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On a Fredholm Type Integro-Differential Equations of Fifth Order with Degenerate Kernel

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Abstract. In this article the questions of one value solvability of the initial value problem for a nonlinear partial Fredholm type integro-differential equation of the fifth order with degenerate kernel are considered. The method of degenerate kernel designed for Fredholm integral equations of the second kind is modified to the case of partial Fredholm type integro-differential equation of the fifth order and three variables. After denoting the Fredholm type integro-differential equation is reduced to a system of functional-algebraic equations. In proof of the existence and uniqueness of solution is used the method of successive approximations combining it with the method of compressing maps.

Key Words and Phrases: Initial value problem, partial integro-differential equation, Fredholm type equation, degenerate kernel, system of functional-algebraic equations, one valued solvability.
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1. Formulation of the problem

Mathematical modelling of many processes occurring in the real world often leads to the study of the initial and boundary value problems for a non-classical equations of mathematical physics. High-order partial differential equations are of great interest from the point of view of physical applications. Many problems of gas dynamics, theory of elasticity, theory of plates and shells are reduced to the consideration of high-order partial differential equations [1], [2], [3], [4]. A large number of works are devoted to the study of initial problems for a high-order partial differential equations (see, for example, [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]). Integro-differential equations with degenerate kernels were considered, in particular, in [15], [16], [17], [18], [19], [20].

In this paper, we propose a method for studying the unique solvability of the initial value problem for a nonlinear integro-differential equation of Fredholm type in partial derivatives of the fifth order with a degenerate kernel. This work is a further development of works [21], [22], [23].

In the domain $\Omega \equiv \Omega_T \times \Omega_l^2$, we consider a fifth-order Fredholm-type integro-differential equation with a degenerate kernel

$$\frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{\partial^3 u(t, x_1, x_2)}{\partial t^3} + \lambda \int_0^T K(t, s) \frac{\partial^2 u(s, x_1, x_2)}{\partial s^2} ds \right) =$$

$$= p(t)\,\beta\left(x_1, x_2, \int_0^T \int_0^l \int_0^l H(s, y_1, y_2) u(s, y_1, y_2) dy_1 dy_2 ds\right) \tag{1}$$

with initial value problem

$$\begin{array}{l} u(0, x_1, x_2) = \varphi_1(x_1, x_2), \quad u_t(0, x_1, x_2) = \varphi_2(x_1, x_2), \\ u_{tt}(0, x_1, x_2) = \varphi_3(x_1, x_2) \quad x_1,\, x_2 \in \Omega_l^2, \end{array} \tag{2}$$

$$u\,(t, 0, 0) = \psi(t), \;\; t \in \Omega_T, \tag{3}$$

where $p(t) \in C\,(\Omega_T)$, $\beta(x_1, x_2, \gamma) \in C\,\left(\Omega_l^2 \times \mathbb{R}\right)$, $\varphi_k(x_1, x_2) \in C^1\left(\Omega_l^2\right)$, $k = 1, 2, 3$, $\psi(t) \in C^2\,(\Omega_T)$, $\psi(0) = \varphi_1(0,0) = \varphi_2(0,0) = \varphi_3(0,0) = 0$, $K(t,s) = \sum_{i=1}^n a_i(t)\, b_i(s)$, $0 < a_i(t),\, b_i(s) \in C(\Omega_T)$, $0 < \int_0^T \int_0^l \int_0^l |\, H(t, x_1, x_2)|\, dx_1 dx_2 dt < \infty$, $\Omega_T \equiv [0, T]$, $\Omega_l^2 \equiv [0, l] \times [0, l]$, λ is spectral parameter.

By the solution of the initial value problem (1)–(3) we mean a function $u(t, x_1, x_2) \in C^{3,1,1}(\Omega)$, satisfying partial integro-differential equation (1) and initial value conditions (2), (3).

2. Reduction of problem (1)-(3) to the integral equation

We introduce a denotation $u_{ttx_1x_2}(t, x_1, x_2) = \vartheta(t, x_1, x_2)$ in differential equation (1). Then this fifth-order integro-differential equation (1) takes a form of first-order integro-differential equation

$$\frac{\partial\,\vartheta(t, x_1, x_2)}{\partial\, t} + \lambda \int_0^T K(t,s)\,\vartheta(s, x_1, x_2) ds =$$

$$= p(t)\,\beta\left(x_1, x_2, \int_0^T \int_0^l \int_0^l H(s, y_1, y_2) u(s, y_1, y_2) dy_1 dy_2 ds\right). \tag{4}$$

By the following designation

$$c_i(x_1, x_2) = \int_0^T b_i(s)\,\vartheta(s, x_1, x_2) ds \tag{5}$$

the integro-differential equation (1) we rewrite as

$$\frac{\partial\,\vartheta(t, x_1, x_2)}{\partial\, t} = -\lambda \sum_{i=1}^n a_i(t) \cdot c_i(x_1, x_2) +$$

$$+ p(t)\,\beta\left(x_1, x_2, \int_0^T \int_0^l \int_0^l H(s, y_1, y_2) u(s, y_1, y_2) dy_1 dy_2 ds\right).$$

By virtue of the conditions of the problem, the right-hand side of this expression are continuous functions of their arguments. By integrating over t from the last integro-differential equation, we obtain the following integral equation

$$\vartheta(t, x_1, x_2) = D_1(x_1, x_2) - \lambda \sum_{i=1}^n c_i(x_1, x_2) \int_0^t a_i(s) ds +$$

$$+q(t)\,\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right), \tag{6}$$

where $D_1(x_1,x_2)\in C^1\left(\Omega_l^2\right)$ is arbitrary continuous function, which is to be determined, $q(t)=\int_0^t p(s)ds$.

Substituting integro-differential equation (6) into designation (5), we have a system

$$c_i(x_1,x_2)=\int_0^T b_i(s)\left[D_1(x_1,x_2)-\lambda\sum_{i=1}^n c_i(x_1,x_2)\int_0^s a_i(\theta)d\theta+\right.$$

$$\left.+q(s)\,\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\right]ds. \tag{7}$$

In (7) we introduce the following designations

$$B_i(x_1,x_2)=\int_0^T b_i(s)\Big[D_1(x_1,x_2)+$$

$$+q(s)\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\Big]ds, \tag{8}$$

$$A_{i\,j}=\int_0^T b_i(s)\int_0^s a_j(\theta)d\theta ds. \tag{9}$$

Then, taking into account the notations (8) and (9), system (7) can be written in the form of the following system of functional algebraic equations (SFAE)

$$c_i(x_1,x_2)+\lambda\sum_{j=1}^n A_{i\,j}\cdot c_j(x_1,x_2)=B_i(x_1,x_2),\quad i=\overline{1,n}. \tag{10}$$

SFAE (10) is uniquely solvable for any finite $B_i(x_1,x_2)$, if the following Fredholm condition is satisfied

$$\Delta(\lambda)=\begin{vmatrix} 1+\lambda A_{1\,1} & \lambda\,A_{1\,2} & \dots & \lambda\,A_{1n} \\ \lambda\,A_{2\,1} & 1+\lambda\,A_{2\,2} & \dots & \lambda\,A_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda\,A_{n1} & \lambda\,A_{n2} & \dots & 1+\lambda\,A_{nn} \end{vmatrix}\neq 0. \tag{11}$$

The determinant $\Delta(\lambda)$ in (11) is a polynomial with respect to λ degree at most n. Equation $\Delta(\lambda)=0$ has at most n distinct real roots. These roots are the eigenvalues of the kernel of the integro-differential equation (1). Other values of λ are regular for which the condition (11) is satisfied. For regular values of λ, the system (10) has a unique solution for any finite right-hand side. In the present paper, for such regular values of the parameter λ, the unique solvability of the initial value problem (1)–(3) is established.

First, we write the solutions of the SFAE (10) in the following form

$$c_i(x_1,x_2)=\frac{\Delta_i(\lambda,x_1,x_2)}{\Delta(\lambda)},\quad i=\overline{1,n}, \tag{12}$$

where

$$\Delta_i(\lambda,x_1,x_2)=\begin{vmatrix} 1+\lambda\,A_{1\,1} & \ldots & \lambda\,A_{1\,(i-1)} & B_1(x_1,x_2) & \lambda\,A_{1(i+1)} & \ldots & \lambda\,A_{1n} \\ \lambda\,A_{21} & \ldots & \lambda\,A_{2(i-1)} & B_2(x_1,x_2) & \lambda\,A_{2(i+1)} & \ldots & \lambda\,A_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda\,A_{n1} & \ldots & \lambda\,A_{n(i-1)} & B_n(x_1,x_2) & \lambda\,A_{n(i+1)} & \ldots & 1+\lambda\,A_{nn} \end{vmatrix}.$$

There are functions $B_i(x_1,x_2)$ between of the elements of the determinants $\Delta_i(\lambda,x_1,x_2)$. In turn, the functions $B_i(x_1,x_2)$ include so far unknown functions $D_1(x_1,x_2)$ and $\beta(x_1,x_2)$. To derive them from the sign of the determinants, the expression in (8) can be written in the following form

$$B_i(x_1,x_2)=D_1(x_1,x_2)\cdot B_{1i}+\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\cdot B_{2i},$$

where $B_{1\,i}=\int_0^T b_i(s)ds,\ B_{2\,i}=\int_0^T b_i(s)q(s)ds.$

In this case, according to the property of the determinant, we obtain

$$\Delta_i(\lambda,x_1,x_2)=D_1(x_1,x_2)\cdot\Delta_{1\,i}(\lambda)+$$
$$+\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\cdot\Delta_{2\,i}(\lambda),$$

where

$$\Delta_{k\,i}(\lambda)=\begin{vmatrix} 1+\lambda\,A_{1\,1} & \ldots & \lambda\,A_{1\,(i-1)} & B_{k\,1} & \lambda\,A_{1\,(i+1)} & \ldots & \lambda\,A_{1\,n} \\ \lambda\,A_{2\,1} & \ldots & \lambda\,A_{2\,(i-1)} & B_{k\,2} & \lambda\,A_{2\,(i+1)} & \ldots & \lambda\,A_{2\,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda\,A_{n\,1} & \ldots & \lambda\,A_{n\,(i-1)} & B_{k\,n} & \lambda\,A_{n\,(i+1)} & \ldots & 1+\lambda\,A_{n\,n} \end{vmatrix},\ k=1,2.$$

Then the solutions (12) of the SFAE (10) can be written as

$$c_i(x_1,x_2)=D_1(x_1,x_2)\cdot\frac{\Delta_{1\,i}(\lambda)}{\Delta(\lambda)}+$$
$$+\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\cdot\frac{\Delta_{2\,i}(\lambda)}{\Delta(\lambda)},\quad i=\overline{1,n}. \tag{13}$$

So, we are solved the following system of functional algebraic equations

$$c_i(x_1,x_2)+\lambda\sum_{j=1}^n A_{i\,j}\cdot c_j(x_1,x_2)=$$
$$=D_1(x_1,x_2)\cdot B_{1i}+\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\cdot B_{2\,i},$$

$i=\overline{1,n}$ and its solutions we present in the form of functions (13).

Substituting the functions (13) into the system (7), we derive the following integro-differential equation

$$u_{ttx_1,x_2}(t,x_1,x_2) = D_1(x_1,x_2) - \lambda \sum_{i=1}^{n} \int_0^t a_i(s)ds \times$$

$$\times \left[D_1(x_1,x_2) \frac{\Delta_{1\,i}(\lambda)}{\Delta(\lambda)} + \beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)u(\theta,y_1,y_2)dy_1dy_2d\theta\right)\frac{\Delta_{2\,i}(\lambda)}{\Delta(\lambda)}\right] +$$

$$+q(t)\,\beta\left(x_1,x_2,\int_0^T\int_0^l\int_0^l H(\theta,y_1,y_2)\,u(\theta,y_1,y_2)dy_1dy_2d\theta\right). \tag{14}$$

By integrating once over every variables of x_1, x_2 and twice over t, from the partial integro-differential equation (14) we obtain

$$u(t,x_1,x_2) = D_2(x_1,x_2)\,t + D_3(x_1,x_2) + \int_0^t (t-s)E(s)ds + t\int_0^{x_1}\int_0^{x_2} D_1(y_1,y_2)dy_1dy_2 -$$

$$-\frac{\lambda}{2}\sum_{i=1}^{n}\int_0^t (t-s)^2 a_i(s)ds \int_0^{x_1}\int_0^{x_2}\left[D_1(y_1,y_2)\frac{\Delta_{1\,i}(\lambda)}{\Delta(\lambda)} + \right.$$

$$\left. +\beta\left(y_1,y_2,\int_0^T\int_0^l\int_0^l H(\theta,z_1,z_2)\,u(\theta,z_1,z_2)dz_1dz_2d\theta\right)\frac{\Delta_{2\,i}(\lambda)}{\Delta(\lambda)}\right]dy_1dy_2 +$$

$$+\bar{q}(t)\int_0^{x_1}\int_0^{x_2}\beta\left(y_1,y_2,\int_0^T\int_0^l\int_0^l H(\theta,z_1,z_2)u(\theta,z_1,z_2)dz_1dz_2d\theta\right)dy_1dy_2, \tag{15}$$

where $D_2(x), D_3(x) \in C^1(\Omega_l^2)$, $E(t) \in C^2\left(\Omega_l^2\right)$ are arbitrary continuous functions, which are to be determined, $\bar{q}(t) = \int_0^t (t-s)q(s)ds$.

Using the given conditions (2) and (3), from the representation (15) we arrive at the following relations:

$$\varphi_1(x_1,x_2) = D_3(x_1,x_2), \;\; \varphi_2(x_1,x_2) = D_2(x_1,x_2);$$

$$\psi(t) = \int_0^t E(s)ds; \;\; \varphi_3(x_1,x_2) = \int_0^{x_1}\int_0^{x_2} D_1(y_1,y_2)dy_1dy_2.$$

Then the integral equation (15) can be rewritten as

$$u(t,x_1,x_2) = \Im(t,x_1,x_2;u) \equiv Q(t,x_1,x_2) +$$

$$+\mu(t)\int_0^x \beta\left(y,\int_0^T\int_0^l\int_0^l H(\theta,z_1,z_2)\,u(\theta,z_1,z_2)dz_1dz_2d\theta\right)dy, \tag{16}$$

where $\mu(t) = \bar{q}(t)$, $Q(t,x_1,x_2) = \varphi_1(x_1,x_2) + t\,\varphi_2(x_1,x_2) - \nu(t)\,\varphi_3(x_1,x_2) + \psi(t)$,

$$\nu(t) = \frac{\lambda}{2}\sum_{i=1}^{n}\frac{\Delta_{1\,i}(\lambda)}{\Delta(\lambda)}\int_0^t (t-s)^2 a_i(s)ds.$$

Consequently, the initial value problem (1)–(3) was reduced to the integral equation (16). Equation (16) with respect to the main unknown function $u(t,x_1,x_2)$ is an integral equation of the Volterra type of the second kind.

3. A theorem on the unique solvability of the initial value problem

For an arbitrary function $l(t, x_1, x_2) \in C(\Omega)$ we consider the following norm

$$\| l(t, x_1, x_2) \|_C = \max \{ | l(t, x_1, x_2) | : (t, x_1, x_2) \in \Omega \} .$$

Theorem 3.1. *Let be fulfilled: the condition* (11) *and*
1). $M_1 = \max\limits_{(t,x_1,x_2)\in\Omega} \left| Q(t, x_1, x_2) + \mu(t) \int_0^{x_1} \int_0^{x_2} \beta(y_1, y_2, \cdot) dy_1 dy_2 \right| < \infty$;
2). $| \beta(E_1, x_2, \gamma_1) - \beta(E_1, x_2, \gamma_2) | \leq L(E_1, x_2) | \gamma_1 - \gamma_2 |$;
3). $\delta_1 = \int_0^T \int_0^l \int_0^l | H(t, x_1, x_2) | dx_1 dx_2 dt < \infty$;
4). $\rho = \delta_1 \max\limits_{(t,x_1,x_2)\in\Omega} | \mu(t) | \int_0^{x_1} \int_0^{x_2} L(y_1, y_2) dy_1 dy_2 < 1$.

Then there exists a unique solution $u(t, x_1, x_2) \in C^{2,1,1}(\Omega)$ *to the initial value problem* (1)–(3).

Proof. Consider the following iterative process for the nonlinear integral equation (16):

$$u_0(t, x_1, x_2) = 0, \;\; u_{k+1}(t, x_1, x_2) = \Im(t, x_1, x_2; \, u_k), \;\; k = 0, \, 1, \, 2, \, \ldots \tag{17}$$

By virtue of the conditions of the theorem, from (17) we obtain the following estimates

$$\| u_{k+1}(t, x_1, x_2) - u_0(t, x_1, x_2) \|_C \leq M_1, \tag{18}$$

$$\| u_{k+1}(t, x_1, x_2) - u_k(t, x_1, x_2) \|_C \leq$$

$$\leq \delta_1 \max_{(t,x_1,x_2)\in\Omega} | \mu(t) | \int_0^{x_1} \int_0^{x_2} L(y_1, y_2) \| u_k(t, y_1, y_2) - u_{k-1}(t, y_1, y_2) \|_C \, dy_1 dy_2 \leq$$

$$\leq \rho \cdot \| u_k(t, x_1, x_2) - u_{k-1}(t, x_1, x_2) \|_C \, , \tag{19}$$

where $\rho = \delta_1 \max\limits_{(t,x_1,x_2)\in\Omega} | \mu(t) | \int_0^{x_1} \int_0^{x_2} L(y_1, y_2) dy_1 dy_2$.

By virtue of the last condition of the theorem, it follows from estimate (19) that the operator on the right-hand side of (16) is contracting. From estimates (18) and (19), we conclude that there is a unique fixed point for operator (16) (see, [24, pp. 389–401]). Consequently, in the domain Ω, the nonlinear integral equation (16) has a unique solution $u(t, x_1, x_2) \in C(\Omega)$. It is assumed here that the fixed point of the operator automatically falls into $C^{2,\,1}(\Omega)$ due to the relation

$$u(t, x_1, x_2) = \Im(t, x_1, x_2; \, u) \in C^{2,1}(\Omega).$$

In addition, the convergence rate estimate is valid

$$\| u_{k+1}(t, x_1, x_2) - u(t, x_1, x_2) \|_C \leq \frac{\rho^{k+1}}{1 - \rho} \cdot (\Delta + M).$$

□

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