Journal of Contemporary Applied Mathematics
V. 12, No 1, 2022, July
ISSN 2222-5498

On Some Characterizations of (s, E)-Convex Functions in the Fourth Sense

W. Saleh

Abstract. A class of function called (s, E)-convex function is defined as a generalization of s-convex and E-convex functions, and some of its basic properties are presented here. Also, some relations between these functions and the other types of convexity functions are given.

Key Words and Phrases: s-convexity; p-convex set; p-convex function; E-convex function..
2010 Mathematics Subject Classifications: 26A51, 26B25, 52A20, 52A41

1. Introduction

Convexity and its generalizations play an important role in optimization theory, convex anlysis and Minkowski space [4, 5, 6, 2, 8].

By relaxing the definitions of convex sets and convex functions, Youness [10] defined E-convex sets and E-convex functions, which have many uses in various branches of mathematical sciences [1, 7, 9].

Furthermore, Youness [11] extended the definitions of E-convex sets and E-convex functions to strongly E-convex sets and strongly E-convex functions.

A key consideration Hudzik and Maligranda explored in their [3] was functions that are s-convex in the second sense. This class is defined in the following way: a function $h : [0, \infty) \longrightarrow \mathbb{R}$ is said to be a s-convex in the second sense if

$$h(ta + (1-t)b) \leq t^s h(a) + (1-t)^s h(b)$$

holds for all $a, b \in [0, \infty)$,$t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

The class of s-convex functions in the second sense is usually denoted by K_s^2. In [12], the s-convex function in the fourth sense is introduced as:

a function $h : U \longrightarrow \mathbb{R}$ is said to be a s-convex in the fourth sense if the inequality

$$h(ta + (1-t)b) \leq t^{\frac{1}{s}} h(a) + (1-t)^{\frac{1}{s}} h(b), \forall a, b \in U.$$

The main aim of this paper is introduced the (s, E)-convex function in the fourth sense and given examples and some characterizations.

2. Main Results

This section deals with the function that is generalization of E-convex functions and s-convex functions in the fourth sense which is called (s,E)-convex function in the fourth sense. Some fundamental properties of this class are given.

Definition 2.1. *Let $E:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a function and $M \subseteq \mathbb{R}^n$ be a nonempty E-convex set. A function $h: M \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a (s,E)-convex function in the fourth sense if*

$$h(\rho_1 E(u_1) + \rho_2 E(u_2)) \le \rho_1^{\frac{1}{s}} h(E(u_1)) + \rho_2^{\frac{1}{s}} h(E(u_2)) \tag{1}$$

holds for all $u_1, u_2 \in M$,$s \in (0,1]$ and $\rho_1, \rho_2 \ge 0$ and $\rho_1 + \rho_2 = 1$.
The inequality (1) is equivalent to the following inequalities:

$$h(\rho_1^s E(u_1) + \rho_2^s E(u_2)) \le \rho_1 h(E(u_1)) + \rho_2 h(E(u_2)),$$

where $\rho_1, \rho_2 \ge 0$ and $\rho_1^s + \rho_2^s = 1$ and

$$h(\rho E(u_1) + (1-\rho)E(u_2)) \le \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)),$$

where $\rho \in [0,1]$.

Remark 2.2. *Defintion 2.1 leads to the following results*

1. *If $s=1$, E-convex function*
2. *If $E=I$, then we have s-convex function in the fourth sense.*
3. *If $s=1$ and $E=I$, then we have convex function.*

On the other hand, the function $h: M \longrightarrow \mathbb{R}$ is said (s,E)-concave function in the fourth sense if

$$h(\rho_1 E(u_1) + \rho_2 E(u_2)) \geqslant \rho_1^{\frac{1}{s}} h(E(u_1)) + \rho_2^{\frac{1}{s}} h(E(u_2))$$

holds for all $u_1, u_2 \in M$,$s \in (0,1]$ and $\rho_1, \rho_2 \ge 0$ and $\rho_1 + \rho_2 = 1$.

Example 2.3. *Let $M \subseteq \mathbb{R}$ and let $h: M \longrightarrow \mathbb{R}$ be defined as $h(x) = -x$ and $E: \mathbb{R} \longrightarrow \mathbb{R}$ such that $E(x) = x^2$,thus for $\rho \in [0,1]$,then*

$$\begin{aligned}
&h(\rho E(u_1) + (1-\rho)E(u_2)) \\
&= h(\rho u_1^2 + (1-\rho)u_2^2) \\
&= -\rho u_1^2 - (1-\rho)u_2^2 \\
&\le -\rho^{\frac{1}{s}} u_1^2 - (1-\rho)^{\frac{1}{s}} u_2^2 \\
&= \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)).
\end{aligned}$$

Hence, h is a (s,E)-convex function in the fourth sense.

Example 2.4. *Let $M \subseteq \mathbb{R}$, $h : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $h(x) = x$, and $E : \mathbb{R} \longrightarrow \mathbb{R}_-$ such as $E(x) = -x$, thus for $\rho \in [0,1]$, it can be written*

$$\begin{aligned} h(\rho E(u_1) + (1-\rho)E(u_2)) &= h(-\rho u_1 - (1-\rho)u_2) \\ &= -\rho u_1 - (1-\rho)u_2 \\ &\leq -\rho^{\frac{1}{s}} u_1 - (1-\rho)^{\frac{1}{s}} u_2 \\ &= \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)), \end{aligned}$$

then h is a (s,E)-convex function in the fourth sense.

Theorem 2.5. *If $h : M \longrightarrow \mathbb{R}$ be a (s,E)- convex function in the fourth sense, then the following inequality is valid for all $x, y \in M$:*

$$h\left(\frac{E(x)+E(y)}{2}\right) \leq \frac{h(E(x)) + h(E(y))}{2^{\frac{1}{s}}}.$$

Proof. It is clear by taking $\rho_1 = \rho_2 = \frac{1}{2}$ in inequality (1). □

Theorem 2.6. *Let $h : M \longrightarrow \mathbb{R}$ be a (s,E)-convex function in the fourth sense, then the inequality*

$$h\left(\frac{E(a)+E(b)}{2}\right) \leq \frac{s}{1+s}\frac{h(E(a)) + h(E(b))}{2}$$

is valid for $0 < s \leq 1$.

Proof. Since h is (s,E)-convex function in the fourth sense on M,then we have

$$\begin{aligned} h\left(\frac{E(a)+E(b)}{2}\right) &= h\left(\frac{tE(a)+(1-t)E(b)}{2} + \frac{(1-t)E(a)+tE(b)}{2}\right) \\ &\leq (\frac{1}{2})^{\frac{1}{s}} \left[h(tE(a)+(1-t)E(b)) + h((1-t)E(a)+tE(b))\right] \\ &\leq (\frac{1}{2})^{\frac{1}{s}} \left[t^{\frac{1}{s}} h(E(a)) + (1-t)^{\frac{1}{s}} h(E(b)) + (1-t)^{\frac{1}{s}} h(E(a)) + t^{\frac{1}{s}} h(E(b))\right] \\ &= (\frac{1}{2})^{\frac{1}{s}} (h(E(a)) + h(E(b)))(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}). \end{aligned} \tag{2}$$

Integrating (2) with respect to t on [0,1], then

$$\begin{aligned} \int_0^1 h\left(\frac{E(a)+E(b)}{2}\right) dt &\leq \int_0^1 (\frac{1}{2})^{\frac{1}{s}} (h(E(a)) + h(E(b)))(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}) dt \\ &= \frac{E(a)+E(b)}{2^{\frac{1}{s}}} \int_0^1 (t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}) dt \\ &= \frac{2s}{1+s}\frac{E(a)+E(b)}{2^{\frac{1}{s}}}. \end{aligned}$$

□

Theorem 2.7. *If $h_i : M \longrightarrow \mathbb{R}$ are (s, E)- convex functions in the fourth sense for $i = 1, 2, ..., m$, then $h = \sum_{i=1}^{m} \mu_i h_i$ is a (s, E)- convex function in the fourth sense where $\mu_i \geq 0$.*

Proof. For $u_1, u_2 \in M$ and $\rho \in [0, 1]$, we have

$$\begin{aligned} h(\rho E(u_1) + (1-\rho)E(u_2)) &= \sum_{i=1}^{m} \mu_i h_i(\rho E(u_1) + (1-\rho)E(u_2)) \\ &\leq \sum_{i=1}^{m} \mu_i(\rho^{\frac{1}{s}} h_i(E(u_1)) + (1-\rho)^{\frac{1}{s}} h_i(E(u_2))) \\ &= \rho^{\frac{1}{s}} \sum_{i=1}^{m} \mu_i h_i(E(u_1)) + (1-\rho)^{\frac{1}{s}} \sum_{i=1}^{m} \mu_i h_i(E(u_2)) \\ &= \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)), \end{aligned}$$

this shows that h is a (s, E)- convex function in the fourth sense . □

Theorem 2.8. *If $h_i : M \longrightarrow \mathbb{R}_-$ are (s, E)- convex functions in the fourth sense for i=1,2,...,m, then $h : M \longrightarrow \mathbb{R}_-$ defined by $h = \max_{1 \leq i \leq m}\{h_i\}$ is a (s, E)- convex function in the fourth sense .*

Proof. For each $u_1, u_2 \in M$ and $\rho \in [0, 1]$, we can write

$$\begin{aligned} & h(\rho E(u_1) + (1-\rho)E(u_2)) = \max_{1 \leq i \leq m} \{h_i(\rho E(u_1) + (1-\rho)E(u_2))\} \\ & \leq \max_{1 \leq i \leq m} \{(\rho^{\frac{1}{s}} h_i(E(u_1)) + (1-\rho)^{\frac{1}{s}} h_i(E(u_2)))\} \\ & \leq \rho^{\frac{1}{s}} \max_{1 \leq i \leq m} \{h_i(E(u_1))\} + (1-\rho)^{\frac{1}{s}} \max_{1 \leq i \leq m} \{h_i(E(u_2))\} \\ & = \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)). \end{aligned}$$

Thus, $h = \max_{1 \leq i \leq m}\{h_i\}$ is a (s, E)- convex function in the fourth sense . □

Theorem 2.9. *Let $0 < s_1 \leq s_2 \leq 1$. If $h : M \longrightarrow \mathbb{R}_-$ is (s_2, E)-convex function in the fourth sense, then h is a (s_1, E)-convex function in the fourth sense.*

Proof. Since h is (s_2, E)-convex funcion in the fourth sense, for all $u_1, u_2 \in M$ and $\rho \in [0, 1]$,then

$$\begin{aligned} & h(\rho E(u_1) + (1-\rho)E(u_2)) \\ & \qquad \leq \rho^{\frac{1}{s_2}} h(E(u_1)) + (1-\rho)^{\frac{1}{s_2}} h(E(u_2)) \\ & \qquad \leq \rho^{\frac{1}{s_1}} h(E(u_1)) + (1-\rho)^{\frac{1}{s_1}} h(E(u_2)). \end{aligned}$$

□

Next, some properties of composition of (s, E)- convex function in the fourth sense are given.

Theorem 2.10. *Let $h_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an increasing function and $h_2 : M \longrightarrow \mathbb{R}_+$ be a function. If h_1 and h_2 are (s, E)-convex function in the fourth sense, then $h_1 \circ h_2$ is also (s, E)-convex function in the fourth sense.*

Proof. Let $u_1, u_2 \in M$ and $\rho \in [0, 1]$,then

$$\begin{aligned}
&(h_1 \circ h_2)(\rho E(u_1) + (1-\rho)E(u_2)) \\
&= h_1(h_2(\rho E(u_1) + (1-\rho)E(u_2)) \\
&\leq h_1(\rho^{\frac{1}{s}} h_2 E(u_1) + (1-\rho)^{\frac{1}{s}} h_2 E(u_2)) \\
&\leq h_1(\rho h_2 E(u_1) + (1-\rho) h_2 E(u_2)) \\
&\leq \rho^{\frac{1}{s}} h_1(h_2(E(u_1))) + (1-\rho)^{\frac{1}{s}} h_1(h_2(E(u_2))) \\
&= \rho^{\frac{1}{s}} (h_1 \circ h_2)(E(u_1)) + (1-\rho)^{\frac{1}{s}} (h_1 \circ h_2)(E(u_2)).
\end{aligned}$$

□

Theorem 2.11. *If $h_1 : M \longrightarrow \mathbb{R}$ is a (s, E)- convex function in the fourth sense and $h_2 : h_1(M) \longrightarrow \mathbb{R}$ is an increasing linear function, then $h_2 \circ h_1 : M \longrightarrow \mathbb{R}$ is a (s, E)-convex function in the fourth sense .*

Proof. Let $u_1, u_2 \in M$ and $\rho \in [0, 1]$.

$$\begin{aligned}
(h_2 \circ h_1)(\rho E(u_1) + (1-\rho)E(u_2)) &= h_2(h_1(\rho E(u_1) + (1-\rho)E(u_2))) \\
&\leq h_2(\rho^{\frac{1}{s}} (h_1(E(u_1))) + (1-\rho)^{\frac{1}{s}} (h_1(E(u_2))) \\
&= \rho^{\frac{1}{s}} h_2(h_1(E(u_1))) + (1-\rho)^{\frac{1}{s}} h_2(h_1(E(u_2))) \\
&= \rho^{\frac{1}{s}} (h_2 \circ h_1)(E(u_1)) + (1-\rho)^{\frac{1}{s}} (h_2 o h_1)(E(u_2)).
\end{aligned}$$

Hence, $h_2 \circ h_1$ is a (s, E)- convex function in the fourth sense. □

Theorem 2.12. *Assume that $h_2 : M_1 \longrightarrow \mathbb{M}_2$ is a linear transformalion and $h_1 : M_2 \longrightarrow \mathbb{R}$ is (s, E)-convex function in the fourth sense, then $h_1 \circ h_2$ is (s, E)-convex function in the fourth sense.*

Proof. Let $u_1, u_2 \in M$ and $\rho \in [0, 1]$, then

$$\begin{aligned}
&(h_1 \circ h_2)(\rho E(u_1) + (1-\rho)E(u_2)) \\
&= h_1(h_2(\rho E(u_1) + (1-\rho)E(u_2))) \\
&= h_1 \rho h_2(E(u_1)) + (1-\rho) h_2(E(u_2)) \\
&\leq \rho^{\frac{1}{s}} h_1(h_2 E(u_1)) + (1-\rho)^{\frac{1}{s}} h_1(h_2 E(u_2)) \\
&= \rho^{\frac{1}{s}} (h_1 \circ h_2)(E(u_1)) + (1-\rho)^{\frac{1}{s}} (h_1 \circ h_2)(E(u_2))
\end{aligned}$$

□

Theorem 2.13. *If $h : M \longrightarrow \mathbb{R}_-$ is a E- convex function , then h is a (s, E)- convex function in the fourth sense*

Proof. Assume that $u_1, u_2 \in M$ and $\rho \in [0, 1]$, we can write

$$\begin{aligned} & h(\rho E(u_1) + (1-\rho)E(u_2)) \\ & \leq \rho h(E(u_1)) + (1-\rho)h(E(u_2))) \leq \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)). \end{aligned}$$

□

Corollary 2.14. *If $h : M \longrightarrow \mathbb{R}$ is a E- concave function, then h is a (s, E)- concave function in the fourth sense*

Theorem 2.15. *If $h : M \longrightarrow \mathbb{R}_+$ is a (s, E)- concave function in the second sense, then h is a (s, E)- concave function in the fourth sense.*

Proof. assume that u_1, u_2 and $\rho \in [0, 1]$, then we have

$$\begin{aligned} h(\rho E(u_1) + (1-\rho)E(u_2)) & \geq \rho^s h(E(u_1)) + (1-\rho)^s h(E(u_2)) \\ & \geq \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)). \end{aligned}$$

□

Theorem 2.16. *If $h : M \longrightarrow \mathbb{R}_+$ is a (s, E)- convex function in the fourth sense, then h is a (s, E)- convex function in the second sense.*

Proof. assume that u_1, u_2 and $\rho \in [0, 1]$, then we have

$$\begin{aligned} h(\rho E(u_1) + (1-\rho)E(u_2)) & \leq \rho^{\frac{1}{s}} h(E(u_1)) + (1-\rho)^{\frac{1}{s}} h(E(u_2)) \\ & \leq \rho^s h(E(u_1)) + (1-\rho)^s h(E(u_2)). \end{aligned}$$

□

Corollary 2.17.

1. *If $h : M \longrightarrow \mathbb{R}_-$ is a (s, E)- convex function in the second sense, then h is a (s, E)- convex function in the fourth sense.*
2. *If $h : M \longrightarrow \mathbb{R}_-$ is a (s, E)- concave function in the fourth sense, then h is a (s, E)-concave function in the second sense.*

Now, let $E : [a, b] \longrightarrow [a, b]$ be a continuous function and $H_E : [0, 1] \longrightarrow \mathbb{R}$ be defined by

$$H_E(t) = \frac{1}{E(b) - E(a)} \int_{E(a)}^{E(b)} h(tx + (1-t)\frac{E(a) + E(b)}{2})dx$$

where $h : [a, b] \longrightarrow \mathbb{R}$ is a (s, E)-convex function in the fourth sense, $[a, b] \subset \mathbb{R}$, $t \in [0, 1]$.

Proposition 2.18. *Let $h : [a, b] \longrightarrow \mathbb{R}$ be as above. Then*

1. *$H_E(t)$ is (s, E)-convex function in the fourth sense on $[0, 1]$,*
2. *We have*

$$\inf_{t\in[0,1]} H_E(t) = H_E(0) = h\left(\frac{E(a) + E(b)}{2}\right).$$

$$\sup_{t\in[0,1]} H_E(t) = H_E(1) = \frac{1}{E(a) + E(b)} \int_{E(a)}^{E(b)} h(x)dx.$$

3. *$H_E(t)$ is monotonic nondecreasing on $[0, 1]$.*

Proof. 1. Let $\rho_1, \rho_2 \geq 0$ with $\rho_1 + \rho_2 = 1$ and $t_1, t_2 \in [0, 1]$.Then

$$H_E(\rho_1 t_1 + \rho_2 t_2) =$$

$$\frac{1}{E(a) + E(b)} \int_{E(a)}^{E(b)} h\left[\rho_1\left(t_1 x + (1 - t_1)\frac{E(a) + E(b)}{2}\right) + \rho_2\left(t_1 x + (1 - t_1)\frac{E(a) + E(b)}{2}\right)\right] dx$$

$$\leq \rho_1^{\frac{1}{s}} \frac{1}{E(a) + E(b)} \int_{E(a)}^{E(b)} h\left[\left(t_1 x + (1 - t_1)\frac{E(a) + E(b)}{2}\right)\right] dx$$

$$+\rho_2^{\frac{1}{s}} \frac{1}{E(a) + E(b)} \int_{E(a)}^{E(b)} h\left[\left(t_2 x + (1 - t_2)\frac{E(a) + E(b)}{2}\right)\right] dx$$

$$= \rho_1^{\frac{1}{s}} H_E(t_1) + \rho_2^{\frac{1}{s}} H_E(t_2),$$

then $H_E(t)$ is (s, E)-convex function in the fourth sense.

2. By Jensn's inequality

$$H_E(t) \geq h\left[\frac{1}{E(a) + E(b)} \int_{E(a)}^{E(b)} \left(tx + (1 - t)\frac{E(a) + E(b)}{2}\right) dx\right]$$

$$= h\left(\frac{E(a) + E(b)}{2}\right). \qquad (3)$$

Since h is (s, E)-convex function in the fourth sense, then

$$H_E(t) \geq \frac{1}{E(a) + E(b)} \int_{E(a)}^{E(b)} \left[t^{\frac{1}{s}} h(x) + (1 - t)^{\frac{1}{s}} h\left(\frac{E(a) + E(b)}{2}\right)\right] dx$$

$$= \frac{t^{\frac{1}{s}}}{E(a) + E(b)} \int_{E(a)}^{E(b)} h(x)dx + (1 - t)^{\frac{1}{s}} h\left(\frac{E(a) + E(b)}{2}\right), \qquad (4)$$

then by combining (3) and(4), we have

$$h\left(\frac{E(a) + E(b)}{2}\right) \leq H_E(t)$$

$$\leq \frac{t^{\frac{1}{s}}}{E(a)+E(b)} \int_{E(a)}^{E(b)} h(x)dx + (1-t)^{\frac{1}{s}} h\left(\frac{E(a)+E(b)}{2}\right)$$
$$\leq \int_{E(a)}^{E(b)} h(x)dx, \forall t \in [0,1].$$

3. Let $t_1, t_2 \in (0,1)$ with $t_1 < t_2$.Then

$$\frac{H_E(t_2) - H_E(t_1)}{t_2 - t_1} \geq H'_{E_+}(t_1)$$
$$= \frac{1}{E(a)+E(b)} \int_{E(a)}^{E(b)} h'_+ \left(t_1 x + (1-t_1)\frac{E(a)+E(b)}{2}\right)\left(x - \frac{E(a)+E(b)}{2}\right) dx.$$

Since h is (s,E)-convex function in the fourth sense, then

$$h\left(\frac{E(a)+E(b)}{2}\right) - h\left(t_1 x + (1-t_1)\frac{E(a)+E(b)}{2}\right)$$
$$\geq t_1 \int_{E(a)}^{E(b)} h'_+ \left(t_1 x + (1-t_1)\frac{E(a)+E(b)}{2}\right)\left(\frac{E(a)+E(b)}{2} - x\right) dx$$

For every $x \in [a,b]$. So

$$\frac{1}{E(a)+E(b)} \int_{E(a)}^{E(b)} h'_+ \left(t_1 x + (1-t_1)\frac{E(a)+E(b)}{2}\right)\left(x - \frac{E(a)+E(b)}{2}\right)$$
$$\geq \frac{1}{t_1}\left[\frac{1}{E(a)+E(b)} \int_{E(a)}^{E(b)} h\left(t_1 x + (1-t_1)\frac{E(a)+E(b)}{2}\right) dx - h\left(\frac{E(a)+E(b)}{2}\right)\right]$$
$$\geq \frac{1}{t_1}\left[H_E(t_1) - h\left(\frac{E(a)+E(b)}{2}\right)\right] \geq 0.$$

Consequently, $H_E(t_2) - H_E(t_1) \geq 0 for 0 \leq t_1 < t_2 \leq 1$.

□

Proposition 2.19. *Suppose that $h : [a,b] \longrightarrow \mathbb{R}$ is (s,E)-convex function in the fourth sense and $J_E : [a,b] \longrightarrow \mathbb{R}$ is defined by*

$$J_E(t) = \frac{1}{E(a)+E(b)} \int_{E(a)}^{E(b)} \left[h\left(\frac{1+t}{2}E(a) + \frac{1-t}{2}x\right) + h\left(\frac{1+t}{2}E(b) + \frac{1-t}{2}x\right)\right] dx.$$

Then

1. *$J_E(t)$is (s,E)-convex function in the fourth sense and monotnic nondecreasing on $[0,1]$,*
2. $$\inf_{t\in[0,1]} J_E(t) = J_E(0) = \frac{1}{E(a)+E(b)} \int_{E(a)}^{E(b)} h(x)dx.$$
$$\sup_{t\in[0,1]} J_E(t) = J_E(1) = \frac{h(E(a)) + h(E(b))}{2}.$$

References

[1] N. Abdulaleem, *V-E -invexity in E-differentiable multiobjective programming.* Numerical Algebra, Control and Optimization, 2021 .

[2] M. Grinalatt and J.T. Linnainmaa, *Jensen's inquality, parameter uncertainty, and multiperiod investment.* Rev. Asset Pricing Stud., 2011 ,1,1, 1–34.

[3] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions.* Aequationes Mathematicae,1994, 48,1, 100–111.

[4] A. Kılıçman and W. Saleh, *Some inequalities for generalized s-convex functions.* JP J. Geom. Topol.,2015,17, 63–82.

[5] A. Kılıçman and W. Saleh, *Generalized convex functions and their applications.* In: Mathematical Analysis and Applications, Selected Topics,2018, 77–99 .

[6] A. Kılıçman and W. Saleh, *Preinvex functions and their applications.* Symmetry, 2018,10,10, 493.

[7] S. Nanda, *On generalization of E-convex and composite functions.* Journal of Information and Optimization Sciences, 2021, 42, 6, 1271-1282.

[8] J.J. Ruel and M.P. Ayres, *Jensen's inequality predicts effects of environmental variation.* Trends Ecol. Evol.,1999, 14,9,361–366.

[9] T. Yan et al., *On Hadamard Type Fractional Inequalities for Riemann-Liouville Integrals via a Generalized Convexity.* Fractal and Fractional, 2022, 6, 1, 28.

[10] E.A. Youness, *E-convex sets, E-convex functions, and E-convex programming.* Journal of Optimization Theory and Applications,1999, 2, 102, 439-450.

[11] E.A. Youness and T. Emam, *Strongly E-convex sets and strongly E-convex functions.* Journal of Interdisciplinary Mathematics 1, 2005, 8,107-117 .

[12] E.K.E.N. Zeynep et al., *s-Convex functions in the fourth sense and some of their properties.* Konuralp Journal of Mathematics (KJM) 2, 2021, 9, 260-267.

W. Saleh
Department of Mathematics, Taibah University, Al-Medina 20012, Saudi Arabia
E-mail: wlehabi@taibahu.edu.sa

Received 04 April 2022
Accepted 22 April 2022