ \ e^{-\tau_2} \int_0^1 f_6 \ (\theta, \cdot) \ e^{\theta \tau_2} d\theta \end{cases}$$

Let's define the following bilinear form:

$$B(U, \ W) = \int_0^l u_{1xx} w_{1xx} dx + \int_0^l u_{3x} \tilde{w}_{1x} dx + \kappa_1 \int_0^l u_1 w_1 dx + \kappa_2 \int_0^l u_3 \tilde{w}_1 dx,$$

and the linear form

$$L(W) = \langle F, W \rangle ,$$

Where $U = (u_1, u_3)$, $W = (w_1,\ \tilde{w}_1)$ and $F = (F_2, F_4\ )$. Let's show that $B(U,\ W)$ and $L(W)$ satisfy the conditions of Lax-Milgram theorem. By using Holder's inequality, we obtain that

$$B(U,\ W) \le 2\int_0^l |u_{1xx}|^2 dx + 2\int_0^l |w_{1xx}|^2 dx + 2\int_0^l |u_{3x}|^2 dx + 2\int_0^l |\tilde{w}_{1x}|^2 dx +$$

$$+2\kappa_1 \int_0^l |u_1|^2 dx + 2\kappa_1 \int_0^l |w_1|^2 dx + 2\kappa_2 \int_0^l |u_3|^2 dx + 2\kappa_2 \int_0^l |\tilde{w}_1|^2 dx \le C \cdot \left(\|U\|^2 + \|\ W\|^2 \right)$$

So, $B(U,\ W)$ is continuous.
In addition, there is such a positive constant C_0 that the following inequality

$$B\,(U, U) = \int_0^l |u_{1xx}|^2 dx + \int_0^l |u_{3x}|^2 dx + \kappa_1 \int_0^l |u_1|^2 dx +$$

$$+\kappa_2 \int_0^l |u_3|^2 dx \ge C_0 \|U\|^2$$

holds, i.e. $B(U,\ W)$ is coercive. These calculations show that B and L satisfy the conditions of Lax-Milgram theorem.
A simple calculation shows that B and L satisfy the conditions of Lax-Milgram theorem. So, there exists a unique $U = (u_1, u_3\ ) \in \widehat{\mathrm{H}}^2 \widehat{\mathrm{H}}^1$ satisfying

$$B(U,\ W) = L(W), \quad \forall\ W \in \widehat{\mathrm{H}}^2 \widehat{\mathrm{H}}^1 \tag{2.16}$$

Consequently, $u_2 = u_1 - f_1 \in \widehat{\mathrm{H}}^2$, $u_4 = u_3 - f_3 \in \widehat{\mathrm{H}}^1$and

$$z_1\,(\cdot, \rho)\,, z_2\,(\cdot, \rho)\,,\ z_{1\rho}\,(\cdot, \rho)\,,\ z_{2\rho}\,(\cdot, \rho) \in L_2(0, l).$$

Using (2.11) and (2.12), we get $z_1\,(\cdot, \rho)$, $z_2\,(\cdot, \rho) \in L_2((0, l) \times (0, l))$. Thus, (2.7) has a unique solution $\omega = (u_1,\ u_2, u_3, u_4,\ z_1, z_2)^T \in \mathcal{H}$. □

Lemma 2.3. *The nonlinear operators $A_1(\cdot)$ and $F_K(\cdot)$ satisfy Lipschitz condition.*

Proof. Since $\left|[\alpha]_+ -\ [\beta]_+\right| \le |\alpha - \beta|$ for any α, β, we get

$$\|A_1(\omega_2)\ -\ A_1(\omega_1)\|_{\ \mathcal{H}}\ \le\ D(A_0).$$

On the other side, by using the inequality

$$|g_1(\hat{u}, \hat{v}) -\ g_1(u, v)| \le c(u, \widehat{u}, v, \hat{v})\,[|\widehat{u}\ - u| + |\hat{v} - v|]$$

we derive that

$$\|F_{iK}(\omega_2)\ -\ F_i(\omega_1)\|_{\ \mathcal{H}} \le c(K)\ \|\omega_2 - \omega_1\|_{\ \mathcal{H}}.$$

□

3. Proof of the thorem

According to the Lemma 1 and Lemma 2, for each K>0, the operator $A_0 + A_1(\cdot) + F_K(\cdot)$ satisfies all conditions of Theorem 4.1 in [19] (see also [20]) . That's why for any $\omega_0 \in \mathcal{H}$, there is a unique solution $\omega(\cdot)\in C([0, +\infty), \mathcal{H})$ of the problem (2.9). So that, if $\omega_0 \in D(A_0)$, then $\omega(,)\in C^1([0, +\infty), \mathcal{H})\cap C([0, +\infty), D(A_0))$.
As $A_0 + A_1(\cdot) + F_K(\cdot)$ is maximal monotone operator, from (2.9) we obtain

$$\|\omega(t)\|^2_{\mathcal{H}} \leq \|\omega_0\|^2_{\mathcal{H}} e^{2[\omega+C(K)]t}.$$

It follows that when $\|\omega_0\|_{\mathcal{H}} < K$, the inequality $\|\omega(t)\|^2_{\mathcal{H}} < K^2$ holds true on the interval $0 \leq t \leq T'$ for $T' = T(\|\omega_0\|_{\mathcal{H}}) = \frac{1}{\omega+C(K)} Ln \frac{K}{\|\omega_0\|_{\mathcal{H}}}$. Thus, on the interval $0 \leq t \leq T'$, $F_K(\omega) = F(\omega)$ is true, i.e., $\omega(t)$ is the solution of the problem (2.7) at the same interval. As can be seen, if $\|\omega(t)\|_{\mathcal{H}} \leq c$ on the interval $0 \leq t \leq T'$, the solution can be continued to the domain $0 \leq t < +\infty$.

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