Journal of Contemporary Applied Mathematics
V. 12, No 1, 2022, July
ISSN 2222-5498

Obtaining the Fundamental Solution of the $\frac{5}{3}$ Order Partial Derivative Equation from the Fundamental Solution of the Two-Dimensional Laplace Equation by the Factorization Method

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Abstract. In this work, a fundamental solution is constructed for the $\frac{5}{3}$ order elliptic type equation by factorization from the fundamental solution of the two-dimensional Laplace equation with $\frac{1}{3}$ step. For this purpose it is necessary to act the fundamental solution of the two-dimensional Laplace equation by the $\frac{1}{3}$ order two-dimensional elliptic type operator. The Riemann-Liouville definition of a fractional derivative was used to obtain $\frac{1}{3}$ order derivatives with respect to each variable.
Key Words and Phrases: Factorization method, Laplace equation, fundamental solution, elliptic type equation, fractional order.

1. Introduction

In the paper, a fundamental solution of the partial derivative elliptic type equation of $\frac{5}{3}$order is obtained with step$\frac{1}{3}$ from the fundamental solution of the two-dimensional Laplace equation by factorization method.

Various boundary value problems with the local boundary conditions for the Laplace equation have been considered in the works [1], [6], [7]. Recently, the problems with the non-local boundary conditions for this equation are in the focus of specialists [8], [9].

The problem of obtaining the solution of low-order equations from the solution of high-order derivative equations by factorization has recently been solved by us in [3], [4], [5].

2. Problem statement.

It is known that the Laplace equation that is a two-dimensional second-order elliptic type equation is as follows

$$\frac{\partial^2 U(x)}{\partial x_2^2} + \frac{\partial^2 U(x)}{\partial x_1^2} = 0, \tag{1}$$

where $x = (x_1, x_2)$.
The fundamental solution of equation (1) is the function [1]

$$U(x) = \frac{1}{2\pi} \cdot \ln|x| \ , \quad |x| = \sqrt{x_1^2 + x_2^2}\,, \tag{2}$$

Thus

$$\frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_1^2} \equiv (D_2^2 + D_1^2)U(x) = \delta(x) = \delta(x_1)\delta(x_2), \tag{3}$$

where $\delta(x_k)$, $(k = 1, 2)$ is the Dirac's "delta" function.
We perform the factorization for thr Laplace operator as follows

$$D_2^2 + D_1^2 = (D_2^{\frac{1}{3}} + aD_1^{\frac{1}{3}})(D_2^{\frac{5}{3}} + bD_2^{\frac{4}{3}}D_1^{\frac{1}{3}} + cD_2D_1^{\frac{2}{3}} + dD_2^{\frac{2}{3}}D_1 + eD_2^{\frac{1}{3}}D_1^{\frac{4}{3}} + fD_1^{\frac{5}{3}}) =$$

$$= D_2^2 + bD_2^{\frac{5}{3}}D_1^{\frac{1}{3}} + cD_2^{\frac{4}{3}}D_1^{\frac{2}{3}} + dD_2D_1 + eD_2^{\frac{2}{3}}D_1^{\frac{4}{3}} + fD_2^{\frac{1}{3}}D_1^{\frac{5}{3}} +$$

$$+aD_2^{\frac{5}{3}}D_1^{\frac{1}{3}} + abD_2^{\frac{4}{3}}D_1^{\frac{2}{3}} + acD_2D_1 + adD_2^{\frac{2}{3}}D_1^{\frac{4}{3}} + aeD_2^{\frac{1}{3}}D_1^{\frac{5}{3}} + afD_1^2. \tag{4}$$

The unknown coeffcients in (4) may be found from the system below

$$\begin{cases} af = 1\,, \\ b + a = 0, \\ c + ab = 0, \\ d + ac = 0, \\ e + ad = 0, \\ f + ae = 0. \end{cases} \tag{5}$$

Solving the last one we get

$$a = i, \ \ b = -i, \quad c = -1, \ \ d = i, \ \ e = 1, \ \ f = -i \tag{6}$$

Subsituting these values into (4) for the factorization of the Laplace operator we obtain

$$D_2^2 + D_1^2 = (D_2^{\frac{1}{3}} + iD_1^{\frac{1}{3}})(D_2^{\frac{5}{3}} - iD_2^{\frac{4}{3}}D_1^{\frac{1}{3}} - D_2D_1^{\frac{2}{3}} + iD_2^{\frac{2}{3}}D_1 + D_2^{\frac{1}{3}}D_1^{\frac{4}{3}} - iD_1^{\frac{5}{3}}), \tag{7}$$

or

$$D_2^2 + D_1^2 = (D_2^{\frac{5}{3}} - iD_2^{\frac{4}{3}}D_1^{\frac{1}{3}} - D_2D_1^{\frac{2}{3}} + iD_2^{\frac{2}{3}}D_1 + D_2^{\frac{1}{3}}D_1^{\frac{4}{3}} - iD_1^{\frac{5}{3}})(D_2^{\frac{1}{3}} + iD_1^{\frac{1}{3}}). \tag{8}$$

From (3) and (8) we get

$$(D_2^{\frac{5}{3}} - iD_2^{\frac{4}{3}}D_1^{\frac{1}{3}} - D_2D_1^{\frac{2}{3}} + iD_2^{\frac{2}{3}}D_1 + D_2^{\frac{1}{3}}D_1^{\frac{4}{3}} - iD_1^{\frac{5}{3}})(D_2^{\frac{1}{3}} + iD_1^{\frac{1}{3}})U(x) = \delta(x)\,. \tag{9}$$

If denote here

$$D_2^{\frac{1}{3}}U(x) + iD_1^{\frac{1}{3}}U(x) = Z(x) \tag{10}$$

then we obtain

$$D_2^{\frac{5}{3}}Z(x) - iD_2^{\frac{4}{3}}D_1^{\frac{1}{3}}Z(x) - D_2D_1^{\frac{2}{3}}Z(x) + iD_2^{\frac{2}{3}}D_1Z(x) + D_2^{\frac{1}{3}}D_1^{\frac{4}{3}}Z(x) - iD_1^{\frac{5}{3}}Z(x) = \delta(x)\,. \tag{11}$$

The function $Z(x)$is a fundamental solution of equation (11). Applying formula (2) in (10) we obtain the fundamental solution for equation (11) as follows

$$Z(x) = D_2^{\frac{1}{3}}U(x) + iD_1^{\frac{1}{3}}U(x) = \frac{1}{2\pi}D_2^{\frac{1}{3}} \ln|x| + \frac{i}{2\pi}D_1^{\frac{1}{3}} \ln|x| =$$

$$= \frac{1}{2\pi}\frac{\partial}{\partial x_2}\int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ \ln\sqrt{x_1^2+t^2}dt +$$

$$+\frac{i}{2\pi}\frac{\partial}{\partial x_1}\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\sqrt{x_2^2+\tau^2}d\tau. \tag{12}$$

In relation (12) denoting

$$I_2 = \frac{1}{2\pi}\frac{\partial}{\partial x_2}\int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ \ln\sqrt{x_1^2+t^2}dt, \tag{13}$$

or

$$I_1 = \frac{i}{2\pi}\frac{\partial}{\partial x_1}\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\sqrt{x_2^2+\tau^2}d\tau, \tag{14}$$

and then calculating these integrals we arrive at$Z(x)$.
Using the integrating by pars formula and the rule to find the special integrals, for I_2 we get [2].

$$I_2 = \frac{1}{2\pi}\frac{\partial}{\partial x_2}\int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ \ln\sqrt{x_1^2+t^2}dt = -\frac{1}{2\pi}\frac{\partial}{\partial x_2}\int_0^{x_2}\frac{d}{dt}\frac{(x_2-t)^{\frac{2}{3}}}{\frac{2}{3}!}\ \ln\sqrt{x_1^2+t^2}dt =$$

$$= -\frac{1}{2\pi}\frac{\partial}{\partial x_2}\left[\frac{(x_2-t)^{\frac{2}{3}}}{\frac{2}{3}!}\ln\sqrt{x_1^2+t^2}\Bigg|_{t=0}^{x_2} - \int_0^{x_2}\frac{(x_2-t)^{\frac{2}{3}}}{\frac{2}{3}!}\frac{t\,dt}{x_1^2+t^2}\right] =$$

$$= \frac{1}{2\pi}\left[\frac{x_2^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\, x_1 + \int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{t\,dt}{x_1^2+t^2}\right] =$$

$$= \frac{1}{2\pi}\left[\frac{x_2^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\, x_1 + M\right], \tag{15}$$

Here we denoted

$$M = \int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{t\,dt}{x_1^2+t^2}\ . \tag{16}$$

To calculate the integral (16) we set $x_2 - t = \eta^3$. Then

$$M = \int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{t\,dt}{x_1^2+t^2} = \int_{\sqrt[3]{x_2}}^{0}\frac{\eta^{-1}}{(-\frac{1}{3})!}\frac{(x_2-\eta^3)\cdot(-3\eta^2)\,d\eta}{x_1^2+(x_2-\eta^3)^2} =$$

$$= \frac{3}{(-\frac{1}{3})!}\int_0^{\sqrt[3]{x_2}}\frac{(x_2-\eta^3)\eta}{\eta^6-2x_2\eta^3+x_1^2+x_2^2}d\eta =$$

$$= \frac{3}{(-\frac{1}{3})!}\int_0^{\sqrt[3]{x_2}}\frac{(x_2-\eta^3)\,\eta}{(\eta^3-(x_2+ix_1))\,(\eta^3-(x_2-ix_1))}d\eta. \tag{17}$$

To calculate the last integral we separate the integrant by the simple fractions. Thus,

$$M=\frac{3}{(-\frac{1}{3})!}\left[-\frac{1}{2}\int_0^{\sqrt[3]{x_2}}\frac{\eta}{\eta^3-(x_2+ix_1)}d\eta-\frac{1}{2}\int_0^{\sqrt[3]{x_2}}\frac{\eta}{\eta^3-(x_2-ix_1)}d\eta\right]=$$

$$=-\frac{3}{2\cdot(-\frac{1}{3})!}\left[M_1+M_2\right]. \qquad (18)$$

Here we denoted

$$M_1=\int_0^{\sqrt[3]{x_2}}\frac{\eta}{\eta^3-(x_2+ix_1)}d\eta, M_2=\int_0^{\sqrt[3]{x_2}}\frac{\eta}{\eta^3-(x_2-ix_1)}d\eta\ . \qquad (19)$$

Then we separate the integrant $M_1$by the simple fraction as follows

$$M_1=\int_0^{\sqrt[3]{x_2}}\frac{\eta}{\eta^3-(x_2+ix_1)}d\eta=$$

$$=\int_0^{\sqrt[3]{x_2}}\frac{\eta}{(\eta-\sqrt[3]{x_2+ix_1})(\eta^2+\eta\sqrt[3]{x_2+ix_1}+\sqrt[3]{(x_2+ix_1)^2})}d\eta=$$

$$=\int_0^{\sqrt[3]{x_2}}\left(\frac{\frac{1}{3\cdot\sqrt[3]{x_2+ix_1}}}{\eta-\sqrt[3]{x_2+ix_1}}+\frac{-\frac{1}{3\cdot\sqrt[3]{x_2+ix_1}}\eta+\frac{1}{3}}{\eta^2+\eta\sqrt[3]{x_2+ix_1}+\sqrt[3]{(x_2+ix_1)^2}}\right)d\eta=$$

$$=\frac{1}{3\cdot\sqrt[3]{x_2+ix_1}}\int_0^{\sqrt[3]{x_2}}\left(\frac{1}{\eta-\sqrt[3]{x_2+ix_1}}-\frac{\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1}-\frac{3}{2}\sqrt[3]{x_2+ix_1}}{(\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2+ix_1})^2}\right)d\eta=$$

$$=\frac{1}{3\cdot\sqrt[3]{x_2+ix_1}}\int_0^{\sqrt[3]{x_2}}\left(\frac{1}{\eta-\sqrt[3]{x_2+ix_1}}-\frac{\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1}}{(\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2+ix_1})^2}+\right.$$

$$\left.+\frac{\frac{3}{2}\sqrt[3]{x_2+ix_1}}{(\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2+ix_1})^2}\right)d\eta.$$

Now we can calculate the integral

$$M_1=\frac{1}{3\cdot\sqrt[3]{x_2+ix_1}}\left[\ln\left|\eta-\sqrt[3]{x_2+ix_1}\right|\Big|_{\eta=0}^{\sqrt[3]{x_2}}-\right.$$

$$-\frac{1}{2}\ln\left|(\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2+ix_1})^2)\right|\Bigg|_{\eta=0}^{\sqrt[3]{x_2}}+$$

$$\left.+\sqrt{3}\,arctg\frac{\eta+\frac{1}{2}\sqrt[3]{x_2+ix_1}}{\frac{\sqrt{3}}{2}\sqrt[3]{x_2+ix_1}}\right|_{\eta=0}^{\sqrt[3]{x_2}}\Bigg]=\frac{1}{3\cdot\sqrt[3]{x_2+ix_1}}\left[\ln\left|\frac{\sqrt[3]{x_2}-\sqrt[3]{x_2+ix_1}}{-\sqrt[3]{x_2+ix_1}}\right|-\right.$$

$$-\frac{1}{2}\ln\left|\frac{(\sqrt[3]{x_2}+\frac{1}{2}\sqrt[3]{x_2+ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2+ix_1})^2}{\sqrt[3]{(x_2+ix_1)^2}}\right|+$$

$$\left.+\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2+ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2+ix_1}}-arctg\frac{1}{\sqrt{3}}\right)\right]=$$

$$= \frac{1}{6\cdot\sqrt[3]{x_2+ix_1}}\left[\ln\left|\frac{(\sqrt[3]{x_2+ix_1}-\sqrt[3]{x_2})^3}{ix_1}\right| + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2+ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2+ix_1}}-\frac{\pi}{6}\right)\right]. \tag{20}$$

Similarly for M_2 we can write

$$M_2 = \int_0^{\sqrt[3]{x_2}} \frac{\eta}{\eta^3-(x_2-ix_1)}d\eta =$$

$$= \int_0^{\sqrt[3]{x_2}} \frac{\eta}{(\eta-\sqrt[3]{x_2-ix_1})(\eta^2+\eta\sqrt[3]{x_2-ix_1}+\sqrt[3]{(x_2-ix_1)^2})}d\eta =$$

$$= \int_0^{\sqrt[3]{x_2}} \left(\frac{\frac{1}{3\cdot\sqrt[3]{x_2-ix_1}}}{\eta-\sqrt[3]{x_2-ix_1}} + \frac{-\frac{1}{3\cdot\sqrt[3]{x_2-ix_1}}\eta+\frac{1}{3}}{\eta^2+\eta\sqrt[3]{x_2-ix_1}+\sqrt[3]{(x_2-ix_1)^2}}\right)d\eta =$$

$$= \frac{1}{3\cdot\sqrt[3]{x_2-ix_1}}\int_0^{\sqrt[3]{x_2}}\left(\frac{1}{\eta-\sqrt[3]{x_2-ix_1}} - \frac{\eta+\frac{1}{2}\sqrt[3]{x_2-ix_1}}{(\eta+\frac{1}{2}\sqrt[3]{x_2-ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2-ix_1})^2} +\right.$$

$$\left.+\frac{\frac{3}{2}\sqrt[3]{x_2-ix_1}}{(\eta+\frac{1}{2}\sqrt[3]{x_2-ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2-ix_1})^2}\right)d\eta.$$

Continuing we can write

$$M_2 = \frac{1}{3\cdot\sqrt[3]{x_2-ix_1}}\left[\ln\left|\eta-\sqrt[3]{x_2-ix_1}\right|\Big|_{\eta=0}^{\sqrt[3]{x_2}} -\right.$$

$$-\frac{1}{2}\ln\left|(\eta+\frac{1}{2}\sqrt[3]{x_2-ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2-ix_1})^2)\right|\Bigg|_{\eta=0}^{\sqrt[3]{x_2}} +$$

$$\left.+\sqrt{3}\,arctg\frac{\eta+\frac{1}{2}\sqrt[3]{x_2-ix_1}}{\frac{\sqrt{3}}{2}\sqrt[3]{x_2-ix_1}}\Bigg|_{\eta=0}^{\sqrt[3]{x_2}}\right] = \frac{1}{3\cdot\sqrt[3]{x_2-ix_1}}\left[\ln\left|\frac{\sqrt[3]{x_2}-\sqrt[3]{x_2-ix_1}}{-\sqrt[3]{x_2-ix_1}}\right| -\right.$$

$$-\frac{1}{2}\ln\left|\frac{(\sqrt[3]{x_2}+\frac{1}{2}\sqrt[3]{x_2-ix_1})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{x_2-ix_1})^2}{\sqrt[3]{(x_2-ix_1)^2}}\right| +$$

$$\left.+\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2-ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2-ix_1}} - arctg\frac{1}{\sqrt{3}}\right)\right] =$$

$$= \frac{1}{6\cdot\sqrt[3]{x_2-ix_1}}\left[\ln\left|\frac{(\sqrt[3]{x_2-ix_1}-\sqrt[3]{x_2})^3}{-ix_1}\right| + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2-ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2-ix_1}}-\frac{\pi}{6}\right)\right]. \tag{21}$$

Considering obtained relations (20) and (21) in (18) and then in (15) for I_2 we get

$$I_2 = \frac{1}{2\pi}\left\{\frac{x_2^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln x_1 - \frac{1}{4\cdot(-\frac{1}{3})!}\left[\frac{1}{\sqrt[3]{x_2+ix_1}}\left(\ln\left|\frac{(\sqrt[3]{x_2+ix_1}-\sqrt[3]{x_2})^3}{ix_1}\right| +\right.\right.\right.$$

$$\left.+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2+ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2+ix_1}}-\frac{\pi}{6}\right)\right) +$$

$$+\frac{1}{\sqrt[3]{x_2-ix_1}}\left(\ln\left|\frac{(\sqrt[3]{x_2-ix_1}-\sqrt[3]{x_2}\,)^3}{-ix_1}\right|+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2-ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2-ix_1}}-\frac{\pi}{6}\right)\right)\Bigg]\Bigg\}. \tag{22}$$

Now by the similar way we calculate I_1

$$I_1=\frac{i}{2\pi}\frac{\partial}{\partial x_1}\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\sqrt{x_2^2+\tau^2}d\tau=-\frac{i}{2\pi}\frac{\partial}{\partial x_1}\int_0^{x_1}\frac{d}{d\tau}\frac{(x_1-\tau)^{\frac{2}{3}}}{\frac{2}{3}!}\ln\sqrt{x_2^2+\tau^2}d\tau=$$

$$=-\frac{i}{2\pi}\frac{\partial}{\partial x_1}\left[\frac{(x_1-\tau)^{\frac{2}{3}}}{\frac{2}{3}!}\ln\sqrt{x_2^2+\tau^2}\Bigg|_{\tau=0}^{x_1}-\int_0^{x_1}\frac{(x_1-\tau)^{\frac{2}{3}}}{\frac{2}{3}!}\frac{\tau d\tau}{x_2^2+\tau^2}\right]=$$

$$=\frac{i}{2\pi}\left[\frac{x_1^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\,x_2+\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{\tau d\tau}{x_2^2+\tau^2}\right]=\frac{i}{2\pi}\left[\frac{x_1^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln\,x_2+N\right], \tag{23}$$

where

$$N=\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{\tau d\tau}{x_2^2+\tau^2}. \tag{24}$$

Let us set here $x_1-\tau=\xi^3$. Then

$$N=\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{\tau d\tau}{x_2^2+\tau^2}=\int_{\sqrt[3]{x_1}}^0\frac{\xi^{-1}}{(-\frac{1}{3})!}\frac{(x_1-\xi^3)\cdot(-3\xi^2)\,d\xi}{x_2^2+(x_1-\xi^3)^2}=$$

$$=\frac{3}{(-\frac{1}{3})!}\int_0^{\sqrt[3]{x_1}}\frac{(x_1-\xi^3)\xi}{\xi^6-2\xi^3x_1+x_1^2+x_2^2}d\xi=$$

$$=\frac{3}{(-\frac{1}{3})!}\int_0^{\sqrt[3]{x_1}}\frac{(x_1-\xi^3)\,\xi}{(\xi^3-(x_1+ix_2))\,(\xi^3-(x_1-ix_2))}d\xi. \tag{25}$$

In (25) separating the integrant by the simple functions we obtain the relation

$$N=-\frac{3}{2\cdot(-\frac{1}{3})!}\left[\int_0^{\sqrt[3]{x_1}}\frac{\xi}{(\xi^3-(x_1+ix_2))}d\xi+\int_0^{\sqrt[3]{x_1}}\frac{\xi}{(\xi^3-(x_1-ix_2))}d\xi\right]=$$

$$=-\frac{3}{2\cdot(-\frac{1}{3})!}\left[N_1+N_2\right]. \tag{26}$$

Here we denoted

$$N_1=\int_0^{\sqrt[3]{x_1}}\frac{\xi}{\xi^3-(x_1+ix_2)}d\xi, N_2=\int_0^{\sqrt[3]{x_1}}\frac{\xi}{\xi^3-(x_1-ix_2)}\,d\xi\,. \tag{27}$$

Separating the integrant by the simple functions we obtain

$$N_1=\int_0^{\sqrt[3]{x_1}}\frac{\xi}{\xi^3-(x_1+ix_2)}d\xi=$$

$$=\int_0^{\sqrt[3]{x_1}}\frac{\xi}{(\xi-\sqrt[3]{x_1+ix_2})(\xi^2+\xi\sqrt[3]{x_1+ix_2}+\sqrt[3]{(x_1+ix_2)^2})}d\xi=$$

$$= \int_0^{\sqrt[3]{x_1}} \left(\frac{\frac{1}{3\cdot\sqrt[3]{x_1+ix_2}}}{\xi - \sqrt[3]{x_1+ix_2}} + \frac{-\frac{1}{3\cdot\sqrt[3]{x_1+ix_2}}\xi + \frac{1}{3}}{\xi^2 + \xi\sqrt[3]{x_1+ix_2} + \sqrt[3]{(x_1+ix_2)^2}} \right) d\xi =$$

$$= \frac{1}{3\cdot\sqrt[3]{x_1+ix_2}} \int_0^{\sqrt[3]{x_1}} \left(\frac{1}{\xi - \sqrt[3]{x_1+ix_2}} - \frac{\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2} - \frac{3}{2}\sqrt[3]{x_1+ix_2}}{(\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1+ix_2})^2} \right) d\xi =$$

$$= \frac{1}{3\cdot\sqrt[3]{x_1+ix_2}} \int_0^{\sqrt[3]{x_1}} \left(\frac{1}{\xi - \sqrt[3]{x_1+ix_2}} - \frac{\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2}}{(\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1+ix_2})^2} + \right.$$

$$\left. + \frac{\frac{3}{2}\sqrt[3]{x_1+ix_2}}{(\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1+ix_2})^2} \right) d\xi.$$

Now we can calculate the integral

$$N_1 = \frac{1}{3\cdot\sqrt[3]{x_1+ix_2}} \left[\ln\left|\xi - \sqrt[3]{x_1+ix_2}\right| \Big|_{\xi=0}^{\sqrt[3]{x_1}} - \right.$$

$$-\frac{1}{2}\ln\left|(\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1+ix_2})^2)\right| \Bigg|_{\xi=0}^{\sqrt[3]{x_1}} +$$

$$\left. +\sqrt{3}\, arctg \frac{\xi + \frac{1}{2}\sqrt[3]{x_1+ix_2}}{\frac{\sqrt{3}}{2}\sqrt[3]{x_1+ix_2}} \Bigg|_{\xi=0}^{\sqrt[3]{x_1}} \right] = \frac{1}{3\cdot\sqrt[3]{x_1+ix_2}} \left[\ln\left| \frac{\sqrt[3]{x_1} - \sqrt[3]{x_1+ix_2}}{-\sqrt[3]{x_1+ix_2}} \right| - \right.$$

$$-\frac{1}{2}\ln\left| \frac{(\sqrt[3]{x_1} + \frac{1}{2}\sqrt[3]{x_1+ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1+ix_2})^2}{\sqrt[3]{(x_1+ix_2)^2}} \right| +$$

$$\left. +\sqrt{3}\left(arctg \frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1+ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1+ix_2}} - arctg\frac{1}{\sqrt{3}} \right) \right] =$$

$$= \frac{1}{6\cdot\sqrt[3]{x_1+ix_2}} \left[\ln\left| \frac{(\sqrt[3]{x_1+ix_2} - \sqrt[3]{x_1})^3}{ix_2} \right| + 2\sqrt{3}\left(arctg \frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1+ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1+ix_2}} - \frac{\pi}{6} \right) \right]. \qquad (28)$$

Similarly, we calculate N_2

$$N_2 = \int_0^{\sqrt[3]{x_1}} \frac{\xi}{\xi^3 - (x_1 - ix_2)} d\xi =$$

$$= \int_0^{\sqrt[3]{x_1}} \frac{\xi}{(\xi - \sqrt[3]{x_1-ix_2})(\xi^2 + \xi\sqrt[3]{x_1-ix_2} + \sqrt[3]{(x_1-ix_2)^2})} d\xi =$$

$$= \int_0^{\sqrt[3]{x_1}} \left(\frac{\frac{1}{3\cdot\sqrt[3]{x_1-ix_2}}}{\xi - \sqrt[3]{x_1-ix_2}} - + \frac{-\frac{1}{3\cdot\sqrt[3]{x_1-ix_2}}\xi + \frac{1}{3}}{\xi^2 + \xi\sqrt[3]{x_1-ix_2} + \sqrt[3]{(x_1-ix_2)^2}} \right) d\xi =$$

$$= \frac{1}{3\cdot\sqrt[3]{x_1-ix_2}} \int_0^{\sqrt[3]{x_1}} \left(\frac{1}{\xi - \sqrt[3]{x_1-ix_2}} - \frac{\xi + \frac{1}{2}\sqrt[3]{x_1-ix_2} - \frac{3}{2}\sqrt[3]{x_1-ix_2}}{(\xi + \frac{1}{2}\sqrt[3]{x_1-ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1-ix_2})^2} \right) d\xi =$$

$$= \frac{1}{3\cdot\sqrt[3]{x_1 - ix_2}}\int_0^{\sqrt[3]{x_1}}\left(\frac{1}{\xi - \sqrt[3]{x_1 - ix_2}} - \frac{\xi + \frac{1}{2}\sqrt[3]{x_1 - ix_2}}{(\xi + \frac{1}{2}\sqrt[3]{x_1 - ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1 - ix_2})^2} + \right.$$

$$\left. + \frac{\frac{3}{2}\sqrt[3]{x_1 - ix_2}}{(\xi + \frac{1}{2}\sqrt[3]{x_1 - ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1 - ix_2})^2}\right) d\xi.$$

Finally, we get

$$N_2 = \frac{1}{3\cdot\sqrt[3]{x_1 - ix_2}}\left[\ln\left|\xi - \sqrt[3]{x_1 - ix_2}\right|\Big|_{\xi=0}^{\sqrt[3]{x_1}} - \right.$$

$$-\frac{1}{2}\ln\left|(\xi + \frac{1}{2}\sqrt[3]{x_1 - ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1 - ix_2})^2)\right|\Bigg|_{\xi=0}^{\sqrt[3]{x_1}} +$$

$$\left. +\sqrt{3}\, arctg\frac{\xi + \frac{1}{2}\sqrt[3]{x_1 - ix_2}}{\frac{\sqrt{3}}{2}\sqrt[3]{x_1 - ix_2}}\Bigg|_{\xi=0}^{\sqrt[3]{x_1}}\right] = \frac{1}{3\cdot\sqrt[3]{x_1 - ix_2}}\left[\ln\left|\frac{\sqrt[3]{x_1} - \sqrt[3]{x_1 - ix_2}}{-\sqrt[3]{x_1 - ix_2}}\right| - \right.$$

$$-\frac{1}{2}\ln\left|\frac{(\sqrt[3]{x_1} + \frac{1}{2}\sqrt[3]{x_1 - ix_2})^2 + (\frac{\sqrt{3}}{2}\sqrt[3]{x_1 - ix_2})^2}{\sqrt[3]{(x_1 - ix_2)^2}}\right| +$$

$$\left. +\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 - ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 - ix_2}} - arctg\frac{1}{\sqrt{3}}\right)\right] =$$

$$= \frac{1}{6\cdot\sqrt[3]{x_1 - ix_2}}\left[\ln\left|\frac{(\sqrt[3]{x_1 - ix_2} - \sqrt[3]{x_1})^3}{-ix_2}\right| + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 - ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 - ix_2}} - \frac{\pi}{6}\right)\right] \tag{29}$$

Putting obtained relations (28) and (29) into (26) we have

$$N = -\frac{1}{4\cdot(-\frac{1}{3})!}\left[\frac{1}{\sqrt[3]{x_1 + ix_2}}\left[\ln\left|\frac{(\sqrt[3]{x_1 + ix_2} - \sqrt[3]{x_1})^3}{ix_2}\right| + \right.\right.$$

$$\left. +2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 + ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 + ix_2}} - \frac{\pi}{6}\right)\right] +$$

$$\left. + \frac{1}{\sqrt[3]{x_1 - ix_2}}\left[\ln\left|\frac{(\sqrt[3]{x_1 - ix_2} - \sqrt[3]{x_1})^3}{-ix_2}\right| + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 - ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 - ix_2}} - \frac{\pi}{6}\right)\right]\right]. \tag{30}$$

Since $I_1 = \frac{i}{2\pi}\left[\frac{x_1^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln x_2 + N\right]$ we can write

$$I_1 = \frac{i}{2\pi}\left\{\frac{x_1^{-\frac{1}{3}}}{(-\frac{1}{3})!}\ln x_2 - \frac{1}{4\cdot(-\frac{1}{3})!}\left[\frac{1}{\sqrt[3]{x_1 + ix_2}}\left(\ln\left|\frac{(\sqrt[3]{x_1 + ix_2} - \sqrt[3]{x_1})^3}{ix_2}\right| + \right.\right.\right.$$

$$\left. +2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 + ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 + ix_2}} - \frac{\pi}{6}\right)\right) + \frac{1}{\sqrt[3]{x_1 - ix_2}}\left(\ln\left|\frac{(\sqrt[3]{x_1 - ix_2} - \sqrt[3]{x_1})^3}{-ix_2}\right| + \right.$$

$$+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1}+\sqrt[3]{x_1-ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1-ix_2}}-\frac{\pi}{6}\right)\Big)\Big]\Big\}. \tag{31}$$

If we consider (22) and (31) in (12) and the fact that $Z(x)=I_2+I_1$ we arrive at the following theorem

Theorem. The fundamental solution of the homogeneous partial differential equation (11) of order $\frac{5}{3}$ with constant coefficients is in the form

$$Z(x)=\frac{1}{2\pi(-\frac{1}{3})!}\left\{x_2{}^{-\frac{1}{3}}\ln x_1-\frac{1}{4}\left[\frac{1}{\sqrt[3]{x_2+ix_1}}\left(\ln\left|\frac{(\sqrt[3]{x_2+ix_1}-\sqrt[3]{x_2})^3}{ix_1}\right|+\right.\right.\right.$$

$$+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2+ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2+ix_1}}-\frac{\pi}{6}\right)\Big)++\frac{1}{\sqrt[3]{x_2-ix_1}}\left(\ln\left|\frac{(\sqrt[3]{x_2-ix_1}-\sqrt[3]{x_2})^3}{-ix_1}\right|+\right.$$

$$+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2-ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2-ix_1}}-\frac{\pi}{6}\right)\Big)\Big]+$$

$$+ix_1^{-\frac{1}{3}}\ln x_2-\frac{i}{4}\left[\frac{1}{\sqrt[3]{x_1+ix_2}}\left(\ln\left|\frac{(\sqrt[3]{x_1+ix_2}-\sqrt[3]{x_1})^3}{ix_2}\right|+\right.\right.$$

$$+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1}+\sqrt[3]{x_1+ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1+ix_2}}-\frac{\pi}{6}\right)\Big)+\frac{1}{\sqrt[3]{x_1-ix_2}}\left(\ln\left|\frac{(\sqrt[3]{x_1-ix_2}-\sqrt[3]{x_2})^3}{-ix_2}+\right|\right.$$

$$+2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1}+\sqrt[3]{x_1-ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1-ix_2}}-\frac{\pi}{6}\right)\Big)\Big]\Big\}. \tag{32}$$

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Received 02 May 2021
Accepted 14 September 2021