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On the Properties of Traces of Functions from Generalized Sobolev-Type Space

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Abstract. In this article, we study the properties of the traces of function $D^{\nu}\left(f-\Phi_{l-1}\right)$, when the function f given in the Sobolev type spaces. In this paper we prove the existence of traces from generalized Sobolev type function spaces. This function spaces is the generalization of classical Sobolev-Slobodetskii and Nikolskii-Besov spaces. We established sufficient conditions under which the trace theorem for these spaces are proved. We reduce the analog of integral representations of functions given by S.L. Sobolev for functions form the Sobolev-type space space.

Key Words and Phrases: Generalized Sobolev space, strong $a\left(h\right)$- horn condition, integral representation, trace theorem.

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1. Introduction

The theory of embedding of spaces of differentiable functions of several variables developed as a new direction of mathematics in the 30s of the 20th century as a result of the works of S.L. Sobolev, which is presented in detail in monograph [5]. This theory studies important connections and relations of differential properties of functions in various metrics. In addition to its independent interest from the point of view of function theory, it also has numerous and effective applications in the theory of partial differential equations. Such applications were given by S.L. Sobolev in [5] (see, also [3]). S.L. Sobolev studied isotropic spaces $W_p^{(l)}\left(G\right)$ of functions f defined on a domain $G\subset R^n$ with the norm

$$\|f\|_{W_p^{(l)}(G)}=\sum_{|\alpha|\le l}\|D^{\alpha}f\|_{L_p(G)}\ ,$$

where l is a non-negative integer and $l\ge 1$. S.L. Sobolev proved embedding theorems for function space $W_p^{(l)}\left(G\right)$ in domains of n-dimensional Euclidean spaces. Namely, theorems on the summability in power q of derivatives $D^{\beta}f$ with respect to a domain G or manifolds of lower dimension belonging to G.

In subsequent years, the theory of embedding developed intensively in various directions and received new interesting and important applications. Among these works, one can note the works of S.M. Nikolskii, O.V. Besov, V.P. Ilin, N. Aronszajn, V.M. Babich, L.N. Slobodetskii, A.S. Jafarov, G. Freud, D. Kralik, V.I. Burenkov, A.J. Jabrailov and others. For more details we refer the readers to [1] and [4].

S.L. Sobolev established embedding theorems using integral representations of functions in terms of their weak derivatives. This method of integral representations was developed in the works of V.P. Ilin and, in particular, was carried over to cases of representation through differences. One of the significant advantages of the method of integral representations is that the representation of a function at a given point is constructed from the values of this function at the points of a bounded cone with vertex at this point. This creates an opportunity to study function spaces of functions defined on an open set of a sufficiently general form.

The remainder of the paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. Our principal assertions, concerning the traces in generalized Sobolev-type spaces with generalized smoothness are formulated and proved in Section 3. We establish sufficient conditions on a domain $G \subset R^n$ for the validity of trace theorem.

2. Prelmininaries.

Let the surface $\Gamma_s \subset \overline{G} \quad (1 \le s \le n-1)$ be defined by the system of equations

$$\left.\begin{array}{l} x_1 = x_1 \\ x_2 = x_2 \\ \dots\dots\dots \\ x_s = x_s \\ x_{s+1} = g_{s_1}(x_1, ..., x_s) \\ \quad \dots\dots\dots \\ x_n = g_n(x_1, ..., x_s) \end{array}\right\}, \tag{1}$$

where the functions $x_k = g_k(x_1, ..., x_s), \left(k = \overline{s+1}; n\right)$ are defined and continuous in some domain $G_s \subset E_s$. It is assumed here that the partial derivatives $\frac{\partial}{\partial x_j} g_k(x_1, ..., x_s) \left(j = \overline{1; s}; k = s+1; ..n\right)$ are continuous and bounded in G_s. That is, there exists the numbers $M_{k,j} = const$ such that

$$\left| \frac{\partial}{\partial x_j} g_k(x_1, ..., x_s) \right| \le M_{k,j} \tag{2}$$

for all $j = \overline{1, s};\ k = \overline{s+1, n}$ and $(x_1, ..., x_s) \in G_s$.

The class of surfaces $\Gamma_s \in G$ satisfying conditions (1) and (2) is denoted by $\prod^1$.

Let $h = (0, .., 0, h_{s+1}, ..., h_n)$ and let

$$h_k > 0.\ \left(k = \overline{s+1; n}\right) \qquad (*)$$

By $F_s + h$ we denote the surface defined by the system of following equations

$$\left.\begin{array}{l}x_1 = x_1\\ x_2 = x_2\\ \dots\dots\dots\\ x_s = x_s\\ x_{s+1} = g_{s+1}\left(x_1,...,x_s\right) + h_{s+1}\\ \quad\dots\dots\dots\\ x_n = g_n\left(x_1,...,x_s\right) + h_n\end{array}\right\}, \tag{3}$$

where $x_{s=}g_k\left(x_1,...,x_s\right) \quad \left(k=\overline{s+1,n}\right)$ the same as in equalities (1) and the vector (*) so that $F_s + h \subset G$.

The traces of functions $D^\nu\left(f-\Phi_{l-1}\right)$ on the surface $\Gamma_s \in \overline{G}$ in $L_q\left(\Gamma_s\right)$ $(1<q<\infty)$are understood as the existence of the limit

$$\left\| D^\nu\left(f-\Phi_{l-1}\right)|_{\Gamma_s}\right\|_{L_q(\Gamma s)} = \lim_{|h|\to 0}\left\|D^\nu\left|\left(f-\Phi_{l-1}\right)\right|_{\Gamma_s}\right\|_{L_q(\Gamma_s+h)}. \tag{4}$$

Under assumptions (2), after corresponding transformations, we have

$$\left\| D^\nu\left(f-\Phi_{l-1}\right)|_{\Gamma_s}\right\|_{L_q(\Gamma_{s+h})} \leq C\left\|D^\nu\left(f-\Phi_{l-1}\right)\left(...g_{s+1}\left(.\right)+h_{s+1},...,g_n\left(.\right)+h_n\right)\right\|_{L_q(G_s)} \tag{5}$$

Therefore, evaluating the expressions on the right-hand side of (5) uniformly with respect to vectors $h=\left(0,...,0,h_{s+1},...,h_n\right)$, we get the opportunity to estimate the integral expressions $\left\| D^\nu\left(f-\Phi_{l-1}\right)|_{\Gamma_s+h}\right\|_{L_q(\Gamma s+h)}$ for $|h| = \sum_{j=s+1}^n h_j \to 0$.

We have

$$\left\|D^\nu\left(f-\Phi_{l-1}\right)\left(...g_{s+1}\left(.\right)+h_{s+1},...,g_n\left(.\right)+h_n\right)\right\|_{L_q(G_s)} \leq$$

$$\leq C\sum_{i=1}^n \left\|D^\nu\left(f-\Phi_{l-1}\right)\left(...g_{s+1}\left(.\right)+h_{s+1},...,g_n\left(.\right)+h_n\right)\right\|_{L_q(\Omega_{s,i})}. \tag{6}$$

Here $\Gamma_s \in \overline{G}$, the subdomains $\Omega_1,...,\Omega_n \subset G$ satisfying the "a_1-horn" condition cover a domain $G \subset E_n$, i.e. $G \subset \bigcup_{i=1}^n \Omega_i$ and $\Gamma_{s,i} = F_s \bigcap \Omega_i \quad \left(i=\overline{1,n}\right)$.
these parts of Γ_s are defined on the corresponding subdomain $\Omega_{s,i}$. So, $G_s \subset \bigcup_{i=1}^n \Omega_{s,i}$.
By inequalities (6) we need to estimate the integral operators

$$\left\|D^\nu\left(f-\Phi_{l-1}\right)\left(...g_{s+1}\left(.\right)+h_{s+1},...,g_n\left(.\right)+h_n\right)\right\|_{L_q(\Omega_{s,i})}$$

for all $i=\overline{1,n}$.
We rewrite the integral representation of the functions $D^{\nu+m}\left(f-\Phi_{l-1}\right)$ at the points $g\left(\overline{x}^s\right) = \left(x_1,...,x_{s,}g_{s+1}\left(x_1,...,x_n\right),g_n\left(x_1,...,x_n\right)\right) == \left(\overrightarrow{x}^s,\ g_{s-1}\left(\overrightarrow{x}^s\right),...,g_n\left(\overrightarrow{x}^s\right)\right)$ of the surface $\Gamma_s \subset \overline{G}$ as

$$D^{\nu+m}\left(f-\Phi_{l-1}\left(x\right)\right) = \sum_{k=1}^n (-1)^{\left|\nu-m+m^k\right|} C_k \int_0^1 W_k\left(G\right)\frac{da_k\left(\nu\right)}{a_k\left(\nu\right)}\times$$

$$\times\int_{E_{\left|E_{n^k}\right|}}dz\int_{E_n}\left\{\Delta^{N^k}\left(\frac{z}{N^k}\right)D^{m^k}f\left(x+y\right)\right\}M_{\delta,k}dy. \tag{7}$$

Also, we rewrite the integral representation of the functions $D^{\nu+m}\left(f-\Phi_{l-1}\right)$ at the points $g\left(\overline{x}^s\right)+h=\left(\overline{x}^s,g_{s+1}\left(\overline{x}^s\right)+h_{s+1},...,g_n\left(\overline{x}^s\right)+h_n\right)$ of the surface $\Gamma_s+h\subset G$ as

$$D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)=\sum_{k=1}^{n}(-1)^{\left|m^k-\nu\right|}C_k\int_0^1\prod_{j=1}^{n}\left(a_j\left(\upsilon\right)\right)^{m_j^k-\nu_j}\frac{da_k\left(\upsilon\right)}{a_k\left(\upsilon\right)}\times$$

$$\int_{E_{\left|E_{N^k}\right|}}\frac{dz}{mes\left(R_{\delta i}..E_{\left|E_{N^k}\right|}\right)}\int_{E_n}\left\{\Delta^{N^k}\left(\frac{z}{N^k}\right)D^{m^k}fg\left(\overline{x}^s\right)+h+y\right\}$$

$$M_{\delta^i},k\left(\frac{y}{a\left(\upsilon\right)};\frac{z}{a\left(\upsilon\right)}\right)\frac{dy}{mes\left(R_{\delta^i}.E_n\right)}. \tag{8}$$

Here in (8), the kernels M_{δ^i},k are the same from (see [1]), while $\delta=\delta^i=\left(\delta_1^i,...,\delta_s^i\right)$.
We observe that $mes\left(R_{\delta^i}.E_{E\left|_{N^k}\right|}\right)=\prod_{j\in E_{N^k}}a_{j(\upsilon)}$, (**)
and

$$mes\left(R_{\delta^i}\bullet E_n\right)=\prod_{j=1}a_{j(\upsilon)}.(***)$$

The indicated notations from (**) and (***) are justified by the fact that the supports of integral representation (8) is an " a_1-horn"

$$R_{ji}=\bigcup_{0<\upsilon\leq 1}\left\{y=\left(y_1,...,y_n\right)\in E_n,C_{j\leq}\frac{y_j\overset{i}{\underset{j}{\delta}}}{a_j\left(\upsilon\right)}\leq C_j^*\right\}$$

with vertex at points $g\left(\overline{x}^s\right)+h$.
Let $G\subset E_n$ be a domain satisfies the "a_1-horn" condition. Thus $G\subset\bigcup_{i=1}^{M}\Omega_i$satisfies the "$a_1$-horn" condition and $\Gamma_{s,i}=\Gamma_s\bigcap\Omega_i\quad(i=1,...,M)$, where $G_s\subset\bigcup_{i=1}^{M}\Omega_{s,i}$.
The collection of auxiliary functions is defined by the equalities

$$B_i\left[D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right]=\sum_{k=1}^{n}(-1)^{\left|m^k-\nu\right|}C_k\int_0^1\prod_{j=1}^{n}a_j\left(\upsilon\right)^{m_j^k-\nu_j}\frac{da_k\left(\upsilon\right)}{a_k\left(\upsilon\right)}\times$$

$$\int_{E_{\left|E_{N^k}\right|}}\frac{dz}{mes\left(R_{\delta i}..E_{\left|E_{N^k}\right|}\right)}\int_{E_n}\left\{\Delta^{N^k}\left(\frac{z}{N^k}\right)D^{m^k}f\left(g\left(\overline{x}^s\right)+h+y\right)\right\}\times$$

$$\times M_{k,\delta^i},k\left(\frac{y}{a\left(\upsilon\right)};\frac{z}{a\left(\upsilon\right)}\right)\frac{dy}{mes\left(R_{\delta^i}.E_n\right)}=$$

$$=\sum_{k=1}^{n}M_{k,\delta^i}\left(D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right). \tag{9}$$

Each function $B_i\left[D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right]$ is defined on $E_s$and on the corresponding $\Omega_{s,i}$coincides with the function $D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)$.

Thus, we have

$$\left\|D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)_{L_q(G_s)}\right\| \leq$$

$$\leq C\sum_{k=1}^{n}\sum_{i=1}^{M}\left\|M_{k,\delta^i}\left(D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right)\right\|_{L_{q(E_s)}}. \tag{10}$$

Therefore, it is necessary to estimate the integral expressions on the right-hand side of inequality (10) for all $k=\overline{1,n}$, $\quad i=\overline{1,M}$.

We observe that

$$\left|M_{k,\delta^i}\left(D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right)\right| \leq$$

$$\leq C\int_0^1 W_k\left(\upsilon\right)\frac{da_k\left(\upsilon\right)}{a_k\left(\upsilon\right)}\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\prod_{j\in E_{N^k}}\frac{dz_j}{z^{1/p_k-\gamma_j}}\times$$

$$\times\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\left|\frac{\Delta^{N^k}\left(\frac{z}{N^k};\Omega_i+R_{\delta^i}\right)D^{m^k}f\left(g\left(\overline{x}^s\right)+h+y\right)}{\prod_{j\in E_{N^k}}\varphi_k^j\left(z_j\right)}\right|dy \tag{11}$$

Here $\overrightarrow{a}\left(G\right)=\left(a_1\left(\upsilon\right),...,a_n\left(\upsilon\right)\right)$,

$$W_k\left(G\right)=\prod_{j=1}^{n}\left(a_j\left(\upsilon\right)^{m_j^k-\nu_j-1}\prod_{j\in E_{N^k}}a_j\left(\upsilon\right)^{\frac{1}{p_k}-1-\gamma_j}\times\right)$$

$$\times\prod_{j\in E_{N^k}}\varphi_j^k\left(a_j\left(G\right)\right)\qquad 0<\gamma_j,<\frac{1}{p_k}\quad\left(j\in E_{N^k}\right), \tag{12}$$

$$g\left(\overline{x}^s\right)+h+y=\left(x_1+y_1,...,x_s+y_s,g_{s+1}\left(x_1,...x_s\right)\right)+$$

$$+h_{s+1}+y_{s+1},...,g_n\left(x_1,...,x_s\right)+h_n+y_n=$$

$$=\left(\overline{x}^s+\overline{y},^s g_{s+1}\left(\overline{x}^s\right)+h_n+y_n\right). \tag{13}$$

3. L_q-estimate of the traces $D^{\nu}\left(f-\Phi_{l-1}\right)$ on the surface Γ_s.

We reduce the main results of our paper.

Theorem 1. Let $f\in\bigcap_{k=1}^{n}\Lambda_{p_k,\theta_k}^{\langle m_j^k N^k\rangle}\left(G,\varphi^k\right)$ and let $1<p_k\leq\theta_k<\infty$, $q\geq p_k$, $\left(k=\overline{1,n}\right)$ .Suppose that $m_k=\left(m_1^k,...,m_n^k\right)$, $N_k=\left(N_1^k,...,N_n^k\right)$ $\left(k=\overline{1,n}\right)$ are “integers and non-negative” vectors such that$\{k\}\subset\sup p\left(m^k+N^k\right)$ $\left(k=\overline{1,n}\right)$ .Let $G\in c\left(\overline{a}\left(\upsilon\right);1\right)$ be a domain and we set $\overrightarrow{a}\left(\upsilon\right)=\left(a_1\left(\upsilon\right),...,a_n\left(\upsilon\right)\right)$, $\left(0<\upsilon\leq1\right)$, where $a_k=a_k\left(\upsilon\right)>0$ $\left(0<\upsilon\leq1\right)$, $a_k\left(\upsilon\right)\rightarrow0$ *for* $\left(\upsilon\rightarrow0+\right)$, $a_k^1\left(\upsilon\right)>0$, $\left(0<\upsilon\leq1\right)$ $\left(k=\overline{1,n}\right)$. Suppose that$\varphi_j^k=\varphi_j^k\left(t^j\right)$ $\left(j=\overline{1,n}\right)$ are positive and continuous vector-functions $\varphi^k\left(t\right)=\left(\varphi_1^k\left(t_1\right),..,\varphi_n^k\left(t_n\right)\right)$such that

$\varphi_j^k(t_j) \to 0 \quad (t_j \to 0) \qquad \left(j = \overline{1,n}\right)$ for all $\left(k = \overline{1,n}\right)$.
Let the multi-index $\nu = (\nu_1, ..., \nu_n)$ satisfy condition
$\begin{cases} \nu_k < N_k^k + m_k \\ \nu_j \geq N_j^k + m_j^k \end{cases}$ $j \neq k$ for all
$k = \overline{1,n}$. Here

$$Q_{k,s}(1) = \int_0^1 \prod_{j=1}^n \left(a_j(\upsilon)^{m_j^k - \nu_j - \frac{1}{p_k}} \prod_{j \in E_{N^k}} a_j(\upsilon)^{\frac{1}{q}} \right) \times$$

$$\times \prod_{j \in E_{N^k}} \varphi_j^k(a_j(\upsilon)) \frac{da_k(\upsilon)}{a_k(\upsilon)}, \tag{14}$$

where $k \in l_n = \{1, 2, ..., n\}$.

Then there is a polynom $\Phi_{l-1} = \Phi_{l-1}(x; f)$ of degree less than $(s_k - 1)$, in each of the variables $x_k \left(k = \overline{1,n}\right)$, while

$$D^\nu (f - \Phi_{l-1})|_{\Gamma_s} \in L_q(\Gamma_s) \tag{15}$$

and $s - 1 = (s_1 - 1, ..., s_n - 1)$,$s_k = m_k^k + N_k^k$ $(k = \overline{1,n})$.
In addition, the following inequality

$$\left\| D^\nu (f - \Phi_{l-1})|_{\Gamma_s} \right\|_{L_q(\Gamma_s)} \leq C \sum_{k=1}^n Q_{k,s}(1) \|f\|_{\Lambda_{p_k,\theta_k}^{\langle m_j^k N^k \rangle}(G;\varphi_k)}$$

holds, where ? is a positive constant independent on $f = f(x)$.

Proof. It follows from the definition of the set of auxiliary functions (9) and integral inequalities (6), (10) and (11) that

$$\|D^\nu (f - \Phi_{l-1})(g(\overrightarrow{x}^s) + h)\|_{L_q(G_s)} \leq$$

$$\leq C \sum_{i=1}^M \|B_i \{D^\nu (f - \Phi_{l-1})(g(\overline{x}^s) + h)\}\|_{L_q(\Omega_{s,i})} \leq \tag{16}$$

$$\leq C \sum_{i=1}^M \sum_{k=1}^n \left\| M_{k,\delta^i} D^\nu (f - \Phi_{l-1})(g(\overline{x}^s) + h) \right\|_{L_q(E_s)}.$$

The resulting integral inequalities show that the proofs of the theorem can be reduced to estimates for the integral operators

$$\left\| M_{k,\delta^i} D^\nu (f - \Phi_{l-1})(g(\overline{x}^s) + h) \right\|_{L_q(E_s)} \tag{17}$$

The estimates for the integral operators (17) are singled out in the form of separate lemmas, and the continuation of the proof of the theorem will be given only after the proof of the auxiliary lemmas.

Lemma 1. Let be satisfy all conditions of Theorem 3.1. Then

$$\left\|M_{k,\delta^i}\left(D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right)\right\|_{L_q(s)}\leq$$

$$\leq CQ_{k,s}\left(1\right)\left\|f\right\|_{\Lambda^{\langle m^k N^k\rangle}_{p_k,\theta_k}\left(\Omega_i+R_{\delta^i};\varphi_k\right)} \tag{18}$$

for all $k\in l_n=\{1,2,...,n\}$ and $i=1,2,...,M$. Here $Q_{k,s}\left(1\right)$ are defined by (14).
Proof. Change the variables on the right-hand side of (11) as

$$g_j\left(\overline{x}^s\right)+h_j+y_j=y_j^* \qquad \left(j=\overline{s+1};n\right).$$

Therefore, we get

$$\left|M_{k,\delta^i}D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overline{x}^s\right)+h\right)\right|\leq$$

$$\leq C\int_0^1 W_k\left(\upsilon\right)\frac{da_k\left(\upsilon\right)}{a_k\left(\upsilon\right)}\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\prod_{j\in E_{N^k}}\frac{dz}{z_j^{1/p_k-\gamma_j}}\times$$

$$\times\int_0^{a_1(\upsilon)}dy_1...\int_0^{a_s(\upsilon)}dy_s\int_{g(\overline{x}^s)+h_s+1}^{g_{s+1}(\overline{x}^s)+h_{s+1}+a_{s+1}(\upsilon)}.....$$

$$...\int_{g_n(\overline{x}^s)+h_s}^{g_n(\overline{x}^s)+h_n+a_{n(\upsilon)}}\left|\frac{\Delta^{N^k}\left(\frac{z}{N^k};\Omega_i+R_{\delta^i}\right)D^{m^k}f\left(\overrightarrow{x}^s+\overrightarrow{y}^s,y_{s+1}^*...y_n^*\right)}{\prod_{j\in E_{N^k}}\varphi_j^k\left(z_j\right)}\right|dy_n^*, \tag{19}$$

where $W_k\left(\upsilon\right)$ are defined by (12).
In first, we set $1<p_k=\theta_k=q<\infty$.
In this case, we apply Hölder's inequalities with exponents p_k and $p_k=\frac{p_k}{p_k-1}$ and integral Minkowskii inequality to the right-hand side of inequality (19), we have

$$\left\|M_{k,\delta^i}\left(D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overrightarrow{x}^s\right)+h\right)\right)\right\|_{L_{p_k}(E_s)}\leq$$

$$\leq C\int_0^1 W_k{}^*\left(\upsilon\right)\frac{da_k\left(\upsilon\right)}{a_k\left(\upsilon\right)}\left\|\left(\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\prod_{j\in E_{N^k}}\frac{dz_j}{1-p_k\gamma_j}\times\int_0^{\overrightarrow{a}(\upsilon)}d\overrightarrow{y}^s\int_{E_{N/s}}|...|^{p_k}dy_{s+1}...dy_n\right)\right\|_{L_{p_k}(E_s)}, \tag{20}$$

where

$$W_k^*\left(\upsilon\right)=W\left(\upsilon\right)\prod_{j\in E_{N^k}}\left(a_j\left(\upsilon\right)\right)^{\frac{1}{p_k^1}}\prod_{j=1}^{n}\left(a_j\left(\upsilon\right)\right)^{\frac{1}{p_k^1}}. \tag{21}$$

Therefore, changing the order of integration and changing the variables$x_j+y_j=x_j^*\quad\left(j=1,\overline{s}\right)$, we obtain

$$\left\|M_{k,\delta^i}\left(D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overrightarrow{x}^s\right)+h\right)\right)\right\|\leq$$

$$\leq \int_0^1 W_k{}^*(\upsilon) \prod_{j=1}^n (a_j(\upsilon))^{\frac{1}{p^k}} \frac{da_k(\upsilon)}{a_k(\upsilon)} \times \left\{ \int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}} \|...\|_{L_{p_k}(E_n)}^{p_k} \prod_{j\in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right\}^{\frac{1}{p_k}}, \tag{22}$$

where

$$\|...\|_{L_{p_k}(E_n)} = \left\| \frac{\Delta^{N^k}\left(\frac{z}{N^k};\Omega_i + R_{\delta^i}\right) D^{m^k} f(.)}{\prod_{j\in E_{N^k}} \varphi_j^k(z_j)} \right\|_{L_{p_k}(E_n)} \tag{23}$$

Let $1 < q = p_k = \theta_k < \infty$.
It follows from (22) and (23) that

$$\left\|M_{k,\delta^i}\right\|_{L_{p_k}(E_s)} \leq C \int_0^1 W_k{}^*(\upsilon) \prod_{j=1}^n (a_j(\upsilon))^{\frac{1}{p^k}} \prod_{j\in E_{N_k}} (a_j(\upsilon))^{\gamma_j} \frac{da_k(\upsilon)}{a_k(\upsilon)} \times$$

$$\times \|f\|_{\Lambda_{p_k,\theta_k}^{\langle m^k;N^k \rangle}} (\Omega_i + R_{\delta^i};\varphi_k). \tag{24}$$

The integral expressions on the right-hand side of the last inequality (24) are equal to the corresponding $Q_{k,s}$ for $q = p_k$. Thus,

$$\left\|M_{k,\delta^i}\right\|_{L_{p_k}(E_s)} \leq C \bullet Q_{k,s}(1) \|f\|_{\Lambda_{p_k,\theta_k}^{\langle m^k;N^k \rangle}(\Omega_i + R_{\delta^i};\phi_k)}. \tag{25}$$

This proved inequalities (18) of the lemma, for $1 < q = p_k = \theta_k < \infty$.

Let $1 < q = p_k = \theta_k < \infty$.

In this case, to the integral expression on the right-hand side of inequality (22), we apply Hölder's inequality with exponents $\lambda_1 = \frac{Q_k}{p_k}$ *and* $\lambda_2 = \frac{Q_k}{Q_k - p_k}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1\right)$. So, we get

$$\left\|M_{k,\delta^i}\right\|_{L_{p_k}(E_s)} \leq C \int_0^1 W_k{}^*(\upsilon) \prod_{j=1}^n (a_j(\upsilon))^{\frac{1}{p^k}} \times$$

$$\prod_{j\in E_{N^k}} \left(\int_0^{a_j(\upsilon)} \frac{dz_j}{z^{1-p_k\gamma_j\lambda_2}} \right)^{\frac{1}{p_k\lambda_2}} \frac{da_k(\upsilon)}{a_k(\upsilon)} \times$$

$$\times \left\{ \int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}} \|...\|_{L_{p_k}(E_n)}^{p_k\lambda_1} \prod_{j\in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right\}^{\frac{1}{p_k\lambda_1}} \tag{26}$$

It is obvious that

$$\left\{ \int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}} \|...\|_{L_{p_k}(E_n)}^{p_k\lambda_1} \prod_{j\in E_{N^k}} \frac{dz_j}{z_j} \right\}^{\frac{1}{p_k\lambda_1}} \leq! \|f\|_{\Lambda_{p_k,\theta_k}^{\langle m^k;N^k \rangle}(\Omega_i + R_{\delta^i};\varphi_k)} \tag{27}$$

and

$$\int_0^1 W_k{}^*(\upsilon)\prod_{j=1}^{s}(a_j(\upsilon))^{\frac{1}{p^k}}\prod_{j\in E_{N^k}}\left(\int_0^{a_j(\upsilon)}\frac{dz_j}{z_j^{1-p_k\gamma_j\lambda_2}}\right)^{\frac{1}{p_k\lambda_2}}\frac{da_k(\upsilon)}{a_k(\upsilon)}=Q_{k,s}(1)\,,\quad ?@8\quad q=p_k. \tag{28}$$

By inequalities (26), (27) and (28), we get

$$\left\|M_{k,\delta^i}\right\|_{L_{p_k}(E_s)}\le C\left\|f\right\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(\Omega_i+R_{\delta^i};\varphi_k\right)} \tag{29}$$

Thus, by (25) the lemma is proved for $1<q=p_k=\theta_k<\infty$.
Let

$$1<q=p_k=\theta_k<\infty, q>p_k. \tag{30}$$

We set $\lambda_1=q,\ \lambda_2=\frac{qp_k}{q-p_k}$ and $\lambda_3=p_1^k=\frac{p_k}{p_k-1}$. It is obvious that $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=1$, while $p_k\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)=1$. Therefore, $\frac{1}{p_k}-\gamma_j=(1-\gamma_jp_k)\left(\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right)$.
Applying Hölder's inequality, with exponents λ_1,λ_2 and $\lambda_3$to the right-hand side of (19), we have

$$\left|M_{k,\delta^i}\left(D^\nu\left(f-\Phi_{l-1}\right)\left(g\left(\overrightarrow{x}^s\right)+h\right)\right)\right|\le C\int_0^1 W_k(\upsilon)\prod_{j\in E_{N_k}}^{n}(a_j(\upsilon))^{\frac{1}{p_k'}}\frac{da_k(\upsilon)}{a_k(\upsilon)}\times$$

$$\Bigg\{\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)}\left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)}d\overrightarrow{y}^s\int_{E_{N\setminus s}}\left|\Delta^{N^k}\left(\tfrac{z}{N^k};\Omega_i+R_{\delta^i}\right).D^{m^k}f\left(\overrightarrow{x}^s+\overrightarrow{y}^s,y_{s+1}^*,...,y_n^*\right)\right|^{p_k}dy...dy^*\right)\times$$

$$\left(\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\left\|\frac{\Delta^{N^k}\left(\frac{z}{N^k};\Omega_i+R_{\delta^i}\right)D^{m^k}f(.)}{\prod_{j\in E_{N^k}}\varphi_j^k(z_j)}\right\|_{L_{p_k}(E_n)}^{p_k}\times\prod_{j\in E_{N^k}}\frac{dz_j}{z_j^{1-p_k\gamma_j}}^{\frac{1}{p_k}-\frac{1}{q}}\right)$$

$$\times\prod_{j\in E_{N^k}}\frac{dz_j}{z_j^{1-p_k\gamma_j}}\Bigg\}^{\frac{1}{q}}. \tag{31}$$

Here $\overrightarrow{0}=(0,...,0)\,,\ \ \overrightarrow{a}(\upsilon)=(a_1(\upsilon),...,a_n(\upsilon))$ and

$$W_k(\upsilon)=\prod_{j=1}^{n}a_j(\upsilon)^{m_j^k-\nu_j-1}\prod_{j\in E_{N^k}}^{n}a_j(\upsilon)^{\frac{1}{p_k}-\nu_j-1}\prod_{j\in E_{N^k}}\varphi_j^k(a_j(\upsilon))\,. \tag{32}$$

By (31), one has

$$\left\|M_{k,\delta^i}\right\|_{Lq(E_s)}\le C\int_0^1\tilde{W}_k(\upsilon)\frac{da_k(G)}{a_k(\upsilon)}\times$$

$$\times\left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)}\left\|\frac{\Delta^{N^k}\left(\frac{z}{N^k};\Omega_i+R_{\delta^i}\right)D^{m^k}f(.)}{\prod_{j\in E_{N^k}}\varphi_j^k(z_j)}\right\|_{L_{p_k}(E_n)}^{p_k}\prod_{j\in E_{N^k}}\frac{dz_j}{z_j^{1-p_k\gamma_j}}\right)^{\frac{1}{p_k}-\frac{1}{q}}\times$$

$$\left\| \left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} \left\{ \int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} dy^s \int_{E_{N\setminus s}} |..|^{p_k} \, dy_{s+1}^* ... dy_n^* \right\} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right)^{\frac{1}{q}} \right\|_{L_q(E_s)}, \tag{33}$$

where

$$|...| = \left| \frac{\Delta^{N^k}\left(\frac{z}{N^k}; \Omega_i + R_{\delta^i}\right) D^{m^k} f\left(\overrightarrow{x}^s + \overrightarrow{y}^s, y_{s+1}^*, ..., y_n^*\right)}{\prod_{j \in E_{N^k}} \varphi_j^k (z_j)} \right|, \tag{34}$$

and

$$\tilde{W}_k(\upsilon) = W_k(\upsilon) \prod_{j \in E_{N^k}}^{n} (a_j(\upsilon))^{\frac{1}{p_k^1}} \prod_{j=1}^{n} (a_j(\upsilon))^{\frac{1}{p_k^1}}. \tag{35}$$

We change the variables $x_j + y_j = x_j^* \quad (j = 1, 2, ..., s)$.Then the integral expressions

$$\left\| \left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} \left[\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} dy^s \int_{E_{N\setminus s}} |...|^{p_k} \, dy_{s+1}^* ... dy_n^* \right] \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right)^{\frac{1}{q}} \right\|_{L_q(E_s)} \tag{36}$$

estimates by

$$\left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} \left\| \frac{\Delta^{N^k}\left(\frac{z}{N^k}; \Omega_i + R_{\delta^i}\right) D^{m^k} f(.)}{\prod_{j \in E_{N^k}} \varphi_j^k (z_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right)^{\frac{1}{q}} \prod_{j=1}^{n} (a_j(\upsilon))^{\frac{1}{q}}. \tag{37}$$

Therefore, by (33), one has

$$\left\| M_{k,\delta^i} \right\|_{L_{p_k}(E_s)} \leq C \int_0^1 W_k{}^*(\upsilon) \prod_{j \in E_{N_k}} (a_j(\upsilon))^{\frac{1}{\lambda_j}} \frac{da_k(\upsilon)}{a_k(\upsilon)} \times$$

$$\left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} \left\| \frac{\Delta^{N^k}\left(\frac{z}{N^k}; \Omega_i + R_{\delta^i}\right) D^{m^k} f(.)}{\prod_{j \in E_{N^k}} \varphi_j^k (z_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right)^{\frac{1}{p_k}}. \tag{38}$$

From (38), in the case of $1 < p_k = \theta_k < \infty$, $q > p_k$, it follows that the inequality

$$\left\| M_{k,\delta^i} \right\|_{L_{p_k}(E_s)} \leq C.Q_{k,s}(1) \, \|f\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k; N^k \right\rangle}\left(\Omega_i + R_{\delta^i}; \varphi^k\right)}. \tag{39}$$

Let $q > p_k$ and let $1 < p_k = \theta_k < \infty$. Applying Hölder's inequality with exponents $\lambda_1 = \frac{\theta_k}{p_k}$ and $\lambda_2 = \frac{\theta_k}{\theta_k - p_k} \quad \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1\right)$ to the integral expressions on the right-hand side of (38), we have

$$\left(\int_{\overrightarrow{0}}^{\overrightarrow{a}(\upsilon)} \left\| \frac{\Delta^{N^k}\left(\frac{z}{N^k}; \Omega_i + R_{\delta^i}\right) D^{m^k} f(.)}{\prod_{j \in E_{N^k}} \varphi_j^k (z_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1-p_k\gamma_j}} \right)^{\frac{1}{p_k}} \leq$$

$$\leq C\left\{\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\|...\|_{L_{p_k}(E_n)}^{p_k\lambda_1}\prod_{j\in E_{N^k}}\frac{dz_j}{z_j}\right\}^{\frac{1}{p_k\lambda_1}}\left(\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\prod_{j\in E_{N^k}}\frac{dz_j}{z_j^{1-p_k\lambda_2}}\right)^{\frac{1}{p_k\lambda_2}}\leq$$

$$\leq C\prod_{j\in E_{N^k}}^{n}(a_j(\upsilon))^{\gamma_j}\left(\int_{\overrightarrow{0}}^{\overrightarrow{a(\upsilon)}}\|...\|_{L_{p_k}(E_n)}^{\theta_k}\prod_{j\in E_{N^k}}\frac{dz_j}{z_j}\right)^{\frac{1}{\theta_k}}\leq$$

$$\leq C\prod_{j\in E_{N^k}}^{n}(a_j(\upsilon))^{\gamma_j}\|f\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(\Omega_i+R_{\delta^i};\varphi^k\right)}. \tag{40}$$

By inequalities (38) and (40), one has

$$\left\|M_{k,\delta^i}\right\|_{L_{p_k}(E_s)}\leq C\left(\int_0^1 W_k^*\frac{da_k(\upsilon)}{a_k(\upsilon)}\right)\|f\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(\Omega_i+R_{\delta^i};\varphi^k\right)}.$$

So, we have the inequality

$$\left\|M_{k,\delta^i}\right\|_{L_{p_k}(E_s)}\leq C\sum_{i=1}^{M}\sum_{k=1}^{n}\left\|M_{k,\delta^i}\bullet D^\nu(f-\Phi_{l-1})(g(\overline{x}^s)+h)\right\|_{L_{q(E_s)}}{}_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(\Omega_i+R_{\delta^i};\varphi^k\right)}. \tag{41}$$

This complete the proof of inequality (18).
Inequalities (16) and the results of Lemma 1 imply the inequalities

$$\|D^\nu(f-\Phi_{l-1})(g(\overrightarrow{x}^s)+h)\|_{L_q(G_s)}\leq$$

$$\leq C\sum_{i=1}^{M}\sum_{k=1}^{n}\left\|M_{k,\delta^i}\bullet D^\nu(f-\Phi_{l-1})(g(\overrightarrow{x}^s)+h)\right\|_{L_{q(E_s)}}\leq$$

$$C\sum_{k=1}^{n}Q_{k,s}(1)\sum_{i=1}^{M}\|f\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(\Omega_i+R_{\delta^i};\varphi^k\right)}\leq$$

$$\leq C\sum_{k=1}^{n}Q_{k,s}(1)\|f\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(\Omega_i+R_{\delta^i};\varphi^k\right)}. \tag{42}$$

In (42) a constant $C>0$ is independent of $h=(0,...,0,h_{s+1},...,h_n)$.
Thus,

$$\lim_{|h-h^*|\to 0}\|D^\nu(f-\Phi_{l-1})(g(\overrightarrow{x}^s)+h)-D^\nu(f-\Phi_{l-1})(g(\overrightarrow{x}^s)+h^*)\|_{L_q(G_s)}=0. \tag{43}$$

Therefore, we have

$$\left\|D^\nu(f-\Phi_{l-1})|_{\Gamma_s}\right\|_{L_q(\Gamma_s)}=\lim_{|h|\to 0}\left\|D^\nu(f-\Phi_{l-1})|_{\Gamma_{s+n}}\right\|_{L_q(\Gamma_s+n)}\leq$$

$$\lim_{|h|\to 0}\left[D^{\nu}\left(f-\Phi_{l-1}\right)\left(g\left(\overrightarrow{x}^{s}\right)+h\right)\right]_{L_q(G_s)}\leq$$

$$\leq C\sum_{k=1}^{n}Q_{k,s}\left(1\right)\|f\|_{\Lambda_{p_k,\theta_k}^{\left\langle m^k;N^k\right\rangle}\left(G;\varphi^k\right)}$$

This complete the proof of Theorem 2.

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