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On a Determination of the Boundary Function in the Initial-Boundary Value Problem for the Second Order Hyperbolic Equation

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Abstract. In the paper the problem of determination of the boundary function is studied in the initial-boundary value problem described by the second order hyperbolic equation. With the help of the additional condition, the functional is constructed, and the problem under consideration is reduced to the optimal control problem. The differential of the function is calculated, a necessary and sufficient condition for optimality is proved.

Key Words and Phrases: Hyperbolic equation, inverse problem, optimal control, optimality condition.

1. Introduction

Recently, the study of the inverse problems for the partial derivative differential equations has become more intensive [6, 17]. The reason for this is their strong application in in medicine, physics, geophysics, astronomy, biology and ect. With the implementation of the modern computer technologies, the application of inverse problems has led to a further expansion of its areas. Different types of inverse and ill-posed problems have been considered in [6, 12, 13, 14]. It is known that there are many methods to solve such problems [1, 2, 6, 12]. One of these methods is to reduce the problem under consideration to the optimal control problem with the help of the inconsistency functional cnstructed using additional condition or conditions and to study the new problem using the methods of optimal control theory [13,6,10]. This way of solution of the inverse problems is traditionally called a variational or optimization solution method. In [1, 6, 16] some inverse problems in variational formulation have been investigated for the partial differential equations. In many cases this method was used to solve such problems for the parabolic type equations [11]. Hyperbolic type eqautions are the less studied by this method class of equations [3, 4, 5, 15].

Therefore in this work we investigate the problem of determination of the boundary function in the mixed problem for the second order hyperbolic type equation.

2. Problem formulation

In the domain $Q = \Omega \times (0, T)$consider the following boundary value problem

$$\frac{\partial^2 u}{\partial t^2} + Au = f(x,t), (x,t) \in Q, \tag{1}$$

$$u(x,0) = \varphi_0(x), \frac{\partial u(x,0)}{\partial t} = \varphi_1(x), x \in \Omega, \tag{2}$$

$$u|_{S^1} = 0, u|_{S^2} = \vartheta(s,t), (s,t) \in S^2, \tag{3}$$

$$\varphi_0|_{S^1} = 0, \varphi_0|_{S^2} = \vartheta|_{t=0}.$$

Here Ω is a bounded domain from $R^n$with smooth boundary Γ; $S = \Gamma \times (0, T)$ is a lateral surface of the cylinder Q; $S = S^1 \bigcup S^2$, $S^1 \bigcap S^2 = \emptyset$, $mes\, S^i > 0$, $i = 1, 2.$, $S^1 = \Gamma^1 \times (0,\ T)$, $S^2 = \Gamma^2 \times (0,\ T)$, $\Gamma = \Gamma^1 \bigcup \Gamma^2$, $\Gamma^1 \bigcap \Gamma^2 = \emptyset$; $\varphi_0 \in W_2^1(\Omega)$, $\varphi_1 \in L_2(\Omega)$, $f \in L_2(Q)$,

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right),$$

the function $a_{ij}(x,\ t) \in C^1(\bar{Q})$ satisfies in Q the conditions:

$$a_{ij}(x,\ t) = a_{ji}(x,\ t),$$

$$\sum_{i,j=1}^{n} a_{ij}(x,\ t)\xi_i\xi_j \geq \alpha \cdot \sum_{k=1}^{n} \xi_k^2, \alpha > 0,\ \alpha = const.$$

In condition (3) $\vartheta(s,t)$ is an unknown function. To find this function we set the following additional condition

$$\frac{\partial u}{\partial \nu_A}|_{S^1} = a(s,t), (s,t) \in S^1. \tag{4}$$

Here

$$\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j}\cos(\nu, x_i)$$

is a conormal derivative, $a(s,t) \in L_2(S^1)$ is a given function.

To solve the considered problem we reduce it to the following minimization problem [1],[6]:

To find a function $\vartheta(s,t) \in V_m \subset W_2^1(S^2)$ that gives minimum to the functional

$$J(\vartheta) = \frac{1}{2}\int_{S^1}\left[\frac{\partial u(s,t;\ \vartheta)}{\partial \nu_A} - a(s,t)\right]^2 dsdt \tag{5}$$

together with the solution of problem (1)-(3).

Here the function $u(x,t;\vartheta)$ is a solution of problem (1)-(3) correcponding to the function $\vartheta(x,t)$. By closed convex set V_m we denote the class of admissible controls [10]. For each control $\vartheta \in V_m$ problem (1)-(3) has a unique solution from $W_2^1(Q)$ [9] and $u \in C\left([0,\ T];\ W_2^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in C([0,\ T];\ L_2(\Omega))$.

As a solution of problem (1)-(3) we take the function $u(x,t;\vartheta) \in W_2^1(Q)$ that satisfies the integral identity

$$\int_Q \left[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,\ t)\frac{\partial u}{\partial x_j}\frac{\partial \eta}{\partial x_i} \right] dxdt - \int_\Omega \varphi_1(x)\eta(x,0)dx = \int_Q f\eta dxdt \tag{6}$$

For $\forall\eta \in W_2^1(Q)$, $\eta(x,T) = 0$ and a.e. the conditions $u(x,0) = \varphi_0(x)$ and $u|_{S^2} = \vartheta(s,t)$.

3. Calculation of the differential of functional (5) and optimality condition

Let us show that functional (5) is differentiable in V_m. Take two admissible controls $\vartheta,\ \vartheta + \delta\vartheta \in V_m$ and denote by $u(x,t;\vartheta)$ and $u(x,t;\vartheta + \delta\vartheta)$ corresponding solutions of problem (1)-(3).

Let $\delta u(x,t) = u(x,t;\vartheta + \delta\vartheta) - u(x,t;\vartheta)$. It is clear that the function $\delta u(x,t)$ is a generalised solution of the boundary value problem

$$\frac{\partial^2 \delta u}{\partial t^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial \delta u}{\partial x_j}\right) = 0, \quad (x,t) \in Q, \tag{7}$$

$$\delta u(x,0) = 0, \qquad \frac{\partial \delta(x,0)}{\partial t} = 0, \qquad x \in \Omega, \tag{8}$$

$$\delta u|_{S^1} = 0, \quad \delta u|_{S^2} = \delta\vartheta(s,\ t),\ \ (s,\ t) \in S^2. \tag{9}$$

As a generalised solution of boundary value problem (7)-(9) we understand the function $\delta u(x,t) \in W_2^1(Q)$ that satisfies for $\forall\eta \in W_2^1(Q)$, $\eta(x,T) = 0$, $\eta|_{S^2} = 0$ the integral identity

$$\int_Q \left[-\frac{\partial \delta u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,\ t)\frac{\partial \delta u}{\partial x_j}\frac{\partial \eta}{\partial x_i} \right] dxdt + \int_{S^1} \frac{\partial \delta u}{\partial \nu_A}\eta dsdt = 0 \tag{10}$$

and a.e. the conditions $\delta u(x,0) = 0$ and $\delta u|_{S^2} = \delta\vartheta(s,\ t)$.

Suppose that the function $\psi = \psi(x,t;\vartheta)$ is a generalised solution from $W_2^1(Q)$ of the adjoint problem [8]

$$\frac{\partial^2 \psi}{\partial t^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x,\ t)\frac{\partial \psi}{\partial x_j}\right) = 0, (x,t) \in Q \tag{11}$$

$$\psi(x,\ T) = 0, \psi_t(x,\ T) = 0, x \in \Omega, \tag{12}$$

$$\psi|_{S^1} = \frac{\partial u(s,t;\vartheta)}{\partial \nu_A} - a(s,t),\ \ \psi|_{S^2} = 0 \tag{13}$$

By virtue of [9] we can state that boundary value problem (11)-(13) has a unique generalised solution for each fixed $\vartheta \in W_2^1\left(S^2\right)$.

As a generalised solution to adjoint problem (11)-(13) we understand the function $\psi(x,t;\vartheta) \in W_2^1(Q)$ that for $\forall g \in W_2^1(Q)$, $g(x,0) = 0$, $g\,|_{S^1} = 0$ satisfies the integral identity

$$\int_Q \left[-\frac{\partial\psi}{\partial t}\frac{\partial g}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial\psi}{\partial x_j}\frac{\partial g}{\partial x_i}\right] dsdt + \int_{S^2}\frac{\partial\psi}{\partial\nu_A} gdsdt = 0 \tag{14}$$

and a.e. the conditions $\psi(x,\ T) = 0$ and

$$\psi\,|_{S^1} = \frac{\partial u(s,t;\vartheta)}{\partial\nu_A} - a(s,t).$$

Now to show the differentiablity of the functional $J(\vartheta)$ we calculate its increment

$$\Delta J(\vartheta) = J(\vartheta+\delta\vartheta) - J(\vartheta) =$$

$$= \frac{1}{2}\int_{S^1}\left\{\left[\frac{\partial u(s,t;\vartheta+\delta\vartheta)}{\partial\nu_A} - a(s,t)\right]^2 - \left[\frac{\partial u(s,t;\vartheta)}{\partial\nu_A} - a(s,t)\right]^2\right\} dsdt =$$

$$= \frac{1}{2}\int_{S^1}\left\{\left[\frac{\partial u(s,t;\vartheta)+\partial\delta u(s,t)}{\partial\nu_A} - a(s,t)\right]^2 - \left[\frac{\partial u(s,t;\vartheta)}{\partial\nu_A} - a(s,t)\right]^2\right\} dsdt =$$

$$= \int_{S^1}\left[\frac{\partial u(s,t;\vartheta)}{\partial\nu_A} - a(s,t)\right]\frac{\partial\delta u(s,t;\vartheta)}{\partial\nu_A} dsdt + \frac{1}{2}\int_{S^1}\left(\frac{\partial\delta u(s,t;\vartheta)}{\partial\nu_A}\right)^2 dsdt,$$

$$\Delta J(\vartheta) = \int_{S^1}\left[\frac{\partial u(s,t;\vartheta)}{\partial\nu_A} - a(s,t)\right]\frac{\partial\delta u(s,t;\vartheta)}{\partial\nu_A} dsdt + \frac{1}{2}\int_{S^1}\left(\frac{\partial\delta u(s,t;\vartheta)}{\nu_A}\right)^2 dsdt \tag{15}$$

If take $\eta(x,t) = \psi(x,t)$in identity (10) and $g(x,t) = \delta u(x,t)$ in (14) and subtract them we get

$$\int_{S^2}\frac{\partial\psi}{\partial\nu_A}\delta\vartheta dsdt - \int_{S^1}\frac{\partial\delta u}{\partial\nu_A}\left[\frac{\partial u(s,t;\vartheta)}{\partial\nu_A} - a(s,t)\right] dsdt = 0 \tag{16}$$

Considering the last one in (15) we obtain

$$\Delta J(\vartheta) = \int_{S^2}\frac{\partial\psi}{\partial\nu_A}\delta\vartheta dsdt + R, \tag{17}$$

where

$$R = \frac{1}{2}\int_{S^1}\left[\frac{\partial\delta u(s,\ t;\ \vartheta)}{\partial\nu_A}\right]^2 dsdt$$

is a remainder term. It is clear that first summand of right side in formula (17) is differential of functional (5)

$$\delta J(\upsilon) = \int_{S^2}\frac{\partial\psi}{\partial\nu_A}\delta\vartheta dsdt$$

and gradient of functional (5) is

$$J'(\vartheta) = \left.\frac{\partial \psi}{\partial \nu_A}\right|_{S_2}.$$

Let

$$\left\| \frac{\partial \delta u}{\partial \nu_A} \right\|^2_{L_2(S^1)} \leq c \left\| \delta \vartheta \right\|^2_{W_2^1(S^2)} \tag{18}$$

Then the functional $J(\vartheta)$ is Frechet differentiable in V_m. If the function $\vartheta_0(s,t) \in V_m$ gives minimum to functional (5), then $\Delta J(\vartheta_0) \geq 0$. From relations (17) and (18) we get

$$\int_{S^2} \frac{\partial \psi(s,t)}{\partial \nu_A} \delta \vartheta ds dt \geq 0$$

or

$$\int_{S^2} \frac{\partial \psi(s,t)}{\partial \nu_A} \left(\vartheta(s,t) - \vartheta_0(s,t) \right) ds dt \geq 0, \forall \vartheta \in V_m. \tag{19}$$

4. Conclusions

Thus the following theorem is proved.

Theorem 4.1. *Let the above conditions on the data of problem* (1)-(3), (5) *are fulfilled. Then the necessary and sufficient condition the control* $\vartheta_0 \in V_m$ *to be optimal control in problem* (1)-(3), (5) *is fulfilment of variational inequality* (19).

References

[1] Alifanov O.M., Artyukhin E.A., Rumyantsev S.V. Extremal methods for solving ill-posed problems. Moscow: Nauka, 1988.

[2] Glasko V.B. Inverse problems of mathematical physics. Moscow: Moscow State University, 1984.

[3] Guliyev H.F., Nasibzadeh V.N. On determination of coefficient at lowest term in the Cauchy problem for string vibrations equation. Journal of Contemporary Applied Mathematics V. 8, No 1, 2018, July ISSN 2222-5498, 9-18.

[4] Guliyev, H.F., Nasibzadeh, V.N. On a determination of the initial functions from the observed values of the boundary functions for the second-order hyperbolic equation // Advanced Mathematical Models & Applications, -2018. V.3, No 3, pp.215-222.

[5] Gunther Uhlmann, Jian Zhai "Inverse problems for nonlinear hyperbolic equations" Jurnal Discrete & Continuous Dynamical Systems, January 2021, Volume 41, Issue 1, 455-469

[6] Kabanikhin S.I. Inverse and ill-posed problems. Novosibirsk, 2009.

[7] Kolmogorov A.N., Fomin S.V. Elements of the theory of functions and functional analysis, Nauka, Moscow, 1981.

[8] Ladyzhenskaya O.A. Boundary value problems of mathematical physics. Moscow: Nauka, 1973.

[9] Lasiecka L., Lions J.-L. and Triggianı R.. Non homogeneos bondary value problems for second order hyperbolic operators. J. Math. pures et appl., 65, 1986, p. 149-192.

[10] Lions J.-L. Optimal control of systems described by partial differential equations. Moscow: Mir, 1972.

[11] Tagiev R. K., Maharramli Sh. I., Variational formulation of an inverse problem for a parabolic equation with integral conditions, Vestn. Yuzhno-Ural. Gos. Un-ta. Ser. Matem. Mekh. Fiz., 2020, Volume 12, Issue 3, 34–40

[12] Tikhonov A.N., Arsenin V.Ya. Methods for solving ill-posed problems. Moscow: Nauka, 1974.

[13] Vasilev F.P. Methods for solving extremal problems. Moscow: Nauka, 1981.

[14] Vladimir G. R, Masahiro Y. (2019). Recovering two coefficients in an elliptic equation via phaseless information. Inverse Problems and Imaging doi:10.3934/ipi.2019005 Volume 13, No. 1, 81–91

[15] Valitov I.R., Kozhanov A.I. (2006). Inverse problems for hyperbolic equations: the case of unknown time-dependent coefficients. Bulletin of the Novosibirsk State University, Series "Mathematics, Mechanics, Informatics", 6(1), 43-59.

[16] Yuldashev T.K. (2019). Determination of the coefficient in the inverse problem for an integro differential equation of hyperbolic type with spectral parameters. The era of science, 17, 134-149.

[17] Zengin M., İbrahimov N., Yaqub G. Existence and uniqueness of a solution of the optimal control problem with boundary functional for nonlinear stationary quasioptical equation with a special gradient term. Lankaran State University Scientific news, Mathematics and Natural Sciences Series, 1/2021, 27-42

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