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Finite-Difference Method for Solving the Optimal Control Problem for the Linear Non-Stationary Equation of Quasi-Optics with a Special Gradient Term

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Abstract. In this paper the problem of the convergence of the finite difference method is studied to solving the optimal control problem for the linear nonstationary equation of quasi-optics with a special gradient term, when the controls are measurable bounded coefficients that depend on the spatial variable. In this case, the estimate is established for the stability of the solution of the difference scheme under consideration. Next, the error of approximation of the difference scheme is studies and the estimate for the error of the considered difference scheme is defined. On the basis of the estimates obtained, the convergence of difference approximations with respect to the functional is proved.

Key Words and Phrases: Non-stationary quasi-optics equation, finite difference method, optimal control, functional convergence.

2010 Mathematics Subject Classifications: 26A51, 26B25, 52A20, 52A41

1. Introduction

In this paper, we study the problem of applying the finite difference method to solving the optimal control problem for the linear non-stationary equation of quasi-optics with a special gradient term, when the controls are bounded measurable coefficients that depend on the spatial variable. The estimates for the convergence of difference approximations with respect to the functional are derived. It should be noted that similar results were previously obtained for the optimal control problems and for system identification problems described by the non-stationary quasi-optics equation or by the non-stationary Schrödinger without a special gradient term, for example, in [1-16] and others.

2. Statement of the problem and its difference approximation

Let $l > 0, L > 0, T > 0$ be given numbers $0 \le x \le l$,$0 \le t \le T$, $0 \le z \le L$, $\Omega_{tz} = (0,l) \times (0,t) \times (0,z)$, $\Omega = \Omega_{TL}$, $\Omega_t = (0,l) \times (0,t)$, $\Omega_z = (0,l) \times (0,z)$, $Q = (0,T) \times (0,L)$.

Consider the problem of minimizing the functional

$$J(v) = \beta_0 \|\psi(.,T,.) - y_0\|^2_{L_2(\Omega_L)} + \beta_1 \|\psi(.,.,L) - y_1\|^2_{L_2(\Omega_T)} \tag{1}$$

on the set

$$V \equiv \{v = v(x) = (v_0(x), v_1(x)) : v_s \in L_2(0,l),$$
$$s = 0,1, |v_0(x)| \le b_0, 0 \le v_1(x) \le b_1, \overset{0}{\forall} x \in (0,l)\Big\}$$

subject to

$$i\frac{\partial\psi}{\partial t} + ia_0\frac{\partial\psi}{\partial z} - a_1\frac{\partial^2\psi}{\partial x^2} + ia_2(x)\frac{\partial\psi}{\partial x}a(x)\psi + v_0(x)\psi + iv_1(x)\psi = f(x,t,z), \ (x,t,z) \in \Omega, \tag{2}$$

$$\psi(x,0,z) = \varphi_0(x,z), \ (x,z) \in \Omega_L, \tag{3}$$

$$\psi(x,t,0) = \varphi_1(x,t), \ (x,t) \in \Omega_T, \tag{4}$$

$$\psi(0,t,z) = \psi(l,t,z) = 0, \ (t,z) \in Q, \tag{5}$$

Where $i = \sqrt{-1}$; $a_0 > 0$, $a_1 > 0, b_0 > 0, b_1 > 0, \beta_0 \ge 0, \beta_1 \ge 0$ are given numbers, such that $\beta_0 + \beta_1 \ne 0$; $a(x), a_2(x)$ are given real valued measurable bounded functions satisfying the following conditions

$$0 \le a(x) \le \mu_0, \ \overset{0}{\forall} x \in (0,l), \ \mu_0 = const > 0; \tag{6}$$

$$0 \le a_2(x) \le \mu_1, \left|\frac{da_2(x)}{dx}\right| \le \mu_2, \overset{0}{\forall} x \in (0,l), \ \mu_1, \mu_2 = const > 0; \tag{7}$$

$\varphi_0(x,z), \varphi_1(x,t), f(x,t,z), y_0(x,z), y_1(x,t)$ are complex valued functions satisfying the conditins

$$\varphi_0 \in \overset{0}{W}{}_2^{2,1}(\Omega_L), \ \varphi_1 \in \overset{0}{W}{}_2^{2,1}(\Omega_T) \ , f \in W_2^{0,1,1}(\Omega); \tag{8}$$

$$y_0 \in W_2^{1,1}(\Omega_L), \ y_1 \in W_2^{1,1}(\Omega_T). \tag{9}$$

For each $v \in V$ the problem of defining the function $\psi = \psi(x,t,z) = \psi(x,t,z;v)$ from conditions (2)-(5) we call a reduced problem for the equation of the form (2). As a solution of this problem we consider almost everywhere solution from the space $\overset{0}{W}{}_2^{2,1,1}(\Omega)$. As follows from [17] for each $v \in V$ direct problem (2)-(5) has unique solution from the space $\overset{0}{W}{}_2^{2,1,1}(\Omega)$ for which is valid the estimation

$$\|\psi\|^2_{\overset{0}{W}{}_2^{2,1,1}(\Omega)} \le c_0 \left(\|\varphi_0\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_L)} + (\|\varphi_1\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_T)} + \|f\|^2_{W_2^{0,1,1}(\Omega)}\right). \tag{10}$$

It follows from [18] optimal control problem (1)-(5) has at least one solution i.e. the set $V_* = \left\{v^* \in V : \ J(v^*) = J_* = \inf_{v \in V} J(v)\right\}$ is not empty.

Consider the difference approximation of problem (1)-(5). To do this first we replace the domain $\overline{\Omega} \equiv [0,l] \times [0,T] \times [0,L]$ by the set of grides

$$\left\{(x_j,\, t_k,\, z_p)_n\right\},\, x_j = jh_n - \tfrac{h_n}{2},\, j = \overline{1, M_n - 1},\, x_0 = x_1 - h_n/2,\, x_{M_n} = x_{M_n - 1} + h_n/2,$$

$$t_k = k\tau_n,\, k = \overline{0, N_n},\, z_p = p\theta_n,\, p = \overline{0, Z_n},\, h_n = \tfrac{l}{M_n - 1},\, \tau_n = \tfrac{T}{N_n},\, \theta_n = \tfrac{L}{Z_n},\, n = 1, 2, ...$$

Let $M = M_n$, $N = N_n$, $Z = Z_n$, $h = h_n$,$\tau = \tau_n$, $\theta = \theta_n$ and

$$\delta_{\bar{t}} \Phi^p_{jk} = \frac{\Phi^p_{jk} - \Phi^p_{jk-1}}{\tau}, \delta_{\bar{z}} \Phi^p_{jk} = \frac{\Phi^p_{jk} - \Phi^{p-1}_{jk}}{\theta},$$

$$\delta_{\bar{x}} \Phi^p_{jk} = \frac{\Phi^p_{jk} - \Phi^p_{j-1k}}{h}, \delta_{\bar{x}} \Phi^p_{1k} = \frac{2\left(\Phi^p_{1k} - \Phi^p_{0k}\right)}{h},$$

$$\delta_x \Phi^p_{jk} = \frac{\Phi^p_{j+1k} - \Phi^p_{jk}}{h}, \delta_x \Phi^p_{M-1k} = \frac{2\left(\Phi^p_{Mk} - \Phi^p_{M-1k}\right)}{h},$$

$$\delta_{x\bar{x}} \Phi^p_{jk} = \frac{\Phi^p_{j+1k} - 2\Phi^p_{jk} + \Phi^p_{j-1k}}{h^2} = \frac{\delta_x \Phi^p_{jk} - \delta_{\bar{x}} \Phi^p_{jk}}{h}.$$

At each $n \geq 1$ consider the problem of minimization of the function

$$I_n\left([v]_n\right) = \beta_0 \theta h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left|\Phi^p_{jN} - y_{0}{}^p_j\right|^2 + \beta_1 \tau h \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left|\Phi^Z_{jk} - y_{1}{}^k_j\right|^2 \quad (11)$$

on the set

$$V_n = \{[v]_n = \left([v_0]_n, [v_1]_n\right) : [v_s]_n = \left(v_{s1},\, v_{s2},\, ...,\, v_{sM-1}\right),$$

$$s = 0, 1, |v_{0j}| \leq b_0, 0 \leq v_{1j} \leq b_1, j = \overline{1, M - 1}\Big\}$$

subject to

$$i\delta_{\bar{t}} \Phi^p_{jk} + ia_0 \delta_{\bar{z}} \Phi^p_{jk} - a_1 \delta_{x\bar{x}} \Phi^p_{jk} + a^j \Phi^p_{jk} +$$

$$+ia_{2j} \delta_{\bar{x}} \Phi^p_{jk} + v_{0j} \Phi^p_{jk} + iv_{1j} \Phi^p_{jk} = f^p_{jk},\; j = \overline{1, M - 1},\; k = \overline{1, N},\; p = \overline{1, Z}, \quad (12)$$

$$\Phi^p_{j0} = \varphi^p_{0j},\; j = \overline{0, M},\; p = \overline{1, Z}, \quad (13)$$

$$\Phi^0_{jk} = \varphi^k_{1j},\; j = \overline{0, M}, k = \overline{1, N}, \quad (14)$$

$$\Phi^p_{0k} = \Phi^p_{Mk} = 0,\; k = \overline{1, N},\; p = \overline{1, Z}, \quad (15)$$

where a^j, a_{2j}, φ^p_{0j}, φ^k_{1j}, f^p_{jk}, y^p_{0j}, y^k_{1j} are grid functions defined by the formulas

$$a^j = \frac{1}{h} \int_{x_j - h/2}^{x_j + h/2} a\left(x\right) dx,\; j = \overline{1, M - 1}\,; a_{2j} = \frac{1}{h} \int_{x_j - h/2}^{x_j + h/2} a_2\left(x\right) dx,\; j = \overline{1, M - 1}; \quad (16)$$

$$\varphi_{0j}^{p}=\frac{1}{h\theta}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\varphi_0\left(x,z\right)dxdz,\ j=\overline{1,M-1}\,,\ p=\overline{1,z},\ \ \varphi_{00}^{p}=\ \varphi_{0M}^{p}=0,\ p=\overline{1,Z}\,; \tag{17}$$

$$\varphi_{1j}^{k}=\frac{1}{\tau\,h}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\varphi_1\left(x,t\right)dxdt,\ j=\overline{1,M-1}\,,\ k=\overline{1,N},\ \ \varphi_{10}^{k}=\ \varphi_{1M}^{k}=0,\ k=\overline{1,N}\,; \tag{18}$$

$$f_{jk}^{p}=\frac{1}{\theta\tau\,h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}f\left(x,t,z\right)dxdtdz,\ \ j=\overline{1,M-1}\,,\ k=\overline{1,N},\ p=\overline{1,Z}; \tag{19}$$

$$y_{0j}^{p}=\frac{1}{\theta\,h}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}y_0\left(x,z\right)dxdz,\ \ j=\overline{1,M-1}\,,\ \ p=\overline{1,Z},$$

$$y_{1j}^{k}=\frac{1}{\theta\tau\,h}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}y_1\left(x,t\right)dxdt,\ \ j=\overline{1,M-1}\,,\ \ k=\overline{1,N}. \tag{20}$$

3. Estimation of stability and error for the solution difference scheme

In this section, we establish stability and error estimates for solving the difference scheme (12)-(15) for each $[v_n]\in V_n$. First, we establish a stability estimate for the solution of the difference scheme.

Theorem 2.1. Let the functions $a\left(x\right),a_2\left(x\right),\varphi_0\left(x,z\right),\ \varphi_1\left(x,t\right),\ f\left(x,t,z\right)$ satisfy conditions (6)-(8). Let moreover for the steps τ and θ are valid: $\tau\leq\frac{1}{2\mu_2},\theta\leq\frac{a_0}{2\mu_2}$. Then for any $[v_n]\in V_n$ for the solution of difference scheme (12)-(15) are valid the estimations

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{q}\right|^2\leq$$

$$\leq c_1\left(\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^2+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^2+\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^2\right) \tag{21}$$

for$\forall m\in\{1,2,...,N\}\,,\ \forall q\in\{1,2,...,Z\}$.

Proof. It is clear that difference scheme (12)-(15) at $z=z_p,\ t=t_k$ is equivalent to the following summator identity

$$h\sum_{j=1}^{M-1}\left(i\delta_{\bar{t}}\Phi_{jk}^{p}\bar{\eta}_{jk}^{p}+ia_0\delta_{\bar{z}}\Phi_{jk}^{p}\bar{\eta}_{jk}^{p}\right)+h\sum_{j=1}^{M}a_1\delta_{\overline{x}}\Phi_{jk}^{p}\delta_{\overline{x}}\bar{\eta}_{jk}^{p}+$$

$$+h\sum_{j=1}^{M-1}ia_{2j}\delta_{\overline{x}}\Phi_{jk}^{p}\delta\bar{\eta}_{jk}^{p}+h\sum_{j=1}^{M-1}iv_{1j}\Phi_{jk}^{p}\delta\bar{\eta}_{jk}^{p}+$$

$$+h\sum_{j=1}^{M-1}\left(a^j\ \Phi_{jk}^{p}+v_{0j}\Phi_{jk}^{p}\right)\bar{\eta}_{jk}^{p}=h\sum_{j=1}^{M-1}f_{jk}^{p}\bar{\eta}_{jk}^{p},\ \ k=\overline{1,N},\ \ p=\overline{1,Z} \tag{22}$$

for any grid function η_{jk}^p defined on the grid $\left\{(x_j,\, t_k,\, z_p)_n\right\}$ and satisfying conditions $\eta_{0k}^p = \eta_{Mk}^p = 0,\, k = \overline{1,N},\; p = \overline{1,Z}$. In this identity choosing $\theta\tau\overline{\Phi}_{jk}^p$ instead of the test function $\bar{\eta}_{jk}^p$we get

$$i\theta h\sum_{j=1}^{M-1}\tau\delta_{\bar{t}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + ia_0\tau h\sum_{j=1}^{M-1}\theta\delta_{\bar{z}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + a_1\theta\tau h\sum_{j=1}^{M}\left|\delta_{\overline{x}}\Phi_{jk}^p\right|^2 + i\theta\tau h\sum_{j=1}^{M-1}a_{2j}\delta_{\overline{x}}\Phi_{jk}^p\bar{\Phi}_{jk}^p +$$

$$+\theta\tau h\sum_{j=1}^{M-1}a^j\left|\Phi_{jk}^p\right|^2 + \theta\tau h\sum_{j=1}^{M-1}v_{0j}\left|\Phi_{jk}^p\right|^2 + i\theta\tau h\sum_{j=1}^{M-1}v_{1j}\left|\Phi_{jk}^p\right|^2 =$$

$$= \theta\tau h\sum_{j=1}^{M-1}f_{jk}^p\bar{\Phi}_{jk}^p,\;\; k = \overline{1,N},\;\; p = \overline{1,Z}. \tag{23}$$

The complex conjugation of this equality takes the form $-i\theta h\sum_{j=1}^{M-1}\tau\delta_{\bar{t}}\bar{\Phi}_{jk}^p\Phi_{jk}^p - ia_0\tau h\sum_{j=1}^{M-1}\theta\delta_{\bar{z}}\bar{\Phi}_{jk}^p\Phi_{jk}^p + a_1\theta\tau h\sum_{j=1}^{M}\left|\delta_{\overline{x}}\Phi_{jk}^p\right|^2 - i\theta\tau h\sum_{j=1}^{M-1}a_{2j}\delta_{\overline{x}}\bar{\Phi}_{jk}^p\Phi_{jk}^p +$

$$+\theta\tau h\sum_{j=1}^{M-1}a^j\left|\Phi_{jk}^p\right|^2 + \theta\tau h\sum_{j=1}^{M-1}v_{0j}\left|\Phi_{jk}^p\right|^2 - i\theta\tau h\sum_{j=1}^{M-1}v_{1j}\left|\Phi_{jk}^p\right|^2 =$$

$$= \theta\tau h\sum_{j=1}^{M-1}\bar{f}_{jk}^p\Phi_{jk}^p,\;\; k = \overline{1,N},\;\; p = \overline{1,Z}.$$

Subtracting this equality from (23) we obtain the validity of the equality

$$i\theta h\sum_{j=1}^{M-1}\tau\left(\delta_{\bar{t}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + \delta_{\bar{t}}\bar{\Phi}_{jk}^p\Phi_{jk}^p\right) + ia_0\tau h\sum_{j=1}^{M-1}\theta\left(\delta_{\bar{z}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + \delta_{\bar{z}}\bar{\Phi}_{jk}^p\Phi_{jk}^p\right) +$$

$$+i\theta\tau h\sum_{j=1}^{M-1}a_{2j}\left(\delta_{\overline{x}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + \delta_{\overline{x}}\bar{\Phi}_{jk}^p\Phi_{jk}^p\right) + i2\theta\tau h\sum_{j=1}^{M-1}v_{1j}\left|\Phi_{jk}^p\right|^2 =$$

$$= \theta\tau h\sum_{j=1}^{M-1}\left(f_{jk}^p\bar{\Phi}_{jk}^p - \bar{f}_{jk}^p\Phi_{jk}^p\right),\;\; k = \overline{1,N},\;\; p = \overline{1,Z}. \tag{24}$$

It is easy to establish the validity of the equalities

$$\tau\left(\delta_{\bar{t}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + \delta_{\bar{t}}\bar{\Phi}_{jk}^p\Phi_{jk}^p\right) = \left|\Phi_{jk}^p\right|^2 - \left|\Phi_{jk-1}^p\right|^2 + \left|\Phi_{jk}^p - \Phi_{jk-1}^p\right|^2, \tag{25}$$

$$\theta\left(\delta_{\bar{z}}\Phi_{jk}^p\bar{\Phi}_{jk}^p + \delta_{\bar{z}}\bar{\Phi}_{jk}^p\Phi_{jk}^p\right) = \left|\Phi_{jk}^p\right|^2 - \left|\Phi_{jk}^{p-1}\right|^2 + \left|\Phi_{jk}^p - \Phi_{jk}^{p-1}\right|^2. \tag{26}$$

Moreover, the equality

$$f_{jk}^p\bar{\Phi}_{jk}^p - \bar{f}_{jk}^p\Phi_{jk}^p = 2iIm\left(f_{jk}^p\bar{\Phi}_{jk}^p\right). \tag{27}$$

is true. With the help of equalities (25)-(27), from equality (24) we obtain an equality, which after multiplication by $(-i)$ will take the form

$$\theta h \sum_{j=1}^{M-1} \left(\left| \Phi_{jk}^{p} \right|^2 - \left| \Phi_{jk-1}^{p} \right|^2 + \left| \Phi_{jk}^{p} - \Phi_{jk-1}^{p} \right|^2 \right) +$$

$$+a_0 \tau h \sum_{j=1}^{M-1} \left(\left| \Phi_{jk}^{p} \right|^2 - \left| \Phi_{jk}^{p-1} \right|^2 + \left| \Phi_{jk}^{p} - \Phi_{jk}^{p-1} \right|^2 \right) +$$

$$+\theta \tau h \sum_{j=1}^{M-1} a_{2j} \left(\delta_{\overline{x}} \Phi_{jk}^{p} \bar{\Phi}_{jk}^{p} + \delta_{\overline{x}} \bar{\Phi}_{jk}^{p} \Phi_{jk}^{p} \right) + 2\theta \tau h \sum_{j=1}^{M-1} v_{1j} \left| \Phi_{jk}^{p} \right|^2 =$$

$$= 2\theta \tau h \sum_{j=1}^{M-1} Im \left(f_{jk}^{p} \overline{\Phi}_{jk}^{p} \right), \ k = \overline{1, N}, \ p = \overline{1, Z}. \tag{28}$$

Now we transform the third term on the left side of this equality. Using equality

$$h \left(\delta_{\overline{x}} \Phi_{jk}^{p} \bar{\Phi}_{jk}^{p} + \delta_{\overline{x}} \bar{\Phi}_{jk}^{p} \Phi_{jk}^{p} \right) = \left| \Phi_{jk}^{p} \right|^2 - \left| \Phi_{j-1k}^{p} \right|^2 + \left| \Phi_{jk}^{p} - \Phi_{j-1k}^{p} \right|^2$$

it is easy to establish the validity of the following equality

$$+\theta \tau h \sum_{j=1}^{M-1} a_{2j} \left(\delta_{\overline{x}} \Phi_{jk}^{p} \bar{\Phi}_{jk}^{p} + \delta_{\overline{x}} \bar{\Phi}_{jk}^{p} \Phi_{jk}^{p} \right) = -\theta \tau h \sum_{j=2}^{M-1} \delta_{\bar{x}} a_{2j} \left| \Phi_{j-1k}^{p} \right|^2 +$$

$$+\theta \tau \sum_{j=1}^{M-1} a_{2j} \left| \Phi_{jk}^{p} - \Phi_{j-1k}^{p} \right| + \theta \tau a_{2M-1} \left| \Phi_{M-1k}^{p} \right|^2. \tag{29}$$

If we take into account this equality on the left side of equality (28), then from there we get

$$\theta h \sum_{j=1}^{M-1} \left(\left| \Phi_{jk}^{p} \right|^2 - \left| \Phi_{jk-1}^{p} \right|^2 + \left| \Phi_{jk}^{p} - \Phi_{jk-1}^{p} \right|^2 \right) +$$

$$+a_0 \tau h \sum_{j=1}^{M-1} \left(\left| \Phi_{jk}^{p} \right|^2 - \left| \Phi_{jk}^{p-1} \right|^2 + \left| \Phi_{jk}^{p} - \Phi_{jk}^{p-1} \right|^2 \right) -$$

$$-\theta \tau h \sum_{j=2}^{M-1} \delta_{\bar{x}} a_{2j} \left| \Phi_{j-1k}^{p} \right|^2 + \theta \tau \sum_{j=1}^{M-1} a_{2j} \left| \Phi_{jk}^{p} - \Phi_{j-1k}^{p} \right|^2 + \theta \tau a_{2M-1} \left| \Phi_{M-1k}^{p} \right|^2 +$$

$$+2\theta \tau h \sum_{j=1}^{M-1} v_{1j} \left| \Phi_{jk}^{p} \right|^2 = 2\theta \tau h \sum_{j=1}^{M-1} Im \left(f_{jk}^{p} \overline{\Phi}_{jk}^{p} \right), \ k = \overline{1, N}, \ p = \overline{1, Z}.$$

Summing these equalities over k from $k=1$ to $k=m\le N$ and over p from $p=1$ to$p=q\le Z$ we obtain

$$\theta h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left(\left|\Phi_{jk}^{p}\right|^{2}-\left|\Phi_{jk-1}^{p}\right|^{2}+\left|\Phi_{jk}^{p}-\Phi_{jk-1}^{p}\right|^{2}\right)+$$

$$+a_0\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left(\left|\Phi_{jk}^{p}\right|^{2}-\left|\Phi_{jk}^{p-1}\right|^{2}+\left|\Phi_{jk}^{p}-\Phi_{jk}^{p-1}\right|^{2}\right)+$$

$$+\theta\tau\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}a_{2j}\left|\Phi_{jk}^{p}-\Phi_{j-1k}^{p}\right|^{2}+\theta\tau\sum_{p=1}^{q}\sum_{k=1}^{m}a_{2M-1}\left|\Phi_{M-1k}^{p}\right|^{2}+2\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}v_{1j}\left|\Phi_{jk}^{p}\right|^{2}=$$

$$=\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=2}^{M-1}\delta_{\bar{x}}a_{2j}\left|\Phi_{j-1k}^{p}\right|^{2}+2\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}Im\left(f_{jk}^{p}\overline{\Phi}_{jk}^{p}\right),\ m=\overline{1,N},\ q=\overline{1,Z}.$$

After summation operations in the first and second terms on the left side of this equality, we obtain the validity of the following equality

$$\theta h\sum_{p=1}^{q}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^{2}-\theta h\sum_{p=1}^{q}\sum_{j=1}^{M-1}\left|\Phi_{j0}^{p}\right|^{2}+\theta h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}-\Phi_{jk-1}^{p}\right|^{2}+$$

$$+a_0\tau h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{q}\right|^{2}-a_0\tau h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{0}\right|^{2}+a_0\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}-\Phi_{jk}^{p-1}\right|^{2}+$$

$$+\theta\tau\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}a_{2j}\left|\Phi_{jk}^{p}-\Phi_{j-1k}^{p}\right|^{2}+\theta\tau\sum_{p=1}^{q}\sum_{k=1}^{m}a_{2M-1}\left|\Phi_{M-1k}^{p}\right|^{2}+2\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}v_{1j}\left|\Phi_{jk}^{p}\right|^{2}=$$

$$=\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=2}^{M-1}\delta_{\bar{x}}a_{2j}\left|\Phi_{j-1k}^{p}\right|^{2}+2\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}Im\left(f_{jk}^{p}\overline{\Phi}_{jk}^{p}\right),\ m=\overline{1,N},\ q=\overline{1,Z}.$$

From this equality, using initial conditions (13), (14), we have

$$\theta h\sum_{p=1}^{q}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^{2}+\theta h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}-\Phi_{jk-1}^{p}\right|^{2}+$$

$$+a_0\tau h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{q}\right|^{2}+a_0\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}-\Phi_{jk}^{p-1}\right|^{2}+$$

$$+\theta\tau\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}a_{2j}\left|\Phi_{jk}^{p}-\Phi_{j-1k}^{p}\right|^{2}+\theta\tau\sum_{p=1}^{q}\sum_{k=1}^{m}a_{2M-1}\left|\Phi_{M-1k}^{p}\right|^{2}+2\theta\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}v_{1j}\left|\Phi_{jk}^{p}\right|^{2}=$$

$$= \theta\tau h \sum_{p=1}^{q} \sum_{k=1}^{m} \sum_{j=2}^{M-1} \delta_{\bar{x}} a_{2j} \left|\Phi_{j-1k}^{p}\right|^2 + \theta h \sum_{p=1}^{q} \sum_{j=1}^{M-1} \left|\varphi_{0j}^{p}\right|^2 + a_0 \tau h \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\varphi_{1j}^{k}\right|^2 +$$

$$+2\theta\tau h \sum_{p=1}^{q} \sum_{k=1}^{m} \sum_{j=1}^{M-1} Im\left(f_{jk}^{p} \overline{\Phi}_{jk}^{p}\right),\ m = \overline{1, N},\ \ q = \overline{1, Z}.$$

From this equality, by virtue of the conditions $v_{1j} \geq 0, a_{2j} \geq 0, j = \overline{1, M-1}$ and due to the non-negativity of the second and fourth terms on the left-hand side, we can write the following inequality

$$\theta h \sum_{p=1}^{q} \sum_{j=1}^{M-1} \left|\Phi_{jm}^{p}\right|^2 + a_0 \tau h \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\Phi_{jk}^{q}\right|^2 \leq$$

$$\leq \theta h \sum_{p=1}^{q} \sum_{j=1}^{M-1} \left|\varphi_{0j}^{p}\right|^2 + a_0 \tau h \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\varphi_{1j}^{k}\right|^2 +$$

$$+\theta\tau h \sum_{p=1}^{q} \sum_{k=1}^{m} \sum_{j=2}^{M-1} \left|\delta_{\bar{x}} a_{2j}\right| \left|\Phi_{j-1k}^{p}\right|^2 + 2\theta\tau h \sum_{p=1}^{q} \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|f_{jk}^{p}\right| \left|\Phi_{jk}^{p}\right|,\ m = \overline{1, N},\ \ q = \overline{1, Z}. \tag{30}$$

By virtue of condition (7), it is easy to establish the following inequalities

$$\left|\delta_{\bar{x}} a_{2j}\right| \leq \mu_2, j = 2, 3, ..., M-1. \tag{31}$$

Then, taking into account these inequalities, from (30) it is easy to obtain the validity of the inequality

$$\theta h \sum_{p=1}^{q} \sum_{j=1}^{M-1} \left|\Phi_{jm}^{p}\right|^2 + a_0 \tau h \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\Phi_{jk}^{q}\right|^2 \leq$$

$$\leq \theta h \sum_{p=1}^{q} \sum_{j=1}^{M-1} \left|\varphi_{0j}^{p}\right|^2 + a_0 \tau h \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\varphi_{1j}^{k}\right|^2 +$$

$$+\mu_2 \theta\tau h \sum_{p=1}^{q} \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\Phi_{jk}^{p}\right|^2 + 2\theta\tau h \sum_{p=1}^{q} \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|f_{jk}^{p}\right| \left|\Phi_{jk}^{p}\right|,\ m = \overline{1, N},\ \ q = \overline{1, Z}. \tag{32}$$

From this for $q = Z$ we obtain

$$\theta h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left|\Phi_{jm}^{p}\right|^2 \leq \theta h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left|\varphi_{0j}^{p}\right|^2 + a_0 \tau h \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left|\varphi_{1j}^{k}\right|^2 +$$

$$+\mu_2 \theta\tau h \sum_{p=1}^{Z} \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|\Phi_{jk}^{p}\right|^2 + 2\theta\tau h \sum_{p=1}^{Z} \sum_{k=1}^{m} \sum_{j=1}^{M-1} \left|f_{jk}^{p}\right| \left|\Phi_{jk}^{p}\right|,\ m = \overline{1, N}.$$

The use of $\tau \le \frac{1}{2\mu_2}$ in the last gives

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2 \le 2\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^2 + 2a_0\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^2 +$$

$$+4\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|\left|\Phi_{jk}^{p}\right|,\ m=\overline{1,N}.$$

Choosing m-th term of the last sum of the right side of this inequality and applying the Cauchy inequality with ε we get

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2 \le 2\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^2 + 2a_0\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^2 + 4\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{m-1}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|\left|\Phi_{jk}^{p}\right| +$$

$$+\frac{1}{\varepsilon}4\theta\tau h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2 + \varepsilon 4\theta\tau h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^2,\ m=\overline{1,N}.$$

Taking in this inequality$\varepsilon = 8\tau$ we obtain the validity of the following inequality

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2 \le 4\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^2 + 4a_0\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^2 + 64T\theta\tau h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|f_{jm}^{p}\right|^2 +$$

$$+8\tau\theta h\sum_{k=1}^{m-1}\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|\left|\Phi_{jk}^{p}\right|,\ \forall m\in\{1,\,2,\,...,N\}.$$

In the last term of this inequality, applying the Cauchy-Bunyakovski inequality, we have

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2 \le 4\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^2 + 4a_0\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^2 + 64T\theta\tau h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|f_{jm}^{p}\right|^2 +$$

$$+4\tau\theta h\sum_{k=1}^{m-1}\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^2 + 4\tau\theta h\sum_{k=1}^{m-1}\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}\right|^2,\ \forall m\in\{1,\,2,\,...,N\}.$$

Combining the third and fourth terms on the right side of this inequality and increasing the summation to $k = N$, we obtain the validity of the inequality

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^2 \le 4\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^2 + 4a_0\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^2 +$$

$$+4\left(1+16T\right)\tau\theta h\sum_{k=1}^{N}\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^2 + 4\tau\theta h\sum_{k=1}^{m-1}\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}\right|^2,\ \forall m\in\{1,\,2,\,...,N\}.$$

Applying in this inequality a discrete analog of the Gronwall lemma (see [19, p.110]), we obtain the following estimate

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^{2}\le c_{2}\left(\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^{2}+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^{2}+\right.$$

$$\left.+\tau\theta h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^{2}\right),\ \forall m\in\{1,\,2,\,...,N\}\,. \tag{33}$$

Similarly to obtaining this estimate from inequality (32) using conditions (31) and $\theta\le\frac{a_0}{2\mu_2}$ it is not difficult to establish the validity of the estimate

$$\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{q}\right|^{2}\le$$

$$\le c_{3}\left(\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^{2}+\tau\, h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^{2}+\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^{2}\right),\ \forall q\in\{1,\,2,\,...,Z\}\,. \tag{34}$$

Summing up (33) and (34), we obtain the following estimate

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^{2}+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{q}\right|^{2}\le$$

$$\le c_{4}\left(\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^{2}+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^{2}+\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^{2}\right) \tag{35}$$

for $\forall m\in\{1,2,...,N\}$, $\forall q\in\{1,2,...,Z\}$. Here taking $A_1=A_4$ we obtain the validity of estimate (21). Along with this estimate, from inequality (29) we can also establish the validity of the following estimate

$$\theta h\sum_{p=1}^{q}\sum_{j=1}^{M-1}\left|\Phi_{jm}^{p}\right|^{2}+\theta h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}-\Phi_{jk-1}^{p}\right|^{2}+$$

$$+a_{0}\tau h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{q}\right|^{2}+a_{0}\tau h\sum_{p=1}^{q}\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\Phi_{jk}^{p}-\Phi_{jk}^{p-1}\right|^{2}\le$$

$$\le c_{5}\left(\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\varphi_{0j}^{p}\right|^{2}+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\varphi_{1j}^{k}\right|^{2}+\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^{2}\right) \tag{36}$$

for$\forall m\in\{1,2,...,N\}$, $\forall q\in\{1,2,...,Z\}$. Theorem 2.1 is proved.

Now let us establish an error estimate for the difference scheme (12)-(15). To this end, consider the following system

$$i\delta_{\bar{t}}W_{jk}^{p}+ia_0\delta_{\bar{z}}W_{jk}^{p}-a_1\delta_{x\bar{x}}W_{jk}^{p}+ia_{2j}\delta_{\bar{x}}W_{jk}^{p}+$$

$$+a^jW_{jk}^{p}+v_{0j}W_{jk}^{p}+iv_{1j}W_{jk}^{p}=F_{jk}^{p},\ j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}, \tag{37}$$

$$W_{j0}^{p}=0,\ j=\overline{0,M},\ p=\overline{1,Z}, \tag{38}$$

$$W_{jk}^{0}=0,\ j=\overline{0,M},\ k=\overline{1,N}, \tag{39}$$

$$W_{0k}^{p}=W_{Mk}^{p}=0,\ k=\overline{1,N},\ p=\overline{1,Z}, \tag{40}$$

where the grid function F_{jk}^{p} is defined as

$$F_{jk}^{p}=\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left(i\frac{\partial\psi}{\partial t}+ia_0\frac{\partial\psi}{\partial z}-a_1\frac{\partial^2\psi}{\partial x^2}+ia_2\left(x\right)\frac{\partial\psi}{\partial x}+a\left(x\right)\psi\right)dxdtdz+$$

$$+\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left(v_0\left(x\right)\psi+iv_1\left(x\right)\psi\right)dxdtdz-$$

$$-i\delta_{\bar{t}}\psi_{jk}^{p}-ia_0\delta_{\bar{z}}\psi_{jk}^{p}+a_1\delta_{x\bar{x}}\psi_{jk}^{p}-a_{2j}\delta_{\bar{x}}\psi_{jk}^{p}-a^j\psi_{jk}^{p}-v_{0j}\psi_{jk}^{p}-iv_{1j}\psi_{jk}^{p},$$

$$j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}, \tag{41}$$

$W_{jk}^{p}=\Phi_{jk}^{p}-\psi_{jk}^{p}$, Φ_{jk}^{p} is a solution to difference scheme (12)- (15) at $[v_n]\in V_n$ and ψ_{jk}^{p} is defined by the formula

$$\psi_{jk}^{p}=\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\psi\left(x,t,z\right)dxdtdz,\ j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z},$$

$$\psi_{j0}^{p}=\ \varphi_{0j}^{p},j=\overline{0,M},\ p=\overline{1,Z}\ ,$$

$$\psi_{jk}^{0}=\ \varphi_{1j}^{k},j=\overline{0,M},\ k=\overline{1,N}\ ,\ \psi_{0k}^{p}=\psi_{Mk}^{p}=0,\ k=\overline{1,N},p=\overline{1,Z}. \tag{42}$$

Let the operator $Q_n=Q_n\left(v\right)$ is given by the formula

$$Q_n\left(v\right)=\left[w\right]_n=\left(\left[w_0\right]_n,\left[w_1\right]_n\right)=\left(w_{01},w_{02},...,w_{0M-1},w_{11},w_{12},...,w_{1M-1}\right), \tag{43}$$

$$w_{sj}=\frac{1}{h}\int_{x_j-h/2}^{x_j+h/2}v_s\left(x\right)dx,\ s=0,1,\ \ j=\overline{1,M-1}. \tag{44}$$

Theorem 2.2. Let the condition of Theorem 2.1 be satisfied. Then for any $m\in\{1,2,...,N\}$, and any $q\in\{1,2,...,Z\}$ it is valid

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|W_{jm}^{p}\right|^2+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|W_{jk}^{q}\right|^2\le c_6\left(\beta_{\tau h}^{\theta}+\left\|Q_n\left(v\right)-\left[v\right]_n\right\|^2\right), \tag{45}$$

where $\beta^{\theta}_{\tau h} > 0$, $\beta^{\theta}_{\tau h} \to 0$ at $\theta \to 0$, $\tau \to 0$, $h \to 0$ and

$$\|Q_n(v) - [v]_n\|^2 = h \sum_{s=0}^{1} \sum_{j=1}^{M-1} |w_{sj} - v_{sj}|^2 . \tag{46}$$

Proof. Using the method of proving estimate (21), as in the proof of Theorem 2.1, we can establish the estimate

$$\theta h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left|W^p_{jm}\right|^2 + \tau h \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left|W^q_{jk}\right|^2 \le c_7 \theta \tau h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left|F^p_{jk}\right|^2 . \tag{47}$$

for $\forall m \in \{1, 2, ..., N\}, \forall q \in \{1, 2, ..., Z\}$. The grid function F^p_{jk} is defined by formula (39). We represent this grid function as

$$F^p_{jk} = F^{p1}_{jk} + F^{p2}_{jk} + F^{p3}_{jk} + F^{p4}_{jk} + F^{p5}_{jk} + F^{p6}_{jk} + F^{p7}_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{48}$$

where

$$F^{p1}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} i \frac{\partial \psi}{\partial t} dx dt dz - i \delta_{\overline{t}} \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{49}$$

$$F^{p2}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} i a_0 \frac{\partial \psi}{\partial z} dx dt dz - i a_0 \delta_{\overline{z}} \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{50}$$

$$F^{p3}_{jk} = -\frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} a_1 \frac{\partial^2 \psi}{\partial x^2} dx dt dz + a_1 \delta_{x\overline{x}} \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{51}$$

$$F^{p4}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} i a_2(x) \frac{\partial \psi}{\partial x} dx dt dz - i a_{2j} \delta_{\overline{x}} \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{52}$$

$$F^{p5}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} a(x) \psi dx dt dz - a^j \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{53}$$

$$F^{p6}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} v_0(x) \psi dx dt dz - v_{0j} \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{54}$$

$$F^{p7}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} i v_1(x) \psi dx dt dz - i v_{1j} \psi^p_{jk},\ j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}. \tag{55}$$

First we estimate $F^{p1}_{jk}, j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}$. By virtue of formula (49) at $j = 1, M-1, k = 2, N, p = 1, Z$ we have

$$F^{p1}_{jk} = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j - h/2}^{x_j + h/2} \left(\int_{-\tau}^{0} \left[\frac{\partial \psi(x, t, z)}{\partial t} - \frac{\partial \psi(x, t + \gamma, z)}{\partial t} \right] d\gamma \right) dx dt dz.$$

From here, applying the Cauchy-Bunyakovsky inequality, we obtain

$$\left|F_{jk}^{p1}\right|^2 \leq \frac{1}{\theta\tau^2 h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{-\tau}^{0}\left|\frac{\partial\psi(x,t,z)}{\partial t} - \frac{\partial\psi(x,t+\gamma,z)}{\partial t}\right|^2 d\gamma dxdtdz,$$

$$j = \overline{1, M-1}, k = \overline{2, N}, p = \overline{1, Z}. \tag{56}$$

ForF_{j1}^{p1} we have

$$F_{j1}^{p1} = \frac{i}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_0}^{t_1}\int_{x_j-h/2}^{x_j+h/2}\frac{\partial\psi\left(x,t,z\right)}{\partial t}dxdtdz-$$

$$-\frac{i}{\theta\tau^2 h}\int_{z_{p-1}}^{z_p}\int_{t_0}^{t_1}\int_{x_j-h/2}^{x_j+h/2}\int_{t_0}^{t}\frac{\partial\psi\left(x,\gamma,z\right)}{\partial\gamma}d\gamma dxdtdz, j = \overline{1, M-1}, p = \overline{1, Z}.$$

The use of Cauchy-Bunyakovski inequality gives

$$\left|F_{j1}^{p1}\right|^2 \leq \frac{4}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_0}^{t_1}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi}{\partial t}\right|^2 dxdtdz,\ j = \overline{1, M-1},\quad p = \overline{1, Z}. \tag{57}$$

Doing in the same way as when obtaining inequalities (56), (57), from formula (50) we can establish the validity of the inequalities $\left|F_{jk}^{p2}\right|^2 \leq \frac{a_0^2}{\theta^2\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{-\theta}^{0}\left|\frac{\partial\psi(x,t,z)}{\partial z} - \frac{\partial\psi(x,t,\eta+z)}{\partial z}\right|^2 d\eta dxdtdz,$

$$j = \overline{1, M-1}, k = \overline{1, N}, p = \overline{2, Z}, \tag{58}$$

$$\left|F_{jk}^{12}\right|^2 \leq \frac{4a_0^2}{\theta\tau h}\int_{z_0}^{z_1}\int_{t_{k-1}}^{t_{1k}}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi}{\partial z}\right|^2 dxdtdz,\ j = \overline{1, M-1},\quad k = \overline{1, N}. \tag{59}$$

Using formula (51) for the grid function F_{jk}^{p3} $j = \overline{2, M-2},\ k = \overline{1, N},\ p = \overline{1, Z}$ we get

$$F_{jk}^{p3} = -\frac{a_1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\frac{\partial^2\psi}{\partial x^2}dxdtdz+$$

$$+\frac{a_1}{\theta\tau h^3}\left\{\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{x}^{x+h}\int_{\xi-h}^{\xi}\frac{\partial^2\psi\left(\beta,t,z\right)}{\partial\beta^2}d\beta d\xi dxdtdz\right\}.$$

From here it is not difficult to obtain the validity of the equality

$$F_{jk}^{p3} = -\frac{a_1}{\theta\tau h^3}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{0}^{h}\int_{-h}^{0}\left(\frac{\partial^2\psi\left(x,t,z\right)}{\partial x^2}-\right.$$

$$\left.-\frac{\partial^2\psi\left(x+\xi+\beta,t,z\right)}{\partial x^2}\right)d\beta d\xi dxdtdz,\ j = \overline{2, M-2},\ k = \overline{1, N},\ p = \overline{1, Z}.$$

Applying here the Cauchy-Bunyakovski inequality we have

$$\left|F_{jk}^{p3}\right|^2 \le \frac{a_1^2}{\theta\tau h^3}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_0^h\int_{-h}^0\left|\frac{\partial^2\psi(x,t,z)}{\partial x^2}-\right.$$

$$\left.-\frac{\partial^2\psi(x+\xi+\beta,t,z)}{\partial x^2}\right|^2 d\beta d\xi dxdtdz,\ j=\overline{2,M-2},\ k=\overline{1,N},\ p=\overline{1,Z}. \qquad (60)$$

Now we estimate $F_{1k}^{p3}, k=\overline{1,N},\ p=\overline{1,Z}$. Using formula (51) for $j=1$ we obtain

$$F_{1k}^{p3} = -\frac{a_1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\frac{\partial^2\psi}{\partial x^2}dxdtdz + \frac{a_1}{\theta\tau h^3}\left\{\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\left[\int_{x_2-h/2}^{x_2+h/2}\psi(x,t,z)\,dx-\right.\right.$$

$$\left.\left.-\int_{x_1-h/2}^{x_1+h/2}\psi(x,t,z)\,dx - 2\int_{x_1-h/2}^{x_1+h/2}\psi(x,t,z)\,dx + 2\int_{x_1-h/2}^{x_1+h/2}\psi(x_1-h/2,t,z)\,dx\right]dtdz\right\} =$$

$$= -\frac{a_1}{\theta\tau h^3}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\int_x^{x+h}\int_\xi^{x_1+h/2}\frac{\partial^2\psi(\beta,t,z)}{\partial\beta^2}d\beta d\xi dxdtdz+$$

$$+\frac{a_1}{\theta\tau h^3}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\int_{x_1-h/2}^{x}\int_{x_1-h/2}^{\xi}\frac{\partial^2\psi(\beta,t,z)}{\partial\beta^2}d\beta d\xi dxdtdz.$$

From this using Cauchy-Bunyakovski inequality we get

$$\left|F_{1k}^{p3}\right|^2 \le \frac{16a_1^2}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\left|\frac{\partial^2\psi}{\partial x^2}\right|^2 dxdtdz, k=\overline{1,N},\ p=\overline{1,Z}. \qquad (61)$$

Similarly for $F_{M-1k}^{p3}, k=\overline{1,N},\ p=\overline{1,Z}$ using (51) we can write

$$\left|F_{M-1k}^{p3}\right|^2 \le \frac{16a_1^2}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_{M-1}-h/2}^{x_{M-1}+h/2}\left|\frac{\partial^2\psi}{\partial x^2}\right|^2 dxdtdz, k=\overline{1,N},\ p=\overline{1,Z}. \qquad (62)$$

To estimate $F_{jk}^{p4}, j=\overline{1,M-1},\ k=\overline{1,N}, p=\overline{1,Z}$ we have

$$F_{jk}^{p4} = \frac{i}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left(a_2(x)\frac{\partial\psi}{\partial x} - a_{2j}\delta_{\overline{x}}\psi_{jk}^p\right)dxdtdz =$$

$$= \frac{i}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}a_{2j}\left(\frac{\partial\psi}{\partial x} - \delta_{\overline{x}}\psi_{jk}^p\right)dxdtdz+$$

$$+\frac{i}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left(a_2(x) - a_{2j}\right)\frac{\partial\psi}{\partial x}dxdtdz =$$

$$= F_{jk}^{p40} + F_{jk}^{p41}, j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}. \qquad (63)$$

The use of formulas for F_{jk}^{p40}, F_{jk}^{p41} implies

$$F_{jk}^{p40} = \frac{i}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} a_{2j} \left(\frac{\partial \psi}{\partial x} - \delta_{\overline{x}} \psi_{jk}^p \right) dxdtdz, j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}, \tag{64}$$

$$F_{jk}^{p41} = \frac{i}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} (a_2(x) - a_{2j}) \frac{\partial \psi}{\partial x} dxdtdz\,, j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}. \tag{65}$$

Now let's estimate F_{jk}^{p41}. Then, using the second formula from (16), we can write the following equality

$$F_{jk}^{p41} = \frac{i}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} (a_2(x) - a_{2j}) \frac{\partial \psi}{\partial x} dxdtdz =$$

$$= \frac{i}{\theta\tau h^2} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \int_{x_j-h/2}^{x_j+h/2} (a_2(x) - a_2(\xi))\, d\xi \frac{\partial \psi}{\partial x} dxdtdz =$$

$$= \frac{i}{\theta\tau h^2} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \int_{x_j-h/2}^{x_j+h/2} \int_{\xi}^{x} \frac{da_2(\eta)}{d\eta} d\eta d\xi \frac{\partial \psi}{\partial x} dxdtdz, j = \overline{1, M-1},$$

$$k = \overline{1, N},\ p = \overline{1, Z}.$$

From this equality, by virtue of condition (7), we have

$$\left| F_{jk}^{p41} \right| \leq \frac{1}{\theta\tau h^2} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \int_{x_j-h/2}^{x_j+h/2} \int_{x_j-h/2}^{x_j+h/2} \left| \frac{da_2(\eta)}{d\eta} \right| d\eta d\xi \left| \frac{\partial \psi}{\partial x} \right| dxdtdz \leq$$

$$\leq \frac{\mu_2}{\theta\tau} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left| \frac{\partial \psi}{\partial x} \right| dxdtdz, j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}.$$

From here, using the Cauchy-Bunyakovski inequality, we obtain the validity of the inequality

$$\left| F_{jk}^{p41} \right| \leq \frac{\mu_2 \sqrt{h}}{\sqrt{\theta\tau}} \left(\int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left| \frac{\partial \psi}{\partial x} \right|^2 dxdtdz \right)^{\frac{1}{2}}, j = \overline{1, M-1},\ k = \overline{1, N},\ p = \overline{1, Z}. \tag{66}$$

Using (64) now estimate F_{jk}^{p40}

$$F_{jk}^{p40} = \frac{ia_{2j}}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left(\frac{\partial \psi}{\partial x} - \delta_{\overline{x}} \psi_{jk}^p \right) dxdtdz =$$

$$= ia_{2j} \left(\frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \frac{\partial \psi}{\partial x} dxdtdz - \delta_{\bar{x}} \psi_{jk}^p \right),$$

$$j=\overline{2,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}.$$

Now let's convert the difference inside the brackets on the right side of this formula. For this purpose, using the formula for ψ_{jk}^{p} we have

$$F_{jk}^{p40}=ia_{2j}\left(\frac{1}{\theta\tau h^{2}}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{-h}^{0}\left(\frac{\partial\psi\left(x,t,z\right)}{\partial x}-\frac{\partial\psi\left(x+\xi,t,z\right)}{\partial x}\right)d\xi dxdtdz\right),$$

$$j=\overline{2,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}.$$

Again applying Cauchy-Bunyakovski inequality we have

$$\left|F_{jk}^{p40}\right|^{2}\leq\tfrac{1}{\theta\tau h^{2}}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{-h}^{0}\left|\tfrac{\partial\psi(x,t,z)}{\partial x}-\tfrac{\partial\psi(x+\xi,t,z)}{\partial x}\right|^{2}d\xi dxdtdz,$$

$$j=\overline{2,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}. \tag{67}$$

Now we consider F_{1k}^{p40}. By the help of formula (64) and conditions $\psi\left(0,t,z\right)=0,\psi_{0k}^{p}=0$ we get

$$F_{1k}^{p40}=\frac{ia_{21}}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\frac{\partial\psi}{\partial x}dxdtdz-ia_{21}\delta_{\overline{x}}\psi_{1k}^{p}=\frac{ia_{21}}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\frac{\partial\psi}{\partial x}dxdtdz-$$

$$-ia_{21}\frac{1}{h}\left(\psi_{1k}^{p}-\psi_{0k}^{p}\right)=ia_{21}\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\frac{\partial\psi\left(x,t,z\right)}{\partial x}dxdtdz-$$

$$-ia_{21}\frac{1}{h}\left(\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\psi\left(x,t,z\right)dxdtdz-\right.$$

$$\left.-\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\psi\left(0,t,z\right)dxdtdz\right)=$$

$$=ia_{21}\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\frac{\partial\psi\left(x,t,z\right)}{\partial x}dxdtdz-$$

$$-ia_{21}\frac{1}{h}\left(\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\int_{x_1-h/2}^{x}\frac{\partial\psi\left(\xi,t,z\right)}{\partial\xi}dxdtdz\right)=$$

$$=ia_{21}\frac{1}{\theta\tau h^{2}}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\int_{x_1-h/2}^{x}\left(\frac{\partial\psi\left(x,t,z\right)}{\partial x}-\frac{\partial\psi\left(\xi,t,z\right)}{\partial\xi}\right)d\xi dxdtdz=$$

$$=ia_{21}\frac{1}{\theta\tau h^{2}}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_1-h/2}^{x_1+h/2}\int_{x_1-h/2}^{x}\int_{\xi}^{x}\frac{\partial^{2}\psi\left(\eta,t,z\right)}{\partial\eta^{2}}d\eta d\xi dxdtdz,\ k=\overline{1,N},\ p=\overline{1,Z}.$$

Using this formula and the Cauchy-Bunyakovski inequality, we obtain the validity of the inequality

$$\left|F_{1k}^{p40}\right|^2 \le \mu_1^2 \frac{h}{\theta\tau} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_1-h/2}^{x_1+h/2} \left|\frac{\partial^2 \psi\left(\eta,t,z\right)}{\partial \eta^2}\right|^2 d\eta dt dz,\ k=\overline{1,N},\ \ p=\overline{1,Z}. \qquad (68)$$

By formula (53) we have

$$F_{jk}^{p5} = \frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} a\left(x\right)\left(\psi\left(x,t,z\right)-\psi_{jk}^p\right) dx dt dz +$$

$$+\frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left(a\left(x\right)-a^j\right)\psi_{jk}^p dx dt dz,\ \ j=\overline{1,M-1},\ \ k=\overline{1,N},\ \ p=\overline{1,Z}.$$

It is clear that, by virtue of the first formula in (16), the second term on the right-hand side of this equality is equal to zero. Therefore, we get the following equality

$$F_{jk}^{p4} = \frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} a\left(x\right)\left(\psi-\psi_{jk}^p\right) dx dt dz,\ \ j=\overline{1,M-1},\ \ k=\overline{1,N},\ \ p=\overline{1,Z}. \qquad (69)$$

Let's consider the difference $\psi-\psi_{jk}^p$. Using formula (42) we have

$$\psi\left(x,t,z\right)-\psi_{jk}^p = -\frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left(\psi\left(x,t,z\right)-\psi\left(\xi,\gamma,\eta\right)\right) d\xi d\gamma d\eta =$$

$$= \frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left(\int_{\xi}^{x} \frac{\partial\psi\left(\beta,t,z\right)}{\partial\beta} + \int_{\gamma}^{t} \frac{\partial\psi\left(\xi,\gamma,z\right)}{\partial\alpha} d\alpha\ + \int_{\eta}^{z} \frac{\partial\psi\left(\xi,\gamma,x\right)}{\partial x} dx\right) d\xi d\gamma d\eta. \qquad (70)$$

From here, using the Cauchy-Bunyakovski inequality, we obtain $\left|\psi\left(x,t,z\right)-\psi_{jk}^p\right| \le$ $\frac{\sqrt{h}}{\sqrt{\theta\tau}}\left(\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(\beta,t,z)}{\partial\beta}\right|^2 d\beta d\gamma d\eta\right)^{1/2} +$

$$+\frac{\sqrt{\tau}}{\sqrt{\theta h}}\left(\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\int_{t_{k-1}}^{t_k}\left|\frac{\partial\psi(\xi,\alpha,z)}{\partial\alpha}\right|^2 d\alpha d\xi d\eta\right)^{1/2} +$$

$$+\frac{\sqrt{\theta}}{\sqrt{\tau h}}\left(\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\int_{z_{p-1}}^{z_p}\left|\frac{\partial\psi\left(\xi,\gamma,\chi\right)}{\partial\chi}\right|^2 d\chi d\xi d\gamma\right)^{1/2}. \qquad (71)$$

By virtue of condition (6) from (69) we have

$$\left|F_{jk}^{p5}\right| \le \frac{\mu_0}{\sqrt{\theta\tau h}}\left(\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\psi-\psi_{jk}^p\right|^2 dx dt dz\right)^{1/2},\ \ j=\overline{1,M-1},\ \ k=\overline{1,N},\ \ p=\overline{1,Z}.$$

Taking into account inequality (71) in this, we obtain the validity of the inequality

$$\left|F_{jk}^{p5}\right|^2 \le \frac{3\mu_0^2 h}{\theta\tau} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|\frac{\partial\psi\left(\beta,t,z\right)}{\partial\beta}\right|^2 d\beta dt dz+$$

$$+\frac{3\mu_0^2 \tau}{\theta h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|\frac{\partial\psi\left(\xi,\alpha,z\right)}{\partial\alpha}\right|^2 d\alpha d\xi dz+$$

$$+\frac{3\mu_0^2 \theta}{h\tau} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|\frac{\partial\psi\left(\xi,\gamma,\chi\right)}{\partial\chi}\right|^2 d\chi d\xi d\gamma,\ j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}. \tag{72}$$

By the help of formula (54) and (43) for F_{jk}^{p6},$j=\overline{1,M-1},\ k=\overline{1,N}$, $p=\overline{1,Z}$ we get

$$F_{jk}^{p6} = \psi_{jk}^{p}\left(w_{0j}-v_{0j}\right) + \frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} v_0\left(x\right)\left(\psi-\psi_{jk}^{p}\right) dx dt dz.$$

From here, by virtue of the Cauchy-Bunyakovski inequality, we obtain the validity of the inequality

$$\left|F_{jk}^{p6}\right| \le \left|\psi_{jk}^{p}\right| \left|w_{0j}-v_{0j}\right| + \frac{1}{\theta\tau h} \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|v_0\left(x\right)\right| \left|\psi-\psi_{jk}^{p}\right| dx dt dz \le$$

$$\le \left|\psi_{jk}^{p}\right| \left|w_{0j}-v_{0j}\right| + \frac{b_0}{\sqrt{\theta\tau h}} \left(\int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|\psi-\psi_{jk}^{p}\right|^2 dx dt dz\right)^{\frac{1}{2}}, j=\overline{1,M-1},$$

$$k=\overline{1,N},\ p=\overline{1,Z}. \tag{73}$$

Using inequality (71), we have

$$\left|\psi\left(x,t,z\right)-\psi_{jk}^{p}\right|^2 \le 3h \int_{x_j-h/2}^{x_j+h/2} \left|\frac{\partial\psi\left(\beta,t,z\right)}{\partial\beta}\right|^2 d\beta+$$

$$+3\frac{\tau}{h} \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} \int_{t_{k-1}}^{t_k} \left|\frac{\partial\psi\left(\xi,\alpha,z\right)}{\partial\alpha}\right|^2 d\alpha d\xi + 3\frac{\theta}{\tau h} \int_{t_{k-1}}^{t_k} \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} \int_{z_{p-1}}^{z_p} \left|\frac{\partial\psi\left(\xi,\gamma,\chi\right)}{\partial\chi}\right|^2 d\chi d\xi d\gamma,$$

$$j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}.$$

Integrating both parts of this inequality over the variables x,t,zon the intervals $\left[x_j-\frac{h}{2},x_j+\frac{h}{2}\right]$,$\left[t_{k-1},t_k\right]$, $\left[z_{p-1},z_p\right]$, correspondingly, we obtain the validity of the inequality

$$\int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|\psi\left(x,t,z\right)-\psi_{jk}^{p}\right|^2 dx dt dz \le 3h^2 \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left|\frac{\partial\psi\left(\beta,t,z\right)}{\partial\beta}\right|^2 d\beta dt dz+$$

$$+3\tau^2 \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} \left|\frac{\partial\psi\left(\xi,\alpha,z\right)}{\partial\alpha}\right|^2 d\xi d\alpha dz+3\theta^2 \int_{z_{p-1}}^{z_p} \int_{t_{k-1}}^{t_k} \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} \left|\frac{\partial\psi\left(\xi,\gamma,\chi\right)}{\partial\chi}\right|^2 d\xi d\gamma d\chi,$$

$$j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}.$$

On the right side of this inequality, changing in the integral variables, we obtain the validity of the inequality

$$\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\psi(x,t,z)-\psi_{jk}^p\right|^2dxdtdz\le 3h^2\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial x}\right|^2dxdtdz+$$

$$+3\tau^2\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}}\left|\frac{\partial\psi(x,t,z)}{\partial t}\right|^2dxdtdz+3\theta^2\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}}\left|\frac{\partial\psi(x,t,z)}{\partial z}\right|^2dxdtdz,$$

$$j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}. \tag{74}$$

By virtue of this inequality, from inequality (73) we have

$$\left|F_{jk}^{p6}\right|^2\le 2\left|\psi_{jk}^p\right|^2\left|w_{0j}-v_{0j}\right|^2+\frac{6b_0^2h}{\theta\tau}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial x}\right|^2dxdtdz+$$

$$+\frac{6b_0^2\tau}{\theta h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial t}\right|^2dxdtdz+\frac{6b_0^2\theta}{\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial z}\right|^2dxdtdz,$$

$$j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}. \tag{75}$$

Similarly to obtaining this inequality using formula (55), for F_{jk}^{p7}we can obtain the validity of the inequality

$$\left|F_{jk}^{p7}\right|^2\le 2\left|\psi_{jk}^p\right|^2\left|w_{1j}-v_{1j}\right|^2+\frac{6b_1^2h}{\theta\tau}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial x}\right|^2dxdtdz+$$

$$+\frac{6b_1^2\tau}{\theta h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial t}\right|^2dxdtdz+\frac{6b_1^2\theta}{\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial\psi(x,t,z)}{\partial z}\right|^2dxdtdz,$$

$$j=\overline{1,M-1},\ k=\overline{1,N},\ p=\overline{1,Z}. \tag{76}$$

Using the Fubini theorem, known from the work (see [19, pp.61-62]), from inequality (56) we have

$$\theta\tau h\sum_{p=1}^{Z}\sum_{k=2}^{N}\sum_{j=1}^{M-1}\left|F_{jk}^{p1}\right|^2\le\frac{1}{\tau}\int_{-\tau}^{0}\left(\int_\Omega\left|\frac{\partial\psi(x,t,z)}{\partial t}-\frac{\partial\psi(x,t+\gamma,z)}{\partial t}\right|^2dxdtdz\right)d\gamma. \tag{77}$$

Take arbitrary $\varepsilon>0$. According to the mean continuity theorem for a function from $L_2(\Omega)$, there exists $\delta>0,$ that

$$\int_\Omega\left|\frac{\partial\psi(x,t,z)}{\partial t}-\frac{\partial\psi(x,t+\gamma,z)}{\partial t}\right|^2dxdtdz<\varepsilon,$$

only if $|\gamma| \leq \tau < \delta$. Therefore for such$\tau$ from (77) we obtain

$$\theta\tau h \sum_{p=1}^{Z} \sum_{k=2}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p1} \right|^2 \leq \omega_\tau^0, \tag{78}$$

where$\omega_\tau^0 > 0$, $\omega_\tau^0 \to 0$ at $\tau \to 0$. From (57) we have

$$\theta\tau h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left| F_{jk}^{p1} \right|^2 \leq 4 \int_0^\tau \left\| \frac{\partial \psi(\cdot, \gamma, \cdot)}{\partial t} \right\|_{L_2(\Omega_L)}^2 d\gamma. \tag{79}$$

Due to the absolute continuity of the integral on the right side of this inequality, we obtain the following relation

$$\theta\tau h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left| F_{jk}^{p1} \right|^2 \leq \tilde{\omega}_\tau^0, \tag{80}$$

where $\tilde{\omega}_\tau^0 > 0$, $\tilde{\omega}_\tau^0 \to 0$ at $\tau \to 0$. Summing (78) and (80) we get the inequality

$$\theta\tau h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p1} \right|^2 \leq \omega_\tau^1, \tag{81}$$

where $\omega_\tau^1 = \omega_\tau^0 + \tilde{\omega}_\tau^0$.
Similarly to obtaining inequality (81), from inequalities (58), (59) we obtain the validity of the inequality

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p2} \right|^2 \leq \omega_\theta^2, \tag{82}$$

where $\omega_\theta^2 > 0$, $\omega_\theta^2 \to 0$ at $\theta \to 0$.
By virtue of inequality (60), based on the Fubini theorem, as in obtaining inequality (78), we have

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=2}^{M-2} \left| F_{jk}^{p3} \right|^2 \leq \tilde{\omega}_h^3, \tag{83}$$

where $\tilde{\omega}_h^3 > 0$, $\tilde{\omega}_h^3 \to 0$ at $h \to 0$. From (61), (62) we get the inequalities

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \left| F_{1k}^{p3} \right|^2 \leq 16 a_1^2 \int_0^h \left\| \frac{\partial^2 \psi(x, \cdot, \cdot)}{\partial x^2} \right\|_{L_2(Q)}^2 dx, \tag{84}$$

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \left| F_{M-1k}^{p3} \right|^2 \leq 16 a_1^2 \int_{l-h}^{l} \left\| \frac{\partial^2 \psi(x, \cdot, \cdot)}{\partial x^2} \right\|_{L_2(Q)}^2 dx, \tag{85}$$

Summing up these inequalities and using the absolute continuity of the integrals, we obtain the validity of the following relation

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \left| F_{1k}^{p3} \right|^2 + \theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \left| F_{M-1k}^{p3} \right|^2 \leq \tilde{\omega}_h^4, \tag{86}$$

where $\tilde{\omega}_h^4 > 0$, $\tilde{\omega}_h^4 \to 0$ at $h \to 0$. From the last and (83) one can get

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p3} \right|^2 \leq \omega_h^3, \tag{87}$$

where $\omega_h^3 = \tilde{\omega}_h^3 + \tilde{\omega}_h^4$.

By virtue of inequalities (66)-(68), from equality (63) we can obtain the validity of the inequality

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p4} \right|^2 \leq \omega_h^4 + c_8 h^2, \tag{88}$$

where $\omega_h^4 > 0$, $\omega_h^4 \to 0$ at $h \to 0$, $A_{12} > 0$ is a constant not depending on θ, τ, h.

By virtue of inequality (72), we obtain the following inequality

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p5} \right|^2 \leq 3\mu_0^2 \left(h^2 \left\| \frac{\partial\psi}{\partial x} \right\|_{L_2(\Omega)}^2 + \tau^2 \left\| \frac{\partial\psi}{\partial t} \right\|_{L_2(\Omega)}^2 + \theta^2 \left\| \frac{\partial\psi}{\partial z} \right\|_{L_2(\Omega)}^2 \right).$$

From this considering estimation (111) we get

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p5} \right|^2 \leq c_9 \left(h^2 + \tau^2 + \theta^2 \right), \tag{89}$$

Using (75) and (76) we obtain

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p6} \right|^2 \leq$$

$$\leq 2\theta\tau h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| \psi_{jk}^{p} \right|^2 |w_{0j} - v_{0j}|^2 + A_{10} \left(h^2 \left\| \frac{\partial\psi}{\partial x} \right\|_{L_2(\Omega)}^2 + \tau^2 \left\| \frac{\partial\psi}{\partial t} \right\|_{L_2(\Omega)}^2 + \theta^2 \left\| \frac{\partial\psi}{\partial z} \right\|_{L_2(\Omega)}^2 \right),$$

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p7} \right|^2 \leq$$

$$\leq 2\theta\tau h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| \psi_{jk}^{p} \right|^2 |w_{1j} - v_{1j}|^2 + A_{11} \left(h^2 \left\| \frac{\partial\psi}{\partial x} \right\|_{L_2(\Omega)}^2 + \tau^2 \left\| \frac{\partial\psi}{\partial t} \right\|_{L_2(\Omega)}^2 + \theta^2 \left\| \frac{\partial\psi}{\partial z} \right\|_{L_2(\Omega)}^2 \right).$$

By the help of these inequalities from (10) we have

$$\theta\tau\, h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| F_{jk}^{p6} \right|^2 \leq c_{12} \left(h^2 + \tau^2 + \theta^2 \right) + 2\theta\tau h \sum_{p=1}^{Z} \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left| \psi_{jk}^{p} \right|^2 |w_{0j} - v_{0j}|^2, \tag{90}$$

$$\theta\tau\, h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|F_{jk}^{p7}\right|^2 \le c_{13}\left(h^2+\tau^2+\theta^2\right)+2\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\psi_{jk}^{p}\right|^2\left|w_{1j}-v_{1j}\right|^2 \quad (91)$$

It is obvious the validity of the following inequality

$$\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\psi_{jk}^{p}\right|^2\left|w_{sj}-v_{sj}\right|^2 \le$$

$$\le \max_{1\le j\le M-1}\left(\theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left|\psi_{jk}^{p}\right|^2\right)\left(h\sum_{j=1}^{M-1}\left|w_{sj}-v_{sj}\right|^2\right), s=0,1. \quad (92)$$

Using (42) we can write

$$\theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left|\psi_{jk}^{p}\right|^2=\theta\tau\sum_{p=1}^{z}\sum_{k=1}^{N}\left|\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\psi\left(x,t,z\right)dxdtdz\right|^2 \le$$

$$\le \theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left(\frac{1}{\theta\tau h}\int_{z_{p-1}}^{z_p}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\psi\left(x,t,z\right)\right|dxdtdz\right)^2, \ j=\overline{1,M-1}.$$

Hence, applying the Cauchy-Bunyakovski inequality, we obtain

$$\theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left|\psi_{jk}^{p}\right|^2 \le \theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left(\frac{1}{\sqrt{\theta\tau h}}\left(\int_{x_j-h/2}^{x_j+h/2}\int_{z_{p-1}}^{z_k}\int_{t_{k-1}}^{t_k}\left|\psi\left(x,t,z\right)\right|^2 dxdtdz\right)^{1/2}\right)^2=$$

$$=\frac{1}{h}\int_{x_j-h/2}^{x_j+h/2}\left(\sum_{p=1}^{z}\sum_{k=1}^{N}\int_{z_{p-1}}^{z_k}\int_{t_{k-1}}^{t_k}\left|\psi\left(x,t,z\right)\right|^2 dtdz\right)dx=$$

$$=\frac{1}{h}\int_{x_j-h/2}^{x_j+h/2}\left\|\psi\left(x,\cdot,\cdot\right)\right\|_{L_2(Q)}^2 dx \le \max_{x\in[0,\ell]}\left\|\psi\left(x,\cdot,\cdot\right)\right\|_{L_2(Q)}^2, \ \forall j\in\{1,2,...,M-1\}. \quad (93)$$

By virtue of an analogue of the well-known inequality (see, for example, [19], p. 412, Lemma 6), we can write the following inequality

$$\max_{x\in[0,l]}\left\|\psi\left(x,\cdot,\cdot\right)\right\|_{L_2(Q)}^2 \le c_{14}\left(\left\|\psi\right\|_{L_2(\Omega)}^2+\left\|\frac{\partial\psi}{\partial x}\right\|_{L_2(\Omega)}^2\right). \quad (94)$$

From this by the help of (10) get

$$\max_{x\in[0,l]}\left\|\psi\left(x,\cdot,\cdot\right)\right\|_{L_2(Q)}^2 \le c_{15}. \quad (95)$$

If to consider this in (93) then we get

$$\theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left|\psi_{jk}^{p}\right|^{2}\leq c_{15},\ \forall j\in\{1,2,...,M-1\}\,. \tag{96}$$

Then, given this inequality, we can write the following

$$\max_{1\leq j\leq M-1}\left(\theta\tau\sum_{p=1}^{Z}\sum_{k=1}^{N}\left|\psi_{jk}^{p}\right|^{2}\right)\leq c_{15}. \tag{97}$$

Consideration of this in (92) gives

$$\theta\tau h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\psi_{jk}^{p}\right|^{2}\left|w_{sj}-v_{sj}\right|^{2}\leq c_{15}h\sum_{j=1}^{M-1}\left|w_{sj}-v_{sj}\right|^{2},\ s=0,1.$$

Using this inequality, from inequalities (90), (91) we have

$$\theta\tau\, h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|F_{jk}^{p6}\right|^{2}\leq c_{12}\left(h^{2}+\tau^{2}+\theta^{2}\right)+2c_{15}h\sum_{j=1}^{M-1}\left|w_{0j}-v_{0j}\right|^{2}, \tag{98}$$

$$\theta\tau\, h\sum_{p=1}^{Z}\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|F_{jk}^{p7}\right|^{2}\leq c_{13}\left(h^{2}+\tau^{2}+\theta^{2}\right)+2c_{15}h\sum_{j=1}^{M-1}\left|w_{1j}-v_{1j}\right|^{2}. \tag{99}$$

Thus, due to estimates (81), (82), (87), (88), (89), (98), (99) and formula (48), using estimate (47), we obtain the following estimate

$$\theta h\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|W_{jm}^{p}\right|^{2}+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|W_{jk}^{q}\right|^{2}\leq c_{16}\left(\omega_{\tau}^{1}+\omega_{\theta}^{2}+\omega_{h}^{3}+\omega_{h}^{4}+h^{2}+\right.$$

$$\left.+\tau^{2}+\theta^{2}+\left\|Q_{n}\left(v\right)-\left[v\right]_{n}\right\|^{2}\right),\ \forall m\in\{1,2,...,N\}\,,\forall q\in\{1,2,...,Z\}\,. \tag{100}$$

If to denote

$$\beta_{\tau h}^{\theta}=\omega_{\tau}^{1}+\omega_{\theta}^{2}+\omega_{h}^{3}+\omega_{h}^{4}+h^{2}+\tau^{2}+\theta^{2}, \tag{101}$$

Then taking $c_6 = c_{16}$ from (100) we get the statement of the theorem. Theorem 2.2 is proved.

4. Convergence of difference approximations with respect to the functional

In this section, we establish the estimate on the convergence of difference approximations with respect to the functional. To this end, we first estimate the difference between functional (1) and function (11).

Theorem 3.1. Let the conditions of Theorem 2.2 and the following conditions for grid step matching be satisfied

$$c_{17} \le \frac{\tau}{h} \le c_{18},\ c_{19} \le \frac{\theta}{h} \le c_{20}, c_{21} \le \frac{\tau}{\theta} \le c_{22}, \tag{102}$$

Where the constants $A_m > 0, m = 6, 7, 8, 9, 10, 11$ do not depend on θ, τ, h. Then for any $v \in V$ and any $[v]_n \in V_n$ the following estimate is valid

$$|J(v) - I_n([v]_n)| \le c_{23}\left(\tilde{\beta}^{\theta}_{\tau h} + \|Q_n(v) - [v]_n\|\right), \tag{103}$$

where $c_{23} > 0$ is a constand not depending on $\theta, \tau,\ h$; $\tilde{\beta}^{\theta}_{\tau h} > 0$, $\tilde{\beta}^{\theta}_{\tau h} \to 0$ at $\theta \to 0$, $\tau \to 0$, $h \to 0$. **Proof.** Consider the difference $J(v) - I_n([v]_n)$. Using formulas (1) and (11) we have

$$J(v) - I_n([v]_n) = \beta_0 \|\psi(., T, .) - y_0\|^2_{L_2(\Omega_L)} + \beta_1 \|\psi(., ., L) - y_1\|^2_{L_2(\Omega_T)} -$$

$$-\beta_0\theta h \sum_{p=1}^{Z}\sum_{j=1}^{M-1}\left|\Phi^p_{jN} - y_0{}^p_j\right|^2 - \beta_1\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\Phi^Z_{jk} - y_1{}^k_j\right|^2 =$$

$$= \beta_0\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left(|\psi(x,T,z) - y_0(x,z)|^2 - \left|\Phi^p_{jN} - y^p_{0j}\right|^2\right)dxdz +$$

$$+\beta_1\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left(|\psi(x,t,L) - y_1(x,t)|^2 - \left|\Phi^Z_{jk} - y^k_{1j}\right|^2\right)dxdt$$

From this due to the inequality $|a|^2 - |b|^2 = (|a| + |b|)(|a| - |b|)$ and inequality $||a| - |b|| \le |a - b|$ we obtain

$$|J(v) - I_n([v]_n)| \le$$

$$= \beta_0\left(\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left(|\psi(x,T,z)| + |y_0(x,z)| + \left|\Phi^p_{jN}\right| + \left|y^p_{0j}\right|\right)^2 dxdz\right)^{\frac{1}{2}} \times$$

$$\times\left(\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left|\psi(x,T,z) - \Phi^p_{jN} + y_0(x,z) - y^p_{0j}\right|^2 dxdz\right)^{\frac{1}{2}} +$$

$$+\beta_1\left(\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left(|\psi(x,t,L)| + |y_1(x,t)| + \left|\Phi^Z_{jk}\right| + \left|y^k_{1j}\right|\right)^2 dxdt\right)^{\frac{1}{2}} \times$$

$$\times\left(\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{t_{k-1}}^{t_k}\int_{x_j-h/2}^{x_j+h/2}\left|\psi(x,t,L) - \Phi^Z_{jk} + y_1(x,t) - y^k_{1j}\right|^2 dxdt\right)^{\frac{1}{2}}.$$

From this inequality using (10), (21) and formulas (20) and also condition $y_0 \in L_2(\Omega_L), y_1 \in L_2(\Omega_T)$ we get

$$|J(v) - I_n([v]_n)| \le$$

$$\le c_{24} \left(\sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \Phi_{jN}^p + y_0(x,z) - y_{0j}^p \right|^2 dxdz \right)^{\frac{1}{2}} +$$

$$+ c_{25} \left(\sum_{k=1}^{N} \sum_{j=1}^{M-1} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,t,L) - \Phi_{jk}^Z + y_1(x,t) - y_{1j}^k \right|^2 dxdt \right)^{\frac{1}{2}} \le$$

$$\le \sqrt{2} c_{24} \left(\sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \Phi_{jN}^p \right|^2 dxdz \right)^{\frac{1}{2}} +$$

$$+ \sqrt{2} c_{24} \left(\sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| y_0(x,z) - y_{0j}^p \right|^2 dxdz \right)^{\frac{1}{2}} +$$

$$+ \sqrt{2} c_{25} \left(\sum_{k=1}^{N} \sum_{j=1}^{M-1} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,t,L) - \Phi_{jk}^Z \right|^2 dxdt \right)^{\frac{1}{2}} +$$

$$+ \sqrt{2} c_{25} \left(\sum_{k=1}^{N} \sum_{j=1}^{M-1} \int_{t_{k-1}}^{t_k} \int_{x_j-h/2}^{x_j+h/2} \left| y_1(x,t) - y_{1j}^k \right|^2 dxdt \right)^{\frac{1}{2}} =$$

$$= \sqrt{2} c_{24} (J_1 + J_2) + \sqrt{2} c_{25} (J_3 + J_4). \qquad (104)$$

First we estimate J_1. Considering its definition we can write

$$(J_1)^2 = \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \Phi_{jN}^p \right|^2 dxdz = \oplus$$

$$= \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \psi_{jN}^p + \psi_{jN}^p - \Phi_{jN}^p \right|^2 dxdz \le$$

$$\le 2 \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \psi_{jN}^p \right|^2 dxdz +$$

$$+ 2 \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi_{jN}^p - \Phi_{jN}^p \right|^2 dxdz = 2 (J_{11} + J_{12}). \qquad (105)$$

By the formula of J_{12} we have

$$J_{12} = \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi_{jN}^p - \Phi_{jN}^p \right|^2 dxdz = \theta h \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \left| \psi_{jN}^p - \Phi_{jN}^p \right|^2.$$

From this equality, by virtue of estimate (45), we obtain

$$J_{12} \leq c_6 \left(\beta_{\tau h}^{\theta} + \| Q_n(v) - [v_n] \|^2 \right). \tag{106}$$

From the formula for J_{11} we get

$$J_{11} = \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \psi_{jN}^p \right|^2 dxdz. \tag{107}$$

Now let's consider the difference $\psi(x,T,z) - \psi_{jN}^p$. Using formula (42), we can write the following equality

$$\psi(x,T,z) - \psi_{jN}^p = \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{N-1}}^{t_N} \int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}} \left(\psi(x,T,z) - \psi(\xi,\alpha,\beta) \right) d\xi d\alpha d\beta =$$

$$= \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{N-1}}^{t_N} \int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}} \left(\psi(x,T,z) - \psi(x,\alpha,z) + \psi(x,\alpha,z) - \psi(\xi,\alpha,z) + \right.$$

$$\left. + \psi(\xi,\alpha,z) - \psi(\xi,\alpha,\beta) \right) d\xi d\alpha d\beta =$$

$$= \frac{1}{\theta \tau h} \int_{z_{p-1}}^{z_p} \int_{t_{N-1}}^{t_N} \int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}} \left[\int_{\alpha}^{T} \frac{\partial \psi(x,\gamma,z)}{\partial \gamma} d\gamma + \int_{\xi}^{x} \frac{\partial \psi(\eta,\alpha,z)}{\partial \eta} + \right.$$

$$\left. + \int_{\beta}^{z} \frac{\partial \psi(\xi,\alpha,\chi)}{\partial \chi} d\chi \right] d\xi d\alpha d\beta, j = \overline{1, M-1}, p = \overline{1, Z}. \tag{108}$$

Using this equality, it is easy to establish the validity of the inequality $\left| \psi(x,T,z) - \psi_{jN}^p \right| \leq \int_{t_{N-1}}^{t_N} \left| \frac{\partial \psi(x,\gamma,z)}{\partial \gamma} \right| d\gamma + \frac{1}{\tau} \int_{t_{N-1}}^{t_N} \int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}} \left| \frac{\partial \psi(\eta,\alpha,z)}{\partial \eta} \right| d\eta +$

$$+ \frac{1}{\tau h} \int_{t_{N-1}}^{t_N} \int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}} \int_{z_{p-1}}^{z_p} \left| \frac{\partial \psi(\xi,\alpha,\chi)}{\partial \chi} \right| d\chi, j = \overline{1, M-1}, p = \overline{1, Z}. \tag{109}$$

Using this and the Cauchy-Bunyakovski inequality from (107), we obtain the validity of the following inequality

$$J_{11} = \sum_{p=1}^{Z} \sum_{j=1}^{M-1} \int_{z_{p-1}}^{z_p} \int_{x_j-h/2}^{x_j+h/2} \left| \psi(x,T,z) - \psi_{jN}^p \right|^2 dxdz \leq 3\tau \int_{t_{N-1}}^{t_N} \left\| \frac{\partial \psi(.,t,.)}{\partial t} \right\|_{L_2(\Omega_L)}^2 dt +$$

$$+3\frac{h^2}{\tau}\int_{t_{N-1}}^{t_N}\left\|\frac{\partial\psi\left(.,t,.\right)}{\partial x}\right\|^2_{L_2(\Omega_L)}dt+3\frac{\theta^2}{\tau}\int_{t_{N-1}}^{t_N}\left\|\frac{\partial\psi\left(.,t,.\right)}{\partial z}\right\|^2_{L_2(\Omega_L)}dt.$$

From this inequality, using the grid step matching condition and increasing in the interval of integration over the variable t, and also due to the estimate for the term J_{11}, we have

$$J_{11}\le A_{26}\left(\theta+\tau+h\right). \tag{110}$$

Then from inequality (104) with the help of inequalities (105), (109) we obtain

$$\left(J_1\right)^2\le c_{27}\left(\beta^{\theta}_{\tau h}+\theta+\tau+h+\left\|Q_n\left(v\right)-\left[v_n\right]\right\|^2\right). \tag{111}$$

Now using the formula of the term J_2 we get

$$\left(J_2\right)^2=\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left|y_0\left(x,z\right)-y^p_{0j}\right|^2dxdz. \tag{112}$$

Consider the difference $y_0\left(x,z\right)-y^p_{0j}$. By the help of the first formula from (20) one can get

$$y_0\left(x,z\right)-y^p_{0j}=\frac{1}{\theta\,h}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left(y_0\left(x,z\right)-y_0\left(\xi,\beta\right)\right)d\xi d\beta=$$

$$=\frac{1}{\theta\,h}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left(y_0\left(x,z\right)-y_0\left(\xi,z\right)+y_0\left(\xi,z\right)-y_0\left(\xi,\beta\right)\right)d\xi d\beta=$$

$$=\frac{1}{\theta\,h}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left(\int_{\xi}^{x}\frac{\partial y_0\left(\eta,z\right)}{\partial\eta}d\eta+\int_{\beta}^{z}\frac{\partial y_0\left(\xi,\chi\right)}{\partial\chi}d\chi\right)d\xi d\beta.$$

This inequality implies

$$\left|y_0\left(x,z\right)-y^p_{0j}\right|\le$$

$$\le\frac{1}{\theta\,h}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial y_0\left(\eta,z\right)}{\partial\eta}\right|d\eta d\xi d\beta+\frac{1}{\theta\,h}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\int_{z_{p-1}}^{z_p}\left|\frac{\partial y_0\left(\xi,\chi\right)}{\partial\chi}\right|d\chi d\xi d\beta=$$

$$=\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial y_0\left(\eta,z\right)}{\partial\eta}\right|d\eta+\frac{1}{h}\int_{x_j-h/2}^{x_j+h/2}\int_{z_{p-1}}^{z_p}\left|\frac{\partial y_0\left(\xi,\chi\right)}{\partial\chi}\right|d\chi d\xi.$$

From this by the help of Cauchy-Bunyakovski inequality we obtain

$$\left|y_0\left(x,z\right)-y^p_{0j}\right|\le\sqrt{h}\left(\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial y_0\left(\eta,z\right)}{\partial\eta}\right|^2d\eta\right)^{\frac{1}{2}}+\frac{\sqrt{\theta}}{\sqrt{h}}\left(\int_{x_j-h/2}^{x_j+h/2}\int_{z_{p-1}}^{z_p}\left|\frac{\partial y_0\left(\xi,\chi\right)}{\partial\chi}\right|^2d\chi d\xi\right)^{\frac{1}{2}},$$

$$j=\overline{1,M-1},p=\overline{1,Z}.$$

By squaring both sides of this inequality, we get the following inequality

$$\left|y_0\left(x,z\right)-y_{0j}^p\right|^2 \le 2h\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial y_0\left(\eta,z\right)}{\partial \eta}\right|^2 d\eta + 2\frac{\theta}{h}\int_{x_j-h/2}^{x_j+h/2}\int_{z_{p-1}}^{z_p}\left|\frac{\partial y_0\left(\xi,\chi\right)}{\partial \chi}\right|^2 d\chi d\xi,$$

$$j=\overline{1,M-1}, p=\overline{1,Z}. \tag{113}$$

Now considering this in the right hand side of (112) we can write

$$\left(J_2\right)^2 = \sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left|y_0\left(x,z\right)-y_{0j}^p\right|^2 dxdz \le$$

$$\le 2h^2\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial y_0\left(\eta,z\right)}{\partial \eta}\right|^2 d\eta dz +$$

$$+2\theta^2\sum_{p=1}^{Z}\sum_{j=1}^{M-1}\int_{z_{p-1}}^{z_p}\int_{x_j-h/2}^{x_j+h/2}\left|\frac{\partial y_0\left(\xi,\chi\right)}{\partial \chi}\right|^2 d\xi d\chi.$$

From this inequality we obtain the validity of the following one

$$\left(J_2\right)^2 \le 2h^2\left\|\frac{\partial y_0}{\partial x}\right\|_{L_2(\Omega_L)}^2 + 2\theta^2\left\|\frac{\partial y_0}{\partial z}\right\|_{L_2(\Omega_L)}^2.$$

If to consider here condition (9) we get

$$\left(J_2\right)^2 \le A_{28}\left(h^2+\theta^2\right). \tag{114}$$

By the similar way we can establish the validity of the inequalities

$$\left(J_3\right)^2 \le c_{29}\left(\beta_{\tau h}^{\theta}+\theta+\tau+h+\left\|Q_n\left(v\right)-\left[v_n\right]\right\|^2\right), \tag{115}$$

$$\left(J_4\right)^2 \le A_{30}\left(h^2+\tau^2\right). \tag{116}$$

With the help of inequalities (111), (114)-(116) from (104) we have

$$\left|J\left(v\right)-I_n\left(\left[v\right]_n\right)\right| \le A_{31}\left(\sqrt{\beta_{\tau h}^{\theta}}+\sqrt{h}+\sqrt{\tau}+\sqrt{\theta}+\left\|Q_n\left(v\right)-\left[v\right]_n\right\|\right). \tag{117}$$

From this if to denote $\tilde{\beta}_{\tau h} = \sqrt{\beta_{\tau h}^{\theta}} + \sqrt{\theta} + \sqrt{\tau} + \sqrt{h}$, then for $c_{23} = c_{31}$we obtain the statement of the theorem. Theorem 3.1 is proved.
We now prove two auxiliary lemmas.
Lemma 3.1. Let the conditions of Theorem 3.1. be satisfied. Let moreover $Q_n\left(v\right)$ is defined by formulas (43), (44) for any $v \in V$. Then $Q_n\left(v\right) \in V_n$ and the estimate

$$|J(\nu) - I_n(Q_n(\nu))| \le c_{23}\tilde{\beta}_{\tau h}, n=1,2,... \tag{118}$$

is valid.
Proof. Let $v \in V$be any element. Using (44) we have

$$w_{sj} = \frac{1}{h}\int_{x_j - h/2}^{x_j + h/2} v_s\left(x\right) dx,\ s = 0,1,\ \ j = \overline{1, M-1}.$$

Using this we can write

$$|w_{sj}| \le \frac{1}{h}\int_{x_j - h/2}^{x_j + h/2} |v_s\left(x\right)| \, dx = b_s,\ s = 0,1,\ w_{1j} \ge 0,\ j = \overline{1, M-1}.$$

From this and from the structure of the set V_n we obtain that $[w]_n = Q_n\left(v\right)$ belongs to the set V_n. Then taking $Q_n\left(v\right)$ instead of the element a $[v]_n \in V_n$ and do as in the proof of Theorem 3.1. the arrive at the validity of the statement of the lemma.
Lemma 3.2. Let the conditions of Theorem 3.1. be satisfied. Let additionally the operator P_n is defined as

$$P_n\left([\nu]_n\right) = \tilde{\nu}(x) = \left(\tilde{\nu}_0(x), \tilde{\nu}_1(x)\right) = \left(\nu_{0j}, \nu_{1j}\right),$$

$$x_j - h/2 \le x \le x_j + h/2,\ j = \overline{1, M-1},\ \forall\, [v]_n \in V_n. \qquad (119)$$

Then $\tilde{\nu} = P_n\left([\nu]_n\right) \in V$ and is valid

$$\left|J\left(P_n\left([\nu]_n\right)\right) - I_n\left([\nu]_n\right)\right| \le c_{23}\tilde{\beta}_{\tau h}, n = 1,2,..., n = 1,2,.... \qquad (120)$$

Proof. Let $[v]_n \in V_n$ is an arbitrary element. By the help of formula (119) we can write

$$|\tilde{\nu}_s(x)| = |\nu_{sj}| \le b_s, s = 0,1, \tilde{\nu}_1(x) = \nu_{1j} \ge 0, x_j - h/2 \le x \le x_j + h/2,\ j = \overline{1, M-1}.$$

From this relation, we obtain that piecewise-constant completions of elements $[v]_n$define the element $\tilde{\nu}(x) = P_n\left([\nu]_n\right), \forall x \in [0,l]$ and this element belongs to the set V. Then taking this element instead of the element $v \in V$ similarly to the proof of Theorem 3.1, we obtain

$$\left|J\left(P_n\left([\nu]_n\right)\right) - I_n\left([\nu]_n\right)\right| \le c_{21}\left(\tilde{\beta}_{\tau h} + \|Q_n(\tilde{\nu}) - [\nu]_n\|\right), n = 1,2,... \qquad (121)$$

Using formulas (44) and (119) one may easily proof that

$$\|Q_n(\tilde{\nu}) - [\nu]_n\| = 0.$$

Taking this into account, from (121) we obtain the statement of the lemma. Lemma 3.2. is proved.
Finally, we prove the following theorem on the convergence of difference approximations with respect to the functional.
Theorem 3.2. Let the conditions of Lemma 3.1 and Lemma 3.2. be satisfied and moreover $v^* \in V$ be a solution to identification problem (1)-(5) and $[v]_n^* \in V_n$ be a solution to

problem (11)-(15) i.e. $J_* = J(v^*) = \inf_{v \in V} J(v)$, $I_{n*} = I([v]_n^*) = \inf_{[v]_n \in V_n} I([v]_n)$. Then problem (11)-(15) approximates problem (1)-(5)

$$\lim_{n \to \infty} I_{n*} = J_* \tag{122}$$

and the estimation

$$|J_* - I_*| \leq c_{10} \tilde{\beta}_{\tau h}, n = 1, 2, ... \tag{123}$$

is valid

Proof. To proof the theorem we use the methodology given in [20]. Let $v^* \in V$and $[v]_n^* \in V_n$ are solutions to problems (1)-(5) and (11)-(15), correspondingly. The using Lemma 3.1. we get

$$|I_{n^*} \leq I_n(Q_n(\nu^*))| \leq c_{23} \tilde{\beta}_{\tau h} + J(\nu^*) = J_* + c_{23} \tilde{\beta}_{\tau h}, n = 1, 2, ...$$

This implies the validity of the inequality

$$J_* - I_* \geq -c_{23} \tilde{\beta}_{\tau h}, n = 1, 2, ... \tag{124}$$

From the other hand due to Lemma 3.2. we can write

$$J_* \leq J(P_n([\nu]_n{}^*)) \leq I_n([\nu]_n{}^*) + c_{23} \tilde{\beta}_{\tau h} = I_{n^*} + c_{23} \tilde{\beta}_{\tau h}, n = 1, 2, ...$$

Then we get

$$J_* - I_* \leq c_{21} \tilde{\beta}_{\tau h}, n = 1, 2, ... \tag{125}$$

Inequalities (124) and (125) lead us the estimate (123). If to use the denotations $\theta = \theta_n$, $\tau = \tau_n$, $h = h_n$ and consider the conditions at$\theta_n \to 0$, $\tau_n \to 0$, $h_n \to 0$ at $n \to \infty$, then from the condition $\tilde{\beta}^{\theta_n}_{\tau_n h_n} \to 0$ at $n \to \infty$ and inequality (123) we obtain the validity of the limit relation (122). Theorem 3.2 is proved.

Note 3.1. The similar result is obtained for identification problem on a minimization of functional (1) on the set V subject to (2)-(4) and boundary conditions

$$\frac{\partial \psi(0, t, z)}{\partial x} = \frac{\partial \psi(l, t, z)}{\partial x} = 0, \ (t, z) \in Q. \tag{126}$$

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