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Invese Problem for the Eıgenvalues of the Paulı Operator

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Abstract. In the paper we consider inverse problem for the eigenvalues of the Pauli operator. By the given set of the defined $s-$functions a scheme is proposed to reconstruct the domain of the problem.

Key Words and Phrases: Pauli operator, eigenvalues, spectral data, support function.

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1. Introduction

The solution of inverse problems is one of the important problems of the mathematical physics. In addition to theoretical interest, these problems also have important applied applications. In traditional inverse spectral problems, by the given experimental data (scattering data, normalization numbers, etc.), the potential is determined or necessary and sufficient conditions are established that provide the unique determination of the sought functions [10, 11].

In contrast to these problems, the inverse spectral problem with respect to the domain has a different specificity. First, in such problems it is required to define not the function, but the domain. Second, the selection of data (observation results) sufficient to define the domain is in itself a very difficult problem [1, 2].

Note that the existence problems of such problems have been studied by many authors. It was shown in that these problems are well-posed if the set of feasible domains satisfies certain geometric requirements, for example, consists of open sets [3].

Here we introduce the definition of s-functions, depending on the spectral data of the Pauli operator and then by the given set of these functions give a scheme to reconstruct the domain of the problem.

Main results. Consider the problem

$$P\varphi = \lambda\varphi \,, \qquad x \in D \,, \tag{1.1}$$

$$\varphi = 0 \,, \qquad x \in S_D \,. \tag{1.2}$$

Here P is the Pauli operator defined as follows [4]

$$P = P(a,\,\nu)\cdot J + \sigma B, \tag{1.3}$$

where D is bounded convex domain from R^2, $S_D \in C^2$ is its smooth boundary. We assume that $q\,(x)$ is positive differentiable function that satisfies the condition $t^2 q\,(xt) = q\,(x)$; $x \in R^2$, $t \in R$; $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P = (a,\,v) = (-i\nabla - a)^2 + V$; i is an imaginary unit; V is smooth enough function; $\nabla = \left\{\frac{\partial}{\partial x},\,\frac{\partial}{\partial y}\right\}$; $a = (a_1,\,a_2) \in R^2$ is a vector potential; B is a magnet field generated by the vector potential a i.e.

$$B = \frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1.$$

If to consider all the above denotations then two dimensional Pauli operator may be written in the following explicit form

$$P = \begin{pmatrix} (-i\nabla - a)^2 + a_2\frac{\partial}{\partial x} - a_1\frac{\partial}{\partial y} + V & 0 \\ 0 & (-i\nabla - a)^2 - a_2\frac{\partial}{\partial x} + a_1\frac{\partial}{\partial y} + V \end{pmatrix} = \tag{1.4}$$

$$= \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

where

$b_{11} = -\Delta + (2ia_1 + a_2)\,\frac{\partial}{\partial x} + (2ia_2 - a_1)\,\frac{\partial}{\partial y} + a^2 + V,$

$b_{22} = -\Delta + (2ia_1 - a_2)\,\frac{\partial}{\partial x} + (2ia_2 + a_1)\,\frac{\partial}{\partial y} + a^2 + V$

Let us denote the set of all bounded convex domains $D \in R^2$ by M. Let

$$K = \left\{D \in M : S_D \in C^2\right\}.$$

Definition 1. The functions $s_j(x)$ defined by the relation

$$s_j\,(x) = \frac{|\nabla u_j\,(x)|^2}{\lambda_j}, \quad x \in S_D, \quad j = 1, 2, ... \tag{1.5}$$

is called to be s- functions of the problem (1.1)-(1.2) in the domain D.
The problem under consideration is: to define a domain $D \in K$ for which holds

$$s_j\,(x) = g_j\,(x)\,, \quad x \in S_D, \quad j = 1, 2, ..., \tag{1.6}$$

where $s_j\,(x)$ are indeed s- functions of the problem (1.1)-(1.2) in the domain D and $g_j\,(x)$,$j = 1,\,2,\,\cdots$ are given in R^n functions.

Other words we have to find a domain from the set M in which the s- functions of the problem (1.1)-(1.2) are equal to the given functions. Since by the definition (1.5) s-functions of the problem (1.1)-(1.2) in the domain $D$are defined by the eigenvalues and eigenfuctions of the eigenvalue problem (1.1)-(1.2), we can consider these functions as a spectral data and considered problem as an inverse spectral problem for the Pauli operator. But in duffer from usual inverse spectral problems here we have to define not functions, but the domain of the problem.

In [8] is proved that forthe the eigenvalues λ_j of problem (1.1)-(1.2) in the domain D is valid the following formula

$$\lambda_j = \eta_j - \frac{1}{4}a^2,$$

where η_j , $j = 1, 2, ...$ are eigenvalues of the Laplace operator Δ in the domain D with the same boundary condition. In [6] the following formula is obtained for the eigenvalues of the Schedinger operator with potential of form $t^2 q(xt) = q(x)$

$$\eta_j = \frac{1}{2}\max_{u_j} \int_{S_D} |\nabla u_j(x)|^2 P_D(n(x))\, ds, \tag{1.7}$$

where $P_D(x) = \max_{l \in D}(l, x)$, $x \in R^n$ is the support function of the domain D and max is taken over all eigenfunctions corresponding to the eigenvalue η_j in the case of its multiplicity. If η_j is a simple eigenvalue, then the max before integral is absent.

Since vector potential $a = (a_1, a_2) \in R^2$ is taken to be constant the term $-\frac{1}{4}a^2$ can be neglected. Considering this after substituting (1.7) into (1.6) we obtain

$$\int_{S_D} g_j(x) P_D(n(x))\, ds = 2, \quad j = 1, 2, \ldots \tag{1.8}$$

This is the basic relation in solving the considered problem – determination of the domain D by the given set of the s- functions.

Below we the lemma that will be used in hereafter.

Lemma 1. [9] Let $f(x)$ s continuous function defined on S_B. Then for any $D_1, D_2 \in K$ is valid

$$\int_{S_{D_1+D_2}} f(n(x))\, ds = \int_{S_{D_1}} f(n(x))\, ds + \int_{S_{D_2}} f(n(x))\, ds, \tag{1.9}$$

where $D_1 + D_2$ is a Minkovski sum of the domains defined as

$$D_1 + D_2 = \{x : x = x_1 + x_2,\, x_1 \in D_1,\, x_2 \in D_2\}.$$

Now let us consider the unit ball $B \subset R^2$with the center at the origin and denote by S_B its boundary. By $\varphi_k(x)$, $k = 1, 2, \ldots$ we denote the basis in $C(S_B)$ in the space of the continuous functions on S_B. These function may be positively-homogeneously extended into the unit ball B. Obtained by this way new functions $\tilde{\varphi}_k(x)$ will be continuous and positively-homogeneous. Without loss the generality, we can redesignate $\tilde{\varphi}_k(x)$ by $\varphi_k(x)$. Thus, we obtain a system of continuous, positively- homogeneous functions defined in B.

It is known that each of the continuous, positively-homogeneous functions $\varphi_j(x)$ may be presented as

$$\varphi_k(x) = \lim_{n \to \infty}\left[g_n^k(x) - h_n^k(x)\right], \tag{1.10}$$

where $g_n^k(x)$, $h_n^k(x)$, $k, n = 1, 2, ...$ are convex, positively-homogeneous, continuous functions [5]. Considering that to each convex, positively-homogeneous, continuous function corresponds bounded convex domain that is called its support function we can state that there exist bounded convex domains G_n^k and H_n^k such that

$$g_n^k(x) = P_{G_n^k}(x), h_n^k(x) = P_{H_n^k}(x).$$

It means that the functions $g_n^k(x)$, $h_n^k(x)$ are support functions for the domains G_n^k and H_n^k, $k, n = 1, 2, \ldots$. Then it is logically to call the domains G_n^k and H_n^k basis domains.

Considering this in (1.10) we obtain

$$\varphi_k(x) = \lim_{n\to\infty}\left[P_{G_n^k}^k(x) - P_{H_n^k}^k(x)\right]. \qquad (1.11)$$

Since $n(x) \in S_B$ for arbitrary $x \in S_D$ we can decompose we can expand$P_D(x)$, $x \in S_B$ in terms of basic functions $\varphi_k(x)$

$$P_D(x) = \sum_{k=1}^{\infty}\alpha_k\varphi_k(x), \;\; x \in S_B, \;\; \alpha \in R. \qquad (1.12)$$

Considering (1.11) from the last we get

$$P_D(x) = \sum_{k=1}^{\infty}\alpha_k \lim_{n\to\infty}\left[P_{G_n^k}^k(x) - P_{H_n^k}^k(x)\right], \;\; x \in S_B. \qquad (1.13)$$

The set of indexes k for which $\alpha \geq 0(\alpha_k < 0)$ denote by I^+ (I^-). Then relation (1.13) may be rewritten in the form

$$P_D(x) = \lim_{n\to\infty}\left(\alpha_k P_{G_n^k}(x) + \textstyle\sum_{k\in I^+}\alpha_k P_{H_n^k}(x)\right) =$$
$$= \lim_{n\to\infty}\left(\textstyle\sum_{k\in I^+}\alpha_k P_{G_n^k}(x) - \sum_{k\in I^-}\alpha_k P_{H_n^k}(x), \quad\right), \quad x \in S_B. \qquad (1.14)$$

From this considering the properties of the support functions we obtain [27]

$$D - \lim_{n\to\infty}\left(\sum_{k\in I^-}\alpha_k G_n^k + \sum_{k\in I^+}\alpha_k H_n^k\right) = \lim_{n\to\infty}\left(\sum_{k\in I^+}\alpha_k G_n^k - \sum_{k\in I^-}\alpha_k H_n^k\right). \qquad (1.15)$$

By virtue of the Lemma 1 we obtain

$$\int_{S_D} g_j(x) P_D(n(x))\,ds + \lim_{n\to\infty}\left(\int_{\sum_{k\in I^-}(-\alpha_k)S_{G^k}} g_j(x) P_D(n(x))\,ds + \right.$$
$$\left. + \int_{\sum_{k\in I^+}\alpha_k S_{H^k}} g_j(x) P_D(n(x))\,ds\right) =$$
$$= \lim_{n\to\infty}\left(\int_{\sum_{k\in I^+}\alpha_k S_{G^k}} g_j(x) P_D(n(x))\,ds + \int_{\sum_{k\in I^-}(-\alpha_k)S_{H^k}} g_j(x) P_D(n(x))\,ds\right).$$

From this taking into account (1.8) we get

$$\int_{S_D} g_j(x) P_D(n(x))\,ds =$$
$$= \sum_{k=1}^{\infty}\alpha_k \lim_{n\to\infty}\left[\int_{S_{G^k}} g_j(x) P_D(n(x))\,ds - \int_{S_{H^k}} g_j(x) P_D(n(x))\,ds\right] = 2. \qquad (1.16)$$

Substituting here (1.13) we finally obtain

$$\sum_{k,m=1}^{\infty} A_{k,m}(j)\,\alpha_k\alpha_m = 2, \quad j = 1, 2, ..., \tag{1.17}$$

where

$$A_{k,m}(j) = \lim_{n\to\infty}\left[\int_{S_{G_n^k}} g_j(x)\left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x))\right]ds - \right.$$

$$\left. - \int_{S_{H_n^k}} g_j(x)\left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x))\right]ds\right].$$

Equation (1.17) has generally speaking, not the only solution. With the help of solutions of this equation by formula (1.13) the support function $P_D(x)$of the desired domain$D$ is constructed. As we noted above, the region is uniquely determined by its support function as its subdifferential at the point 0.

Suppose that there exists the only solution of (1.17) that provides convexity for the function $P_D(x)$ defined by relation (1.13). Let us show that the expressions$\frac{|\nabla u_j(x)|^2}{\lambda_j}$, $x \in S_D$, $j = 1, 2, ...$ for problem (1.1), (1.2) in the domain Ddetermined by taking the subdifferential at the point 0 of the support function constructed by formula (1.13) with the help of this solution are in fact s- functions. Indeed, if $\bar{D}$ is a domain in which problem (1.1), (1.2) has the s-functions given by formula (1.5), then expanding by formula (1.13) and arguing as above, we arrive at equation (1.17) with the same coefficients. From the assumption of the uniqueness of the solution to this equation, it follows that $\overline{D} = D$.If equation (1.17) has not a unique solution, then the desired domain is among the domains determined by taking subdifferntial of the support function constructed by formula (1.13) using these solutions, taking into account the convexity condition.

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