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The Jost Solutions to the Schrödinger Equation with an Additional Complex Periodic Potential

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Abstract. We consider the one-dimensional Schrodinger equation with an additional complex periodic potential. Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.
Key Words and Phrases: Schrödinger equation, non-self-adjoint differential operator; the space$L_2(-\infty,+\infty)$; transformation operator; the Jost solution.
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1. Introduction and main result

We consider the differential equation

$$-y''+p(x)y+q\left(x\right)y=k^2y, \tag{1}$$

where $p\left(x\right)=\sum\limits_{n=1}^{\infty}p_ne^{inx}$, $\sum\limits_{n=1}^{\infty}\left|p_n\right|<\infty$ and $q(x)$ is a continuously differentiable function with bounded support. When $q(x)=0$, the unperturbed equation

$$-y''+p(x)y=k^2y \tag{2}$$

was first considered by M.G.Gasymov in [1],(see also [2]-[4]) which found sufficient conditions for the solvability of the inverse problem. He showed that equation (2) has a Floquet solution of the form

$$f_0\left(x,k\right)=e^{ikx}\left(1+\sum_{n=1}^{\infty}\frac{1}{n+2k}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right),$$

where the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha\left(\alpha-n\right)\left|V_{n\alpha}\right|,\ \sum_{n=1}^{\infty}n\left|V_{n\alpha}\right|$$

converge. It was proved in [2], for any k with $Imk > 0$, the function $f_0(x,k)$ belongs to $L_2(0,+\infty)$ and the function $f_0(x,-k)$ belongs to $L_2(-\infty,0)$. Moreover, the functions $f_0(x,k)$ and $f_0(x,-k)$ are linearly independent, and their Wronslian is equal to $2ik$.

This paper is devoted to the study of the solutions of (1) with asymptotic conditions

$$f_{\pm}(x,k) = f_0(x,\pm k)\left[1+o(1)\right], x \to \pm\infty.$$

We shall derive the integral representation, which is usually called the Jost translation representation between $f_{\pm}(x,k)$ and $f_0(x,\pm k)$. The obtained results can be used to study the spectral properties of the non-self-adjoint differential operator L, generated by the differential expression $l(y) = -y'' + p(x)y + q(x)y$ in the space $L_2(-\infty,+\infty)$.

The main result of the present paper is as follows.

Theorem 1.1. *For any $k \neq \mp\frac{n}{2}, n = 1,2,...$ from the complex plane, equation* (1) *has solutions $f_{\pm}(x,k)$, which can be represented in the form*

$$f_{\pm}(x,k) = f_0(x,\pm k) \pm \int_x^{\pm\infty} K^{\pm}(x,t)\, f_0(t,\pm k)\, dt, \tag{3}$$

where the kernels $K^{\pm}(x,t)$ are continuous functions and satisfy the following conditions:

$$K^{\pm}(x,t) = \pm\frac{1}{2}\int_x^{\pm\infty} q(t)dt. \tag{4}$$

2. Proof of the theorem

Without loss of generality, we consider the case $''+''$ and assume that $x \geq 0$. We shall use the following notation

$$\sigma(x) = \frac{1}{2}\int_x^{+\infty} |q(t)|\, dt.$$

We first consider the following lemmas before turning to the proof of the theorem.

Lemma 2.1. *If $q(x)$ is a continuously differentiable function with bounded support, then the integral equation*

$$U(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^{+\infty} q(\xi)\, d\xi + \int_0^{\eta_0}\int_{\xi_0}^{+\infty} \left[p(\xi-\eta) - p(\xi+\eta) + q(\xi-\eta)\right] U(\xi,\eta)\, d\xi d\eta, \tag{5}$$

has one and only one solution $U(\xi_0,\eta_0)$. Furthermore, if $q(x) = 0$ when $x > a$, then

$$U(\xi_0,\eta_0) = 0 \text{ when } \xi_0 \geq a. \tag{6}$$

Proof. Using the method of successive approximation, let

$$U_0(\xi_0,\eta_0)=\frac{1}{2}\int\limits_{\xi_0}^{+\infty} q(\xi)\,d\xi, \tag{7}$$

$$U_n(\xi_0,\eta_0)=\int\limits_0^{\eta_0}\int\limits_{\xi_0}^{+\infty}\left[p(\xi-\eta)-p(\xi+\eta)+q(\xi-\eta)\right]U_{n-1}(\xi,\eta)\,d\xi d\eta. \tag{8}$$

Because the function $q(x)$ with bounded support, there exists an $a>0$ such that $q(x)=0$ for $x>a$. By induction with respect to n, we have

$$U_n(\xi_0,\eta_0)=0 \ \ for \ \ \xi_0>a, n=0,1,2,... \tag{9}$$

For any $R>0$, suppose that $0<\eta_0<R$, $0<\xi_0<+\infty$. By (7), we have

$$|U_0(\xi_0,\eta_0)|\le\sigma(\xi_0).$$

Taking the notation

$$M=\max_{\substack{0\le\xi\le 2a\\ 0\le\eta\le R}}|p(\xi-\eta)-p(\xi+\eta)+q(\xi-\eta)|$$

into account, we obtain

$$|U_1(\xi_0,\eta_0)|\le\sigma(\xi_0)(M\eta_0).$$

Using induction, by (8) we next prove that

$$|U_n(\xi_0,\eta_0)|\le\sigma(\xi_0)\frac{1}{n!}(M\eta_0)^n. \tag{10}$$

Hence the series

$$U(\xi_0,\eta_0)=\sum_{n=0}^{\infty}U_n(\xi_0,\eta_0) \tag{11}$$

is uniformly and absolutely convergent, so $U(\xi_0,\eta_0)$ is the solution of the integral equation (5). From (10) and (11), it follows that

$$|U(\xi_0,\eta_0)|\le\sigma(\xi_0)\exp(M\eta_0). \tag{12}$$

This implies obviously the uniqueness of the solution to the equation (5). The assertion (6) is justified by (9) and (11). □

Lemma 2.2. *Suppose $q(x)$ is a continuously differentiable function with bounded support. Then the solution $U(\xi_0,\eta_0)$ of the integral equation* (5) *satisfies the following differential equation*

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}+[p(\xi-\eta)-p(\xi+\eta)+q(\xi-\eta)]\,U(\xi_0,\eta_0)=0 \tag{13}$$

and

$$U(\xi_0,0)=\frac{1}{2}\int\limits_{\xi_0}^{+\infty} q(\xi)\,d\xi. \tag{14}$$

Proof. From (5) the differentiability of $U(\xi_0,\eta_0)$ is evident. Differentiating equation (5) directly, we get the equation (14). Putting $\eta_0=0$ in (6), we get the result (14).

We now let $\xi_0=\frac{t+x}{2}$, $\eta_0=\frac{t-x}{2}$ and express the function $K(x,t)=U(\xi_0,\eta_0)$ as a function of x,t. Then the function $K(x,t)$ is twice continuously differentiable. Moreover, from the two preceding lemmas we get the following lemma. □

Lemma 2.3. *Suppose $q(x)$ is a continuously differentiable function with bounded support. Then the function $K(x,t)=U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ satisfies both the differential equation*

$$\frac{\partial^2 K(x,t)}{\partial x^2}-[p(x)+q(x)]\,K(x,t)=\frac{\partial^2 K(x,t)}{\partial t^2}-p(t)\,K(x,t) \tag{15}$$

and the condition

$$K(x,x)=\frac{1}{2}\int\limits_{x}^{+\infty} q(t)\,dt.$$

Furthermore, if $q(x)=0$ when $x>a$, then $K(x,t)=0$ when $x+t>2a$.

Now the theorem can be proved. By differentiation from (1), we have

$$f'_+(x,k)=f'_0(x,k)-K(x,x)\,f_0(x,k)+\int\limits_{x}^{+\infty} K_x(x,t)\,f_0(t,k)\,dt, \tag{16}$$

$$\begin{aligned} f''_+(x,k)&=f''_0(x,k)-\tfrac{dK(x,x)}{dx}f_0(x,k)-K(x,x)\,f'_0(x,k)-\\ &-K_x(x,x)\,f_0(x,k)+\int\limits_{x}^{+\infty} K''_{xx}(x,t)\,f_0(t,k)\,dt. \end{aligned} \tag{17}$$

From Lemma 2.3, it is easily seen that when t sufficiently large, $K(x,t)=0$, so the last terms of (3), (16),(17) are integrable. From

$$-f''_0(x,k)+p(x)\,f_0(x,k)=k^2 f_0(x,k) \tag{18}$$

and (3), we have

$$\begin{aligned} k^2 f_+(x,k)&=k^2 f_0(x,k)+\int\limits_{x}^{+\infty} K(x,t)\,p(t)\,f_0(t,k)\,dt-\\ &-\int\limits_{x}^{+\infty} K(x,t)\,f''_0(t,k)\,dt. \end{aligned} \tag{19}$$

Hence, integrating by parts, we obtain

$$\begin{aligned}&\int\limits_{x}^{+\infty} K(x,t)\, f_0''(t,k)\, dt = -K(x,x)\, f_0'(x,k) - \int\limits_{x}^{+\infty} K_t'(x,t)\, f_0'(t,k)\, dt = \\ &= -K(x,x)\, f_0'(x,k) + K_t'(x,x)\, f_0(x,k) + \int\limits_{x}^{+\infty} K_{tt}''(x,t)\, f_0(t,k)\, dt.\end{aligned} \tag{20}$$

By virtue of (3) and (17)-(20),we have

$$\begin{aligned}&-f_+''(x,k) + p(x)\, f_+(x,k) - k^2 f_+(x,k) = \\ &= \int\limits_{x}^{+\infty} \left[K_{tt}''(x,t) - K_{xx}''(x,t) + K(x,t)\,(p(x) + q(x) - p(t))\right] f_0(t,k)\, dt + \\ &+ \left[2\frac{dK(x,x)}{dx} + q(x)\right] f_0(x,k).\end{aligned}$$

From the lemma 3 and the last relation, $f_+(x,k)$, when $K^+(x,t) = K(x,t)$ satisfies equation (1). Furthermore, by virtue of (12)-(18), it follows that $f_+(x,k) = f_0(x,k)$ when x sufficiently large. Hence, the $f_+(x,k)$ is a Jost solution. Thus, the proof of the theorem is complete.

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