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On Some Spectral Properties of Differential Operator with Unbounded Operator Coefficient

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Abstract. In this article we research asymptotic eigenvalue distrubtion of boundary value problem dependent on spectral parameter. Further we calculate the first trace formula of the same problem.

Key Words and Phrases: Hilbert space, eigenvalue parameter, the first regularized trace.

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1. Introduction

In the Hilbert space $L_2(H,(0,1))$ (where H is separable Hilbert space) we consider the spectral problem

$$l[y] \equiv -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t) \tag{1}$$

$$y'(0) = 0 \tag{2}$$

$$y(1)\,(1+\lambda) = y'(1)(1+h+\lambda). \tag{3}$$

Here A is a self-adjoint positive-definite operator in $H(A > E\,,\ E$ is an identity operator in H), $A^{-1} \in \sigma_\infty$ and $h \in R$. Under that conditions the operator A is discrete operator. Denote the eigenvalues and eigenvectors of the operator A by $\gamma_1 \le \gamma_2 \le \ldots$ and $\varphi_1, \varphi_2, \ldots$, respectively. $q(t)$ is a self-adjoint operator-valued function in H for each t. Suppose that $q(t)$ is weakly measurable and the following conditions are satisfied:

1. $\| q(t) \| \le C, t \in [0,1]$;
2. $\sum\limits_{k=1}^{\infty} |(q\,(t)\,\varphi_k, \varphi_k)| < const;$ for each t
3. $\int\limits_0^1 (q\,(t)\,\varphi_k, \varphi_k)\,dt = 0$ for $k = \overline{1,\infty}.$
4. $q'\,(0) = q'\,(1) = 0.$

Denote a scalar product and a norm in H by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Define the scalar product in $L_2 = L_2(H,(0,1)) \oplus H$ as

$$(Y,Z)_{L_2} = \int\limits_0^1 (y(t),z(t))dt + \frac{1}{h}(y_1,z_1). \tag{4}$$

Here $Y = \{y(t), y_1\} \in L_2$, $Z = \{z(t), z_1\} \in L_2$, $y(t), z(t) \in L_2(H,(0,1))$,$y_1, z_1 \in H$.

For $q(t) \equiv 0$ in space L_2 one can associate with problem (1)-(3) the operator L_0 defined as

$$D(L_0) = \big\{Y : Y = \{y(t), y_1\} / - y''(t) + Ay(t) \in L_2(H,(0,1)),$$
$$y'(0) = 0, y_1 = -y(1) + y'(1)\big\}$$
$$L_0 Y = \{-y''(t) + Ay(t),\ y(1) - (1+h)\, y'(1)\} \tag{5}$$

The operator corresponding to the case $q(t) \not\equiv 0$ is denoted by $L = L_0 + Q$, where $Q : Q\{y(t), -y(1) + y'(1)\} = \{q(t)y(t), 0\}$ is a bounded self-adjoint operator in L_2.

First we prove some facts about operator L_0 and L . Then study the asymptotic eigenvalue distribution of the operator L_0. After all, we calculate the first trace formula for the corresponding to problem operator L_0.

The study of boundary value problems for differential operator equations is important for the applications to boundary value problems for partial differential equations. This kind of problems were investigated in works of a lot of author's. As it is well known for operator Sturm-Liouville equation that problem was studied by Kostyuchinko-Levitan [10]. A formula for the regularized trace was first considered by I.M. Gelfand and B.M Levitan [8]. In that direction we may also refer to [1, 2, 3, 4, 5, 11, 14, 13, 12]. One of the boundary conditions of considered problem (1)-(3) contains eigenvalue parameter. Such problems arise in boundary value problems, when boundary condition contains partial derivative with respect to time. We may cite Fulton [7] which include comprehensive list of references on related problems. In [15] consider a Sturm-Liouville problem with turning points in a finite interval. In [9] investigated the problem with a spectral parameter in the boundary condition.

In comparison with indicated above works we consider the boundary value problem for differential equation with unbounded operator coefficient containing first degree polynomial depending on spectral parameter .

2. Asymptotic distribution of eigenvalues

Lemma 2.1. *Operator L_0 is symmetric in L_2.*

Proof. Let Y be from domain of L_0

$$(L_0 Y, Y)_{L_2} = \int_0^1 \left(-y''(t) + Ay(t), y(t)\right) dt +$$

$$+\frac{1}{h}\left(y(1)-(h+1)y'(1),-y(1)+y'(1)\right)=-\int_0^1\left(y''(t),y(t)\right)dt+$$

$$+\int_0^1 (Ay(t),y(t))\,dt+\frac{1}{h}\left(-(y(1),y(1))+\left(y(1),y'(1)\right)+\right.$$

$$\left.+(h+1)\left(y'(1),y(1)\right)-(h+1)\left(y'(1),y'(1)\right)\right)=$$

$$=-\left(y'(1),y(1)\right)+\int_0^1\left(y'(t),y'(t)\right)dt+\int_0^1 (y(t),Ay(t))\,dt+$$

$$+\frac{1}{h}\left(-(y(1),y(1))+\left(y(1),y'(1)\right)+(h+1)\left(y'(1),y(1)\right)-(h+1)\left(y'(1),y'(1)\right)\right)=$$

$$=-\left(y'(1),y(1)\right)+\left(y(1),y'(1)\right)+\int_0^1\left(y(t),-y''(t)+Ay(t)\right)dt+$$

$$+\frac{h+1}{h}\left(y'(1),y(1)\right)+\frac{1}{h}\left(y(1),y'(1)\right)+$$

$$+\frac{1}{h}\left(-(y(1),y(1))-(h+1)\left(y'(1),y'(1)\right)\right)=\int_0^1\left(y(t),-y''(t)+Ay(t)\right)dt+$$

$$+\frac{1}{h}\left(-(y(1),y(1))+(h+1)\left(y(1),y'(1)\right)+\left(y'(1),y(1)\right)-(h+1)\left(y'(1),y'(1)\right)\right)=$$

$$=\int_0^1\left(y(t),-y''(t)+Ay(t)\right)dt+\frac{1}{h}\left(-y(1)+y'(1),y(1)-(h+1)y'(1)\right)=(Y,L_0Y)_{L_2}$$

that completes the proof. □

Using that lemma it might be shown that L_0 is self-adjoint positive definite operator. That is why in according to Relliche theorem, L_0 also is a discrete operator. Since $q(t)$ is bounded then L is also discrete operator because of relation : $R_\lambda(L)-R_\lambda(L_0)=R_\lambda(L)QR_\lambda(L_0)$.

Suppose that the eigenvalues of A behave like

$$\gamma_k\sim gk^\alpha, k\to\infty, g>0, \alpha>0 \tag{6}$$

The following theorem about eigenvalue of L_0 is true.

Theorem 2.2. *The eigenvalues of the operator L_0 form two sequences:*

$$\lambda_k=\gamma_k+\alpha_{k,0}^2, \left|\alpha_{k,0}\right|<M=const; \lambda_{k,n}\sim\gamma_k+\alpha_n^2, \alpha_n=\pi n+O\left(\frac{1}{n}\right), n\in Z.$$

Proof. In virtue of spectral expansion of the self-adjoint operator A for coefficients $y_k(t)=(y(t),\varphi_k)$ we get the following problem

$$-y_k''(t)=(\lambda-\gamma_k)\,y_k(t) \tag{7}$$

$$y_k'(0)=0 \tag{8}$$

$$y_k(1) - (1+h)\, y_k'(1) = \lambda(-y_k(1) + y_k'(1)) \tag{9}$$

The solution of problem (7), (8) is $y_k(t) = \cos\sqrt{\lambda - \gamma_k}t$. By virtue of condition (9) eigenvalues of the operator L_0 consist of those real $\lambda \neq \gamma_k$, which satisfy at least for one k the equality

$$\cos\sqrt{\lambda - \gamma_k} + (1+h)\sqrt{\lambda - \gamma_k}\sin\sqrt{\lambda - \gamma_k} =$$
$$= \lambda\left(-\cos\sqrt{\lambda - \gamma_k} - \sqrt{\lambda - \gamma_k}\sin\sqrt{\lambda - \gamma_k}\right) \tag{10}$$

Search for imaginary roots of equation (10). Denote $z = \sqrt{\lambda - \gamma_k}$ .Then equation (10) becomes like

$$\cos z + (1+h)\, z\sin z = -\left(z^2 + \gamma_k\right)(\cos z + z\sin z) \tag{11}$$

By taking in (11) $z = iy$, $y > 0$ we get

$$chy - (1+h)\, yshy = -\left(\gamma_k - y^2\right)(chy - yshy) \tag{12}$$

Let us write the equation (12) in the form of series:

$$\sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} - (1+h)\, y \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} = -\left(\gamma_k - y^2\right)\left(\sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} - y\sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!}\right)$$

Therefore we get the following expression

$$\sum_{n=0}^{\infty} \frac{1+\gamma_k}{(2n)!} y^{2n} - \sum_{n=0}^{\infty} \frac{2+h+2n+\gamma_k}{(2n+1)!} y^{2n+2} + \sum_{n=0}^{\infty} \frac{y^{2n+4}}{(2n+1)!} = 0$$

$$\sum_{n=0}^{\infty} \frac{1+\gamma_k}{(2n)!} y^{2n} = 1 + \gamma_k + \frac{1+\gamma_k}{2} y^2 + \sum_{n=2}^{\infty} \frac{1+\gamma_k}{(2n)!} y^{2n+2} =$$

$$= 1 + \gamma_k + \frac{1+\gamma_k}{2} y^2 + \sum_{n=0}^{\infty} \frac{1+\gamma_k}{(2n+4)!} y^{2n+4}$$

and

$$\sum_{n=0}^{\infty} \frac{2+h+2n+\gamma_k}{(2n+1)!} y^{2n+2} = (2+h+\gamma_k)\, y^2 + \sum_{n=1}^{\infty} \frac{2+h+2n+\gamma_k}{(2n+1)!} y^{2n+2} =$$

$$= (2+h+\gamma_k)\, y^2 + \sum_{n=0}^{\infty} \frac{4+h+2n+\gamma_k}{(2n+3)!} y^{2n+4}$$

Then we get

$$1 + \gamma_k + \frac{1+\gamma_k}{2} y^2 + \sum_{n=0}^{\infty} \frac{1+\gamma_k}{(2n+4)!} y^{2n+4} - (2+h+\gamma_k)\, y^2 -$$

$$-\sum_{n=0}^{\infty}\frac{4+h+2n+\gamma_k}{(2n+3)!}y^{2n+4}+\sum_{n=0}^{\infty}\frac{y^{2n+4}}{(2n+1)!}=0$$

We can write the last in the following form

$$1+\gamma_k-\left(\frac{3}{2}+h+\frac{\gamma_k}{2}\right)y^2+$$

$$+\sum_{n=0}^{\infty}\frac{1+\gamma_k-(2n+4)\left(4+h+2n+\gamma_k\right)+(2n+2)\left(2n+3\right)\left(2n+4\right)}{(2n+4)!}y^{2n+4}=0$$

or

$$1+\gamma_k-\left(\frac{3}{2}+h+\frac{\gamma_k}{2}\right)y^2+$$

$$+\sum_{n=0}^{\infty}\frac{8n^3+32n^2+36n-2nh-2n\gamma_k-4h-3\gamma_k+9}{(2n+4)!}y^{2n+4}=0 \qquad (13)$$

Let's investigate the function $a(x)=8x^3+32x^2+36x-2xh-2x\gamma_k-4h-3\gamma_k+9$. This function is continuous on the positive semiaxis and $a(0)=-4h-3\gamma_k+9<0$, $a(x)\to+\infty$ as $x\to+\infty$. On the other hand the function $a(x)$ has only one sign change of terms. Using Descartes principle we get there is only one positive root. Denote this root by $x=M$. It is clear that $a(x)<0$ when $x<M$ and $a(x)>0$ when $x>M$. Therefore, coefficients of series (13) are negative when $n<[M]$and positive when $n>[M]$. Because there is only one change of sign of coefficients in (13). On the other hand for great y values the following equation has no roots:

$$thy=\frac{1+\gamma_k-y^2}{\left(1+h+\gamma_k-y^2\right)y}.$$

Because left hand side of this equation goes to 1,but right hand side goes to 0 when $y\to\infty$. Let's write the bounded set of eigenvalues corresponding to existing root $\lambda_k=\gamma_k+\alpha_{k,0}^2$.

Let's research the asymptotics of eigenvalues which the real roots of equation (11), in other words greater than γ_k. Write this equation in the form

$$1+\gamma_k+z^2+\left(1+h+\gamma_k+z^2\right)ztgz=0, z\in(0,\infty),$$

or

$$tgz=-\frac{1+z^2+\gamma_k}{z\left(1+h+z^2+\gamma_k\right)}. \qquad (14)$$

From (14) we find the following

$$z=\pi n+O\left(\frac{1}{n}\right). \qquad (15)$$

The eigenvalues of the operator L_0 corresponding to this roots are

$$\lambda_{k,n}\sim\gamma_k+\alpha_n^2,\quad \alpha_n=\pi n,\quad n\in Z.$$

The theorem is proved. □

Lemma 2.3. *Suppose that* $A = A^* > E$ *in* H *,*A^{-1} *be compact and relation* (6) *hold. Then for the eigenvalues of* L_0 *we can write*

$$\lambda_n (L_0) \sim dn^{\frac{2\alpha}{\alpha+2}}, d = const.$$

Proof. The proof of Lemma 2.3 is similar to the proof of Lemma 2 from [**?**].
Let $N(\lambda, L_0)$ be the distribution function of L_0 . Then

$$N(\lambda, L_0) = N_1 (\lambda) + N_2 (\lambda), \quad N_1 (\lambda) = \sum_{\lambda_k < \lambda} 1, \quad N_2 (\lambda) = \sum_{\lambda_{m,k} < \lambda} 1.$$

By using statement of theorem 1 we get

$$N_1 \sim C_1 \lambda^{\frac{2}{\alpha}}, N_2 (\lambda) \sim C_2 \lambda^{\frac{2+\alpha}{2\alpha}}$$

For $\alpha > 0$ we get $N(\lambda, L_0) \sim C_2 \lambda^{\frac{2+\alpha}{2\alpha}}$.

The lemma is proved. □

Eigenvalues of the operator L_0 are $\{\cos (x_{k,n} t)\, \varphi_k, (-x_{k,n} \sin x_{k,n} - \cos x_{k,n})\, \varphi_k\}$, denote the orthonormal eigenvectors of this operator by $\{\Psi_n\}$, $n = 1, 2, \ldots$ $\{\varphi_k\}$ are the eigen-vectors of the operator A, $x_{k,n}$ is the root of the equation (11) .

3. Trace formula

Denote, for convenience ,real and imaginary roots of equation (9) , by $x_{k,n}$ $\left(k = \overline{1, \infty}\right)$ and $x_{k,0}$, respectively.

$$\|\psi_n\|^2 = \int_0^1 \cos^2 (x_{k,n}\, t)\; dt + \frac{1}{h} (-x_{k,n} \sin x_{k,n} - \cos x_{k,n})^2 =$$

$$= \int_0^1 \frac{1 + \cos (2x_{k,n}\, t)}{2} dt + \frac{1}{h} (x_{k,n} \sin x_{k,n} + \cos x_{k,n})^2 =$$

$$= \frac{1}{2} + \frac{\sin 2x_{k,n}}{4x_{k,n}} + \frac{x_{k,n}^2 \sin^2 x_{k,n} + 2x_{k,n} \sin x_{k,n} \cos x_{k,n} + \cos^2 x_{k,n}}{h} =$$

$$= \frac{2hx_{k,n} + h \sin 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n} + 4x_{k,n}^2 \sin 2x_{k,n} + 4x_{k,n} \cos^2 x_{k,n}}{4hx_{k,n}} .$$

Therefore , we get

$$\psi_n = \mathrm{B}_{k,n} \{\cos (x_{k,n} t)\, \varphi_k, (-x_{k,n} \sin x_{k,n} - \cos x_{k,n})\, \varphi_k\}$$

$$\mathrm{B}_{k,n} = \sqrt{\frac{4hx_{k,n}}{2hx_{k,n} + h \sin 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n} + 4x_{k,n}^2 \sin 2x_{k,n} + 4x_{k,n} \cos^2 x_{k,n}}}$$

$$\begin{pmatrix} n=\overline{0,\infty}, k=\overline{N,\infty} \\ n=\overline{1,\infty}, k=\overline{1,N-1} \end{pmatrix} \tag{16}$$

From Lemma 2.3 in similar way as in [11] we get

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=\lim_{m\to\infty}\sum_{n=1}^{n_m}(Q\psi_n,\psi_n).$$

Prove the following lemma.

Lemma 3.1. *Provided that for operator-valued function $q(t)$ hold the conditions 1)-3), then*

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left|\frac{2hx_{k,n}\int\limits_0^1\cos(2x_{k,n}t)\,q_k(t)\,dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}\right|<\infty \tag{17}$$

where $q_k(t)=(q(t)\varphi_k,\varphi_k)$.

Proof. For large k and from asymptotics of $x_{k,n}$, we get

$$\frac{2hx_{k,n}}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}<$$

$$<\frac{1}{1+\frac{\sin 2x_{k,n}}{2x_{k,n}}}=1+O\left(\frac{1}{x_{k,n}}\right)\quad. \tag{18}$$

Integrating by parts twice and using condition 4) for $q(t)$ we get

$$\int\limits_0^1\cos(2x_{k,n}t)\;q_k(t)\,dt=$$

$$=\frac{1}{2x_{k,n}}\sin(2x_{k,n})\;q_k\;(1)-\frac{1}{(2x_{k,n})^2}\int\limits_0^1\cos(2x_{k,n}t)\;q_k''\;(t)\,dt. \tag{19}$$

In virtue of asymptotics $x_{k,n}$, λ_k and (18), (19) it follows from the last relation, that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2hx_{k,n}\int\limits_0^1\cos(2x_{k,n}t)\;q_k(t)\,dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(1+O\left(\frac{1}{x_{k,n}}\right)\right)\left(O\left(\frac{1}{n^2}\right)q_k\,(1)+\int\limits_0^1 O\left(\frac{1}{n^2}\right)\cos(2x_{k,n}t)\;q_k''\;(t)\;dt\right)\leq$$

$$\leq const\sum_{k=1}^{\infty}\left(q_k\ (1)+\int\limits_0^1 q_k''\ (t)\ dt\right) \quad .$$

From condition 1) and the last relation it follows absolute convergence of the series

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{2hx_{k,n}\int\limits_0^1\cos\left(2x_{k,n}\,t\right)\,q_k\left(t\right)dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}$$

The lemma is proved. □

Theorem 3.2. *If the operator function $q(t)$ satisfy conditions 1)-4). Then under the conditions of lemma 2.1, we get the following regularized trace formula*

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}\left(\lambda_n-\mu_n\right)=\frac{tr\,q\left(0\right)+tr\,q\left(1\right)}{8}$$

Proof. Let us calculate the following using condition 3)

$$\sum_{n=1}^{n_m}\left(Q\psi_n,\psi_n\right)=$$

$$=\sum_{n=1}^{n_m}\frac{4hx_{k,n}\int\limits_0^1\cos^2\left(x_{k,n}t\right)\left(q\left(t\right)\varphi_{i_n},\varphi_{i_n}\right)dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}=$$

$$=\sum_{n=1}^{n_m}\frac{4hx_{k,n}\int\limits_0^1\frac{1+\cos\left(2x_{k,n}\,t\right)}{2}\left(q\left(t\right)\varphi_{i_n},\varphi_{i_n}\right)dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}=$$

$$=\sum_{n=1}^{n_m}\frac{2hx_{k,n}\int\limits_0^1\cos\ \left(2x_{k,n}\,t\ \right)\ \left(q\left(t\right)\varphi_{i_n},\varphi_{i_n}\right)\ dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}\ .$$

According to lemma 3.1 series (17) converges absolutely, so we will have

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(Q\psi_n,\psi_n)=$$

$$=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\int_0^1\frac{2hx_{k,n}\cos\ \left(2x_{k,n}\,t\right)\ q_k\ \left(t\right)\ dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}} \tag{20}$$

Let us compute the value of the series

$$\sum_{n=0}^{\infty}\int_{0}^{1}\frac{2hx_{k,n}\cos\left(2x_{k,n}\,t\right)\,q_k\left(t\right)\,dt}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}. \tag{21}$$

It is equivalent to investigate the asymptotics behavior of the function $M_N\left(t\right)$ as $N\to\infty$:

$$M_N\left(t\right)=\sum_{n=0}^{N}\frac{2hx_{k,n}\,\cos\left(2x_{k,n}t\right)}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}+4x_{k,n}^2\sin 2x_{k,n}+4x_{k,n}\cos^2 x_{k,n}}\quad .$$

Express the $k-$th term of the sum $M_N\left(t\right)$ as a residue at the pole $x_{k,n}$ of the following function of complex variable z :

$$G\left(z\right)=\frac{hz\cos 2zt}{\left(\frac{\cos\,z+z(1+h)\sin z}{\cos z+z\sin z}+z^2+\gamma_k\right)\left(\cos z+z\sin z\right)^2}\quad .$$

This function has simple poles at the points $x_{k,n}$ and ω_n .
We have

$$\underset{z=x_{k,n}}{res}\;G\left(z\right)=\frac{hx_{k,n}\,\cos 2x_{k,n}\,t}{\left(\cos x_{k,n}+x_{k,n}\sin x_{k,n}\right)^2\,\left(\frac{\cos\,z+z(1+h)\sin z}{\cos z+z\sin z}+z^2+\gamma_k\right)'_{z=x_{k,n}}}$$

Let us calculate the following

$$\left(\frac{\cos\;z+z\left(1+h\right)\sin z}{\cos z+z\sin z}+z^2+\gamma_k\right)'=2z+$$

$$+\frac{\left(-\sin\;z+\left(1+h\right)\sin z+z\left(1+h\right)\cos z\right)\left(\cos z+z\sin z\right)}{\left(\cos z+z\sin z\right)^2}-$$

$$-\frac{\left(-\sin z+\sin z+z\cos z\right)\left(\,\cos\;z+z\left(1+h\right)\sin z\right)}{\left(\cos z+z\sin z\right)^2}=2z+$$

$$+\frac{\left(h\sin z+z\left(1+h\right)\cos z\right)\left(\cos z+z\sin z\right)-z\cos z\left(\,\cos\;z+z\left(1+h\right)\sin z\right)}{\left(\cos z+z\sin z\right)^2}=$$

$$=2z+\frac{h\sin z\left(\cos z+z\sin z\right)+z\left(1+h\right)\cos^2 z-z\cos^2 z}{\left(\cos z+z\sin z\right)^2}=$$

$$=2z+\frac{h\sin z\cos z+hz\sin^2 z+hz\cos^2 z}{\left(\cos z+z\sin z\right)^2}=$$

$$=2z+\frac{\frac{h}{2}\sin 2z+hz}{\left(\cos z+z\sin z\right)^2}.$$

Then

$$\underset{z=x_{k,n}}{res}\ G(z)=\frac{hx_{k,n}\ \cos 2x_{k,n}\,t}{\left(\cos x_{k,n}+x_{k,n}\sin x_{k,n}\right)^{2}\left(2x_{k,n}+\frac{\frac{h}{2}\sin 2x_{k,n}+hx_{k,n}}{\left(\cos x_{k,n}+x_{k,n}\sin x_{k,n}\right)^{2}}\right)}=$$

$$=\frac{hx_{k,n}\ \cos 2x_{k,n}\,t}{2x_{k,n}\left(\cos x_{k,n}+x_{k,n}\sin x_{k,n}\right)^{2}+\ \frac{h}{2}\sin 2x_{k,n}+hx_{k,n}}=$$

$$=\frac{2hx_{k,n}\ \cos 2x_{k,n}\,t}{2hx_{k,n}+h\sin 2x_{k,n}+4x_{k,n}^{3}\sin^{2}x_{k,n}+4x_{k,n}^{2}\sin 2x_{k,n}+4x_{k,n}\cos^{2}x_{k,n}}.$$

Take into consideration $\cos\omega_n+\omega_n\sin\omega_n=0$, let us find the residue at the point $z=\omega_n$:

$$\underset{z=\omega_n}{res}\ G(z)=\frac{h\omega_n\cos 2\omega_n t}{\left(\frac{\cos\,\omega_n+\omega_n(1+h)\sin\omega_n}{\cos\omega_n+\omega_n\sin\omega_n}+\omega_n^2+\gamma_k\right)2\left(\cos\omega_n+\omega_n\sin\omega_n\right)\omega_n\cos\omega_n}=$$

$$=\frac{h\cos 2\omega_n t}{\left(\cos\,\omega_n+\omega_n\left(1+h\right)\sin\omega_n+\left(\omega_n^2+\gamma_k\right)\left(\cos\omega_n+\omega_n\sin\omega_n\right)\right)2\cos\omega_n}=$$

$$=\frac{h\cos 2\omega_n t}{\left(\cos\,\omega_n+\omega_n\left(1+h\right)\sin\omega_n\right)2\cos\omega_n}=$$

$$=\frac{h\cos 2\omega_n t}{\left(\cos\,\omega_n+\omega_n\sin\omega_n+\omega_n h\sin\omega_n\right)2\cos\omega_n}=$$

$$=\frac{h\cos 2\omega_n t}{2\omega_n h\sin\omega_n\cos\omega_n}=\frac{\cos 2\omega_n t}{\omega_n\sin 2\omega_n}.$$

Take as a contour of integration the rectangle with vertices at $\pm iB\,, A_N\pm iB$, which has cut at $ix_{k,0}$ and will bypass it on the right and the point $-ix_{k,0}$and the origin on the left. Take also $B>x_{k,0}$.Then B will go to infinity and take $A_N=\pi N+\frac{\pi}{2}$. For such choice of A_N we have $x_{N-1,k}<A_N<x_{N,k}$. Since $G(z)$ is an odd function of z, then the integrals along the part of contours on imaginary axis, and the integral along semicircles centered at $\pm ix_{k,0}$ vanish. If $z=u+iv$, then for large v and for $u\geq 0$ $G(z)$ is of order $O\left(\frac{e^{2|v|\,(t-1)}}{|v|^3}\right)$ that is why for the given value of A_N the integrals along upper and lower sides of the contour also go to zero when $B\to\infty$.

Hence,we get the formula

$$M_N(t)+S_N(t)=\frac{1}{2\pi i}\lim_{B\to\infty}\int\limits_{A_N-iB}^{A_N+iB}G(z)\,dz+\frac{1}{2\pi i}\lim_{r\to 0}\int\limits_{|z|=r,-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}G(z)\,dz,$$

$$S_N(t)=\sum_{n=1}^{N}\frac{\cos 2\omega_n t}{\omega_n\sin 2\omega_n}\qquad(22)$$

As $N \to \infty$

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int\limits_{A_N-iB}^{A_N+iB} G(z)\ dz \sim \frac{1}{2\pi i}\int\limits_{A_N-i\infty}^{A_N+i\infty}\frac{\cos 2zt}{z^3\sin^2 z}dz =$$

$$= \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\cos\ (2\pi Nt+\pi t+2ivt)}{(A_N+iv)^3\ (1-\cos(2A_N+2iv))}dv =$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\cos\ ((2N+1)\ \pi t\,)\cos 2ivt-\sin\ ((2N+1)\ \pi t\,)\sin 2ivt}{(A_N+iv)^3\ (1+\cos 2iv)}dv =$$

$$= \frac{1}{2\pi}\cos\ ((2N+1)\ \pi t\,)\int_{-\infty}^{+\infty}\frac{ch\,2vtdv}{(A_N+iv)^3\ (1+\cos 2iv)}+$$

$$+\frac{1}{2\pi i}\sin\ ((2N+1)\ \pi t\,)\int_{-\infty}^{+\infty}\frac{sh2vtdv}{(A_N+iv)^3\ (1+\cos 2iv)}. \tag{23}$$

Denote the integrals on the right hand side of (23) by K_1 and K_2, respectively. Then,

$$|K_1| = \left|\frac{1}{2\pi}\cos\ ((2N+1)\ \pi t\,)\int\limits_{-\infty}^{+\infty}\frac{ch\,2vt}{(A_N+iv)^3\ (1+\cos 2iv)}dv\right| <$$

$$< \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{ch2vt}{\sqrt{\left(A_N^2+v^2\right)^3}\,(1+ch2v)}dv < \frac{1}{A_N^2\pi}\int\limits_{-\infty}^{+\infty}\frac{ch2vt}{1+ch2v} =$$

$$= \frac{1}{A_N^2\cos\frac{\pi t}{2}}\ \underset{N\to\infty}{\to} 0. \tag{24}$$

The similar estimate is obtained also for K_2 .So, by using (23) and (24) in (22),we get

$$\lim_{N\to\infty}\int_0^1 M_N(t)\,q_k(t)\ dt = -\lim_{N\to\infty}\int_0^1 S_N(t)\ q_k(t)\ dt+$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 q_k(t)\ dt\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} G(z)\,dz \tag{25}$$

By condition 3) the second term in the right hand side of (25) can be written as

$$\frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 q_k(t)\int\limits_{|z|=r,-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}\frac{hz\cos 2zt}{\left(\frac{\cos\ z+z(1+h)\sin z}{\cos z+z\sin z}+z^2+\gamma_k\right)(\cos z+z\sin z)^2}dzdt =$$

$$= \frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 q_k(t)\int\limits_{|z|=r,-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}\frac{hz\,(\cos 2zt-1)}{\left(\frac{\cos\ z+z(1+h)\sin z}{\cos z+z\sin z}+z^2+\gamma_k\right)(\cos z+z\sin z)^2}dzdt =$$

$$= \frac{1}{\pi i} \lim_{r\to 0} \int_0^1 q_k(t) \int\limits_{|z|=r, -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}$$

$$\frac{-hz\sin^2 zt}{\left(\frac{\cos\, z+z(1+h)\sin z}{\cos z+z\sin z} + z^2 + \gamma_k\right)(\cos z + z\sin z)^2} dzdt$$

$$\sim \frac{1}{\pi i} \lim_{r\to 0} \int_0^1 q_k(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-2hz\sin^2 zt}{\left(\frac{\cos\, z+z(1+h)\sin z}{\cos z+z\sin z} + z^2 + \gamma_k\right)(\cos z + z\sin z)^2} dzdt \sim$$

$$\sim -\frac{1}{\pi i} \lim_{r\to 0} \int_0^1 q_k(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-2hz\sin^2 zt}{\left(\frac{\cos\, z+z(1+h)\sin z}{\cos z+z\sin z} + z^2 + \gamma_k\right)(\cos z + z\sin z)^2} dzdt \sim \lim_{r\to 0} \int_0^1 q_k(t) \times$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-\frac{2}{\pi i} hi \left(re^{i\varphi}\right)^4 d\varphi dt}{1 + (1+h)\left(re^{i\varphi}\right)^2 + (1+h)\left(re^{i\varphi}\right)^4 + \left(\left(re^{i\varphi}\right)^2 + \gamma_k\right)\left(1 + \left(re^{i\varphi}\right)^2 + \left(re^{i\varphi}\right)^4\right)} = 0.$$

Using the above calculation (25) gets the following form

$$\lim_{N\to\infty} \int_0^1 M_N(t)\, q_k(t)\, dt = -\lim_{N\to\infty} \int_0^1 S_N(t)\, q_k(t)\, dt; S_N(t) = \sum_{n=1}^{N} \frac{\cos 2\omega_n t}{\omega_n \sin 2\omega_n}$$

Express the $k-$th term of the sum $S_N(t)$ as a residue at the pole $x_{k,n}$ of some function of complex variable z : $F(z) = \frac{\cos 2zt}{2(\cos z+z\sin z)\sin z}$.

$$\underset{z=\omega_n}{res}\, F(z) = \frac{\cos 2\omega_n\, t}{2\left(\cos z + z\sin z\right)'_{z=\omega_n} \sin \omega_n} = \frac{\cos 2\omega_n\, t}{2\omega_n \cos \omega_n \sin \omega_n} = \frac{\cos 2\omega_n t}{\omega_n \sin 2\omega_n}$$

$$\underset{z=\pi n}{res}\, F(z) = \frac{\cos 2\pi n\, t}{2\left(\cos \pi n + \pi n \sin \pi n\right)\cos \pi n} = \frac{\cos 2\pi n\, t}{2\cos^2 \pi n} = \frac{1}{2} \cos 2\pi n\, t$$

We can write the following

$$S_N(t) + R_N(t) = \frac{1}{2\pi i} \lim_{B\to\infty} \int_{A_N - iB}^{A_N + iB} F(z)\, dz + \frac{1}{2\pi i} \lim_{r\to 0} \int\limits_{\substack{|z| = r \\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}}} F(z)\, dz \quad ,$$

So, as the above opinion and by substitution $\pi t = z$ we have

$$\lim_{N\to\infty} \int_0^1 S_N(t)\, q_k(t)\, dt = -\lim_{N\to\infty} \int_0^1 R_N(t)\, q_k(t)\, dt = -\sum_{n=1}^{\infty} \int_0^1 \frac{1}{2} \cos 2\pi nt\, q_k\, (t)\, dt =$$

$$= -\frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^1 \cos 2\pi nt\, q_k\, (t)\, d\pi t = -\frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^1 \cos 2nz\, q_k \left(\frac{z}{\pi}\right) dz =$$

$$
= -\frac{1}{8}\sum_{n=1}^{\infty}\left[\cos n\cdot 0\cdot\frac{2}{\pi}\int_0^{\pi}\cos nz\, f_k\left(\frac{z}{\pi}\right)dz+\right.
$$

$$
\left.+\cos n\cdot\pi\cdot\frac{2}{\pi}\int_0^{\pi}\cos nz\, f_k\left(\frac{z}{\pi}\right)dz\right] = -\frac{q_k\ (0)+q_k\ (1)}{8}
$$

Summing by all k , we get

$$
\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n) = \sum_{k=1}^{\infty}\frac{q_k\,(0)+q_k\,(1)}{8} = \frac{tr\, q\,(0)+tr\, q\,(1)}{8}
$$

Theorem is proved. □

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