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K-Bessel, K-Hilbert Systems and the Relationship between them in Non-separable Banach Spaces

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Abstract. This work deals with the analogs of known results for Bessel and Hilbert systems in separable Banach spaces for the case of non-separable Banach spaces. The concept and criterion of uncountable unconditional basis for non-separable Banach spaces are presented. The concepts of K-Bessel and K-Hilbert systems in non-separable Banach spaces with respect to non-separable Banach space of scalars K are given. In case where K is a space of sequences of coefficients of an uncountable unconditional basis, K-Besselness and K-Hilbertness criteria are proved and the relationship between them is studied. The concept of K-basis in non-separable Banach spaces is introduced. The relationship between K-bases and K-Bessel and K-Hilbert systems is established.

Key Words and Phrases: non-separable Banach space, uncountable unconditional basis, K-Besselness and K-Hilbertness, K-basis.

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1. Introduction

The concepts of Bessel and Hilbert sequences in the Hilbert space $L_2(a,b)$ have been introduced by N.K.Bari ([1]) in 1951 for a pair of biorthogonal sequences. Besselness and Hilbertness criteria for sequences have also been proved in [1] and the relationship between them studied. Besides, it was shown in [1] that the complete minimal sequence which is simultaneously Bessel and Hilbert sequence forms a basis with l_2 as a space of coefficients. Such basis is called a Riesz basis. Later, these concepts have been extended to abstract Hilbert spaces. Note that to establish Riesz basis one may use the known theorems of Bari and Paley-Wiener (for more details see, e.g., [2-6]). Riesz bases are the special cases of frames in Hilbert spaces. The concept of frame has been introduced by Duffin and Schaeffer [7] in 1952 in the study of some problems of non-harmonic Fourier series in the context of perturbed exponential systems. Some properties of frames consisting of exponents have been studied in [7]. Also, the concept of abstract frame in separable Hilbert space has been introduced in [7] and some properties of frames from perturbed exponential systems have been extended to this case. Recall that the system $\{\varphi_n\}_{n\in N}$ of nonzero elements of the separable Hilbert space H is called a frame in H, if there exist the constants $A > 0$ and $B > 0$ such that for every $f \in H$

$$A \|f\|_H^2 \leq \sum_{n=1}^{\infty} |(f, \varphi_n)|^2 \leq B \|f\|_H^2 ,$$

where A is a norm in H generated by the scalar product $(\cdot, \cdot)$. Frames have important applications in various fields of natural sciences such as signal and image processing and coding (human speech, satellite images, internal organs radiography, etc.), image recognition, properties of crystal surfaces and nano-objects, data compression, etc. This theory dates back to the 80's of the last century, when the concept of wavelets emerged (see [8-11]). There exist different generalizations of frames and the number of researches in this field is rapidly growing (see [12-14]). Frames in Banach were first treated by K.H. Gröchenig [15] in 1991, who introduced the concepts of atomic decomposition and Banach frame. Note that, dislike Hilbert case, Banach frame is not a representation system. L^p case has been considered by A. Aldroubi, Q. Sun, W.-Sh. Tang in [16], where the concept of p-frame has been introduced and the atomic decomposition with respect to the subspaces of L_p invariant with regard to shift operator has been obtained. This idea has been extended to the general Banach case in [17]. The concept of q-Riesz basis with respect to the Banach space generalizing a Riesz basis has also been introduced in [16]. The analogs of these results for a Banach space of sequences of scalars have been considered in [18-22].

Different versions of Bessel and Hilbert sequences in Banach spaces have been studied in [23-28]. In [27], Bessel and Hilbert sequences have been considered in separable Banach spaces with respect to the Banach space of sequences of scalars. All results obtained in [1] have been extended to this case. This work is a continuation of [29] and is dedicated to the Bessel and Hilbert systems in non-separable Banach spaces. The analogs of the results of [27] for non-separable Banach spaces are obtained. The concept of a pair of K-Bessel and K-Hilbert biorthogonal systems in non-separable Banach space is introduced. K-Besselness and K-Hilbertness criteria are proved in case where the space K is a space of systems of scalars consisting of coefficients of an uncountable unconditional basis ([29, 30]). The relationship between these concepts is established. The concept of K-basis in non-separable case is defined and a criterion for K-basicity is proved.

2. Needful concepts and facts

Throughout this work, X and Y will be non-separable Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. By $L(X, Y)$ we will denote a space of linearly bounded operators from X to Y, $L(M)$ will be a linear span of the set $M \subset X$, $\overline{M}$ will denote a closure of the set M in X. A space conjugate to X will be denoted by X^*, $\{x_\alpha\}_{\alpha \in I} \subset X$ and $\{x_\alpha^*\}_{\alpha \in I} \subset X^*$ will be a pair of biorthogonal systems.

Let I be an uncountable set of indices, I^a be a set of at most countable subsets of I, I_0 be a set of finite subsets of I, and $\{\varphi_\alpha\}_{\alpha \in I}$ be some system in X. Let K be some non-separable Banach space of systems $\{\lambda_\alpha\}_{\alpha \in I}$ of scalars.

The concepts below are the uncountable generalizations of the pair of biorthogonal Bessel and Hilbert sequences in Banach spaces (see [27]).

Definition 2.1 ([29]). *A pair $\{x_\alpha; x_\alpha^*\}$ is called K-Bessel in X if for $\forall x \in X$ the inclusion $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$ holds. In case where the system $\{x_\alpha\}_{\alpha\in I}$ is complete in X and the pair $\{x_\alpha; x_\alpha^*\}$ is K-Bessel in X, then the system $\{x_\alpha\}_{\alpha\in I}$ is called K-Bessel in X.*

Definition 2.2 ([29]). *A pair $\{x_\alpha; x_\alpha^*\}$ is called K-Hilbert in X if for $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ there exists $x \in X$ such that $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. In case where the system $\{x_\alpha\}_{\alpha\in I}$ is complete in X and the pair $\{x_\alpha; x_\alpha^*\}$ is K-Hilbert in X, then the system $\{x_\alpha\}_{\alpha\in I}$ is called K-Hilbert in X.*

As space K, we will consider a space of coefficients with respect to uncountable unconditional basis of some non-separable Banach space. Let's give a definition for an uncountable unconditional basis.

Definition 2.3 ([29, 30]). *A system $\{\varphi_\alpha\}_{\alpha\in I}$ is called an uncountable unconditional basis for X if for $\forall x \in X$ there exists a unique system $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ of scalars such that $I_\lambda = \{\alpha \in I : \lambda_\alpha \neq 0\} \in I^a$ and the series $\sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$ is unconditionally converging in X to x.*

We will need the following criterion for uncountable unconditional basis in non-separable Banach spaces.

Theorem 2.1 ([29]). *Let X be a non-separable Banach space and $\{\varphi_\alpha\}_{\alpha\in I}$ be some system in X. In order for the system $\{\varphi_\alpha\}_{\alpha\in I}$ to be an uncountable unconditional basis for X, it is necessary and sufficient that the following conditions hold:*

1) the system $\{\varphi_\alpha\}_{\alpha\in I}$ is complete in X;

2) the system $\{\varphi_\alpha\}_{\alpha\in I}$ is minimal in X;

$$3)\exists M > 0 : \forall J \in I_0, \forall x \in X \left\|\sum_{\alpha\in J} \varphi_\alpha^*(x)\varphi_\alpha\right\|_X \leq M\,\|x\|_X\,,$$

where $\{\varphi_\alpha^*\}_{\alpha\in I}$ is a system biorthogonal to $\{\varphi_\alpha\}_{\alpha\in I}$.

As in the case of usual basis, Theorem 2.1 has the following direct corollary.

Corollary 2.1. *Let X be a non-separable Banach space, $\{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for X, and $\{\varphi_\alpha^*\}_{\alpha\in I}$ be a system biorthogonal to $\{\varphi_\alpha\}_{\alpha\in I}$. Then $\{\varphi_\alpha^*\}_{\alpha\in I}$ forms an uncountable unconditional basis for $\overline{L\left(\{\varphi_\alpha^*\}_{\alpha\in I}\right)}$.*

3. Uncountable unconditional K-basis

Let the Banach space Y have an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$. Denote by K_φ a space of systems of coefficients with respect to the uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$, i.e. a space of systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ such that $I_\lambda = \{\alpha \in I : \lambda_\alpha \neq 0\} \in I^a$ and the series $\sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$ is unconditionally convergent. The space K_φ is a Banach space with the usual linear operations and the norm

$$\|\lambda\|_{K_\varphi} = \sup_{J\in I_0} \left\|\sum_{\alpha\in J} \lambda_\alpha\varphi_\alpha\right\|. \tag{1}$$

The following K_φ-Besselness criterion is true.

Theorem 3.1. ([29]). *In order for the pair $\{x_\alpha; x^*_\alpha\}$ to be K_φ-Bessel in X, it is necessary and, in case of completeness of $\{x_\alpha\}_{\alpha\in I}$ in X, sufficient that there exist the operator $T \in L(X, Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$.*

Proof. Necessity. Let the pair $\{x_\alpha; x^*_\alpha\}$ be K_φ-Bessel in X. Then $\forall x \in X$ the inclusion $\{x^*_\alpha(x)\}_{\alpha\in I} \in K_\varphi$ holds. Consider the linear operator $T : X \to Y$ defined by the formula

$$T(x) = \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha.$$

Obviously, $\forall\alpha \in I$ we have $Tx_\alpha = \varphi_\alpha$. It remains to show the boundedness of the operator T. For $\forall\omega \in I^a$, consider the operator

$$T_\omega(x) = \sum_{\alpha\in\omega} x^*_\alpha(x)\varphi_\alpha.$$

From the results of [27] it follows that $T_\omega \in L(X, K)$. Taking into account (1), for $\forall x \in X$ we obtain

$$\|T_\omega(x)\|_{K_\varphi} = \left\|\{x^*_\alpha(x)\}_{\alpha\in\omega}\right\|_{K_\varphi} \le \left\|\{x^*_\alpha(x)\}_{\alpha\in I}\right\|_{K_\varphi}.$$

Hence,

$$\sup_{\omega\in I^a} \|T_\omega(x)\|_{K_\varphi} \le \left\|\{x^*_\alpha(x)\}_{\alpha\in I}\right\|_{K_\varphi} < +\infty.$$

Consequently, by Banach-Steinhaus theorem we obtain $\sup\limits_{\omega\in I^a} \|T_\omega\| < +\infty$. As the boundedness of the operator T is equivalent to the condition $\sup\limits_{\omega\in I^a} \|T_\omega\| < +\infty$, the operator T is bounded.

Sufficiency. Let the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X and there exist the operator $T \in L(X, Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Let $\{\varphi^*_\alpha\}_{\alpha\in I}$ be a system biorthogonal to $\{\varphi_\alpha\}_{\alpha\in I}$. Then for $\forall\alpha, \beta \in I$ we obtain

$$\delta_{\alpha\beta} = \varphi^*_\alpha(\varphi_\beta) = \varphi^*_\alpha(Tx_\beta) = T^*\varphi^*_\alpha(x_\beta).$$

Hence, due to the completeness of $\{x_\alpha\}_{\alpha\in I}$, we obtain $T^*\varphi^*_\alpha = x^*_\alpha$. For $\forall x \in X$ we have

$$x^*_\alpha(x) = T^*\varphi^*_\alpha(x) = \varphi^*_\alpha(Tx).$$

Thus, for $\forall x \in X$ we have $\{x^*_\alpha(x)\}_{\alpha\in I} = \{\varphi^*_\alpha(Tx)\}_{\alpha\in I} \in K_\varphi$, i.e. the system $\{x_\alpha\}$ is K_φ-Bessel in X. The theorem is proved.

The following theorem is also true.

Theorem 3.2. ([29]). *Let the system $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for Y. In order for the pair $\{x_\alpha; x^*_\alpha\}$ to be K_φ-Hilbert in X, it is sufficient and, in case of completeness of $\{x^*_\alpha\}_{\alpha\in I}$ in X^*, necessary that there exist the operator $T \in L(Y, X)$ such that $T\varphi_\alpha = x_\alpha$, $\forall\alpha \in I$.*

Proof. Necessity. Let the pair $\{x_\alpha; x^*_\alpha\}$ be K_φ-Hilbert in X and the system $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^*. Consider $\forall y \in Y$ and let $y = \sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$, be its decomposition with respect to the uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ in Y. By the K_φ-Hilbertness of the pair $\{x_\alpha; x_\alpha^*\}$ in X, there exists $x \in X$ such that $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. The uniqueness of such element follows from the completeness of the system $\{x_\alpha^*\}_{\alpha\in I}$ in X^*. Define the linear operator $T : Y \to X$ by the formula $T(y) = x$. Obviously, we have $T\varphi_\alpha = x_\alpha$ for $\forall\alpha \in I$. The operator T is a closed operator. In fact, let $\{y_n\}_{n\in N} \subset Y$ be an arbitrary sequence such that $\{y_n\}_{n\in N}$ converges to $y \in Y$ and the sequence $\{Ty_n\}_{n\in N}$ converges to z as $n \to \infty$. Let $Ty = x$ and $Ty_n = z_n$ for $\forall n \in N$. Taking into account the decompositions $y = \sum_{\alpha\in I} x_\alpha^*(x)\varphi_\alpha$ and $y_n = \sum_{\alpha\in I} x_\alpha^*(z_n)\varphi_\alpha$, for $\forall\alpha \in I$ we obtain

$$|x_\alpha^*(x) - x_\alpha^*(z_n)| = \left|\varphi_\alpha^*\left(\sum_{\beta\in I} x_\beta^*(x)\varphi_\beta - \sum_{\alpha\in I} x_\beta^*(z_n)\varphi_\beta\right)\right| =$$

$$= |\varphi_\alpha^*(y - y_n)| \le \|\varphi_\alpha^*\| \, \|y - y_n\|_Y \,. \tag{2}$$

As $\{y_n\}_{n\in N}$ converges to $y \in Y$, from (2) it follows that $x_\alpha^*(z_n) \to x_\alpha^*(x)$ as $n \to \infty$. On the other hand, by the continuity of the functional x_α^*, the convergence of $\{z_n\}_{n\in N}$ to z implies $x_\alpha^*(z_n) \to x_\alpha^*(z)$ as $n \to \infty$, and, consequently, $x_\alpha^*(x) = x_\alpha^*(z)$, $\forall\alpha \in I$. Then from the completeness of $\{x_\alpha^*\}_{\alpha\in I}$ in X^* it follows that $Ty = x = z$, i.e. the operator T is closed. By the closed graph theorem, the operator T is bounded.

Sufficiency. Suppose there exists $T \in L(Y, X)$ such that $T\varphi_\alpha = x_\alpha$, $\forall\alpha \in I$. For $\forall\alpha, \beta \in I$ we obtain

$$\delta_{\alpha\beta} = x_\alpha^*(x_\beta) = x_\alpha^*(T\varphi_\beta) = T^*x_\alpha^*(\varphi_\beta).$$

It follows that $T^*x_\alpha^* = \varphi_\alpha^*$. Consider $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$ and let $y = \sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$. Denote $T(y) = x$. Then for $\forall\alpha \in I$ we obtain

$$x_\alpha^*(x) = x_\alpha^*(Ty) = T^*x_\alpha^*(y) = \varphi_\alpha^*(y) = \lambda_\alpha,$$

i.e. the pair $\{x_\alpha; x_\alpha^*\}$ is K_φ-Hilbert in X. The theorem is proved.

The theorem below establishes the relationship between K-Besselness and K-Hilbertness of the system.

Theorem 3.3. *Let X be a reflexive space and the system $\{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for Y. The following assertions are true:*

1) if the pair $\{x_\alpha; x_\alpha^*\}$ is K_φ-Hilbert in X and $\{x_\alpha^*\}_{\alpha\in I}$ is complete in X^*, then $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ^*-Bessel in X^*;

2) if the pair $\{x_\alpha; x_\alpha^*\}$ is K_φ-Bessel in X and $\{x_\alpha\}_{\alpha\in I} \subset X$ is complete in X, then $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ^*-Hilbert in X^*,
where K_φ^* is a space conjugate to K_φ.

Proof. 1). Let $\{x_\alpha; x_\alpha^*\}$ be K_φ-Hilbert in X. Then, by Theorem 3.2, there exists the operator $T \in L(X, Y)$ such that $T\varphi_\alpha = x_\alpha$, $\forall\alpha \in I$. It is obvious that $T^*x_\alpha^* = \varphi_\alpha^*$, $\forall\alpha \in I$. From Corollary 2.1 and the completeness of the system $\{x_\alpha^*\}_{\alpha\in I}$ in X^* it follows that $\{\varphi_\alpha^*\}_{\alpha\in I}$ is an uncountable unconditional basis for X^* with the space of coefficients K_φ^*. From the relations $T^*x_\alpha^* = \varphi_\alpha^*$, $\forall\alpha \in I$, and the completeness of $\{x_\alpha^*\}_{\alpha\in I}$ inX^*, by Theorem 3.1, it follows that the system $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ^*-Bessel inX^*.

2) Let the pair $\{x_\alpha; x_\alpha^*\}$ be K_φ-Bessel in X. By Theorem 3.1, there exists the operator $T \in L(X,Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall \alpha \in I$. We have $T^*\varphi_\alpha^* = x_\alpha^*$, $\forall \alpha \in I$. By the condition of theorem, the system $\{x_\alpha\}_{\alpha\in I}$ is complete in X. Consequently, by Theorem 3.2, the system $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ^*-Hilbert in X^*. The theorem is proved.

The following K_φ-Besselness criterion is also true.

Theorem 3.4. *Let the systems $\{x_\alpha\}_{\alpha\in I}$ and $\{x_\alpha^*\}_{\alpha\in I}$ be complete, and the system $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for Y. Then, in order for $\{x_\alpha\}_{\alpha\in I}$ to be K_φ-Bessel in X, it is necessary and sufficient that the linear operator $A_\lambda : K_\varphi \to K_\varphi$ defined by the formula*

$$A_\lambda(\lambda) = \left\{ \sum_{\alpha\in I} x_\beta^*(\varphi_\alpha)\lambda_\alpha \right\}_{\beta\in I}, \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi, \tag{3}$$

be bounded in K_φ.

Proof. Necessity. Let the system $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Bessel in X. By Theorem 3.1, the following linear bounded operator $T : X \to Y$ is defined:

$$Tx = \sum_{\alpha\in I} x_\alpha^*(x)\varphi_\alpha.$$

Consider the operator $F^{-1}TF$, where F is an isomorphism of the spaces K_φ and Y, defined by the formula $F\lambda = \sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$, $\forall \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$. Obviously, for $\forall x \in X$ we have $F^{-1}x = \{\varphi_\alpha^*(x)\}_{\alpha\in I}$, where $\{\varphi_\alpha^*\}_{\alpha\in I}$ is a system conjugate to $\{\varphi_\alpha\}_{\alpha\in I}$. From $Tx_\alpha = \varphi_\alpha$, $\forall \alpha \in I$, it follows that $T^*\varphi_\alpha^* = x_\alpha^*$, $\forall \alpha \in I$. Then

$$TF(\lambda) = T\left(\sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha\right) = \sum_{\alpha\in I} \lambda_\alpha T\varphi_\alpha =$$

$$= \sum_{\alpha\in I} \lambda_\alpha \left(\sum_{\beta\in I} \varphi_\beta^*(T\varphi_\alpha)\varphi_\beta\right) = \sum_{\alpha\in I} \lambda_\alpha \left(\sum_{\beta\in I} T^*\varphi_\beta^*(\varphi_\alpha)\varphi_\beta\right) =$$

$$= \sum_{\alpha\in I} \lambda_\alpha \left(\sum_{\beta\in I} x_\beta^*(\varphi_\alpha)\varphi_\beta\right) = \sum_{\beta\in I} \left(\sum_{\alpha\in I} x_\beta^*(\varphi_\alpha)\lambda_\alpha\right)\varphi_\beta.$$

Thus,

$$F^{-1}TF(\lambda) = \left\{\sum_{\alpha\in I} x_\beta^*(\varphi_\alpha)\lambda_\alpha\right\}_{\beta\in I},$$

i.e. $A_\lambda = F^{-1}TF$, and therefore, $A_\lambda \in L(K_\varphi)$.

Sufficiency. Let the operator A_λ, defined by (3), be bounded in K_φ. Consider the operator $T = FA_\lambda F^{-1}$. Obviously, $T \in L(X,Y)$. $\forall \gamma \in I$ we obtain

$$Tx_\gamma = FA_\lambda F^{-1}x_\gamma = FA_\lambda(\{\varphi_\alpha^*(x_\gamma)\}_{\alpha\in I}) = F\left(\left\{\sum_{\alpha\in I} x_\beta^*(\varphi_\alpha)\varphi_\alpha^*(x_\gamma)\right\}_{\beta\in I}\right) =$$

$$= F\left(\left\{x^*_\beta\left(\sum_{\alpha\in I}\varphi^*_\alpha(x_\gamma)\varphi_\alpha\right)\right\}_{\beta\in I}\right) = F(\{x^*_\beta(x_\gamma)\}_{\beta\in I}) = F(\{\delta_{\beta\gamma}\}_{\beta\in I}) = \varphi_\gamma.$$

Then, by Theorem 3.1, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel in X. The theorem is proved.

Similar theorem is true for K_φ-Hilbert system.

Theorem 3.5. *Let X be a reflexive Banach space, the systems $\{x_\alpha\}_{\alpha\in I}$ and $\{x^*_\alpha\}_{\alpha\in I}$ be complete, and the system $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for Y. Then, in order for the system $\{x_\alpha\}_{\alpha\in I}$ to be K_φ-Hilbert in X, it is necessary and sufficient that the operator $A_\lambda : K^*_\varphi \to K^*_\varphi$, defined by*

$$A_\lambda(\lambda) = \left\{\sum_{\alpha\in I}\varphi^*_\alpha(x_\beta)\lambda_\alpha\right\}_{\beta\in I}, \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K^*_\varphi, \tag{4}$$

be bounded in K^*_φ.

Proof. Necessity. Let $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Hilbert in X. By Theorem 3.3, the system $\{x^*_\alpha\}_{\alpha\in I}$ is K^*_φ-Bessel in X^*. Then, by Theorem 3.4, the linear operator $A_\lambda : K^*_\varphi \to K^*_\varphi$: $A_\lambda(\lambda) = \left\{\sum_{\alpha\in I}\varphi^*_\alpha(x_\beta)\lambda_\alpha\right\}_{\beta\in I}$ is bounded in K^*_φ.

Sufficiency. Let the operator $A_\lambda : K^*_\varphi \to K^*_\varphi$ defined by the formula (4) be bounded in K^*_φ. By Theorem 3.4, the system $\{x^*_\alpha\}_{\alpha\in I}$ is K^*_φ-Bessel in X^*, and, consequently, by Theorem 3.3, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Hilbert in X. The theorem is proved.

The next theorem establishes the relationship between K-Besselness of the system and the existence of the uncountable unconditional basis with the space of coefficients K.

Theorem 3.6. *Let X, Y be Banach spaces, and the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X. In order for Y to have an uncountable unconditional basis $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ such that $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel in X, it is necessary and sufficient that there exist the operators $T \in L(X, Y)$ and $A : Y \to Y^*$ with $D_A = L(\{Tx_\alpha\}_{\alpha\in I})$ satisfying the following conditions:*

$$1) Ker T^* = \{0\}, T^*ATx_\alpha = x^*_\alpha, \forall\alpha \in I;$$

$$2) \exists M > 0 : \left\|\sum_{\alpha\in J}(ATx_\alpha)(\varphi)Tx_\alpha\right\|_Y \le M\|\varphi\|_Y, \forall J \in I_0.$$

Proof. Necessity. Let Y have an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ with the space of sequences of coefficients K_φ and $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Bessel in X. By Theorem 3.1, there exists the operator $T \in L(X, Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Obviously, $\overline{ImT} = Y$. Consequently, $KerT^* = \{0\}$. In fact, if $T^*\varphi^* = 0$, then $\forall x \in X$ we obtain

$$0 = T^*\varphi^*(x) = \varphi^*(Tx).$$

Hence, from $\overline{ImT} = Y$ and the continuity of φ^* it follows that $\varphi^*(y) = 0$ for $\forall y \in Y$, i.e. $\varphi^* = 0$. Define the linear operator $A : Y \to Y^*$ on $L(\{Tx_\alpha\}_{\alpha\in I})$ by the formula $ATx_\alpha = \varphi^*_\alpha$. Then $A\varphi_\alpha = \varphi^*_\alpha$ and $T^*ATx_\alpha = T^*\varphi^*_\alpha = x^*_\alpha$. Further, by Theorem 2.1,

$$\exists M > 0, \forall J \in I_0 \left\|\sum_{\alpha\in J}\varphi^*_\alpha(\varphi)\varphi_\alpha\right\|_Y \le M\|\varphi\|_Y. \tag{5}$$

Using (5), we obtain

$$\left\| \sum_{\alpha\in J} (ATx_\alpha)(\varphi) Tx_\alpha \right\|_Y = \left\| \sum_{\alpha\in J} \varphi_\alpha^*(\varphi)\varphi_\alpha \right\| \leq M \left\|\varphi\right\|_Y .$$

Sufficiency. Let there exist the operators $T \in L(X,Y)$ and $A : Y \to Y^*$ with $D_A = L(\{Tx_\alpha\}_{\alpha\in I})$, satisfying the conditions 1) and 2). Put $Tx_\alpha = \varphi_\alpha$ and $A\varphi_\alpha = \varphi_\alpha^*$. From $KerT^* = \{0\}$ it follows that the system $\{\varphi_\alpha\}_{\alpha\in I}$ is complete in X. In fact, if otherwise, then, by the corollary of the Hahn-Banach theorem, there exists a nonzero functional $\varphi^* \in Y^*$ such that $\varphi^*(Tx_\alpha) = 0$, $\forall\alpha \in I$. Therefore, $T^*\varphi^*(x_\alpha) = 0$, $\forall\alpha \in I$, and from the completeness of $\{x_\alpha\}_{\alpha\in I}$ in X it follows $T^*\varphi^* = 0$, which contradicts $KerT^* = \{0\}$. Further, we have

$$\varphi_\alpha^*(\varphi_\beta) = A\varphi_\alpha(Tx_\beta) = T^*A\varphi_\alpha(x_\beta) = x_\alpha^*(x_\beta) = \delta_{\alpha\beta},$$

i.e. the systems $\{\varphi_\alpha\}_{\alpha\in I}$ are $\{\varphi_\alpha^*\}_{\alpha\in I}$ biorthogonal. By the condition 2), for $\forall J \in I_0$ and $\forall\varphi \in Y$ we obtain

$$\left\| \sum_{\alpha\in J} \varphi_\alpha^*(\varphi)\varphi_\alpha \right\|_Y = \left\| \sum_{\alpha\in J} (ATx_\alpha)(\varphi) Tx_\alpha \right\|_Y \leq M \left\|\varphi\right\|_Y .$$

Consequently, by Theorem 2.1, the system $\{\varphi_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis for Y. Let K_φ be a space generated by the system $\{\varphi_\alpha\}_{\alpha\in I}$. As $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$, by Theorem 3.1, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel in X. The theorem is proved.

In case of K-Hilbertness of the system, the following theorem holds.

Theorem 3.7. *Let X, Y be reflexive Banach spaces, and the systems $\{x_\alpha\}_{\alpha\in I}$ and $\{x_\alpha^*\}_{\alpha\in I}$ be complete. Then, in order for Y to have an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ such that $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Hilbert in X, it is necessary and sufficient that there exist the operators $T \in L(X^*,Y^*)$ and $A : Y^* \to Y$ with $D_A = L(\{Tx_\alpha^*\}_{\alpha\in I})$ satisfying the following conditions:*

$$1) KerT^* = \{0\}, T^*ATx_\alpha^* = x_\alpha, \forall\alpha \in I;$$

$$2) \exists M > 0 : \left\| \sum_{\alpha\in J} \varphi^*(ATx_\alpha^*)Tx_\alpha^* \right\|_{Y^*} \leq M \left\|\varphi^*\right\|_{Y^*}, \forall J \in I_0, \forall\varphi^* \in Y^*.$$

Proof. Necessity. Let $\{\varphi_\alpha\}_{\alpha\in I}$ be an uncountable unconditional basis for Y and $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Hilbert in X. By Theorem 3.3, the pair $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ^*-Bessel in X^*. Consequently, by Theorem 3.6, there exist the operators whish satisfy the required conditions.

Sufficiency. Let there exist the operators which satisfy the conditions 1) and 2). By Theorem 3.6, there exists an uncountable unconditional basis $\{\varphi_\alpha^*\}_{\alpha\in I}$ for the space Y^*, and $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ^*-Bessel in X^*. Then, by Theorem 3.3, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Hilbert in X. The theorem is proved.

The concept below is an analog of the generalization of Riesz basis in Banach spaces for the case of non-separable Banach spaces.

Definition 3.1. *Let $\{\varphi_\alpha\}_{\alpha\in I}$ be an uncountable unconditional basis for Y. Uncountable unconditional basis $\{x_\alpha\}_{\alpha\in I} \subset X$ is called an uncountable unconditional K_φ-basis for X, if there exists an isomorphism $T : X \to Y$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$.*

The next theorem establishes a criterion for uncountable unconditional K_φ-basis.

Theorem 3.8. *Let $\{x_\alpha\}_{\alpha\in I} \subset X$ and $\{x_\alpha^*\}_{\alpha\in I}$ be a pair of complete biorthogonal systems, and $\{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for Y. In order for the system $\{x_\alpha\}_{\alpha\in I}$ to form an uncountable unconditional K_φ-basis for X, it is necessary and sufficient that $\{x_\alpha\}_{\alpha\in I}$ be simultaneously K_φ-Bessel and K_φ-Hilbert in X.*

Proof. Necessiy. Let the system $\{x_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for X and the operator $T : X \to Y$ perform an isomorphism such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. From Theorems 3.1 and 3.2 it directly follows that $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel and K_φ-Hilbert in X.

Sufficiency. Let the system $\{x_\alpha\}_{\alpha\in I}$ be simultaneously K_φ-Bessel and K_φ-Hilbert in X. Then, by Theorems 3.1 and 3.2, there exist the operators $T \in L(X,Y)$ and $S \in L(Y,X)$ such that for $\forall\alpha \in I$ we have $Tx_\alpha = \varphi_\alpha$ and $S\varphi_\alpha = x_\alpha$. From these equalities we obtain $STx_\alpha = x_\alpha$ and $TS\varphi_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Consequently, due to the completeness of the systems $\{x_\alpha\}_{\alpha\in I}$ and $\{\varphi_\alpha\}_{\alpha\in I}$, we have $ST = I_X$, $TS = I_Y$. Therefore, the operator T is boundedly invertible and $T^{-1} = S$. Thus, from the equality $x_\alpha = T^{-1}\varphi_\alpha$ it follows that the system $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis with the space of sequences of coefficients of K_φ. The theorem is proved.

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