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On the existence of a solution to the inverse problem of finding the right side of parabolic equation in a domain with a moving

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Abstract. The paper considers to investigate corrects of the inverse problem for finding the unknown right side, which depends on the time variable. It is considered the Neumann mixed boundary value problem on domain which the boundary depends on the time variable, an additional condition for finding the unknown function is given in the integral form. A theorem on the existence of a generalized solution of the considered inverse problem is proved.

Key Words and Phrases: Parabolic equation, domain with moving boundaries, inverse problem, the existence of a solution.

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1. Introduction and main result

Let $x = \gamma(t), 0 < a = \gamma(0) \leq \gamma(t) \leq \gamma(T) = b < \infty$, $0 < T = const$, $a, b = const$, $t \in [0, T]$ be the given function, (x, t) be an arbitrary point in the bounded domain $D = (0, \gamma(t) \times (0, T]$. The spaces $C^l(\cdot), C^{l+\alpha}(\cdot), C^{l,l/2}(\cdot), C^{l+\alpha,(1+\alpha)/2}(\cdot)$, $l = 0, 1, 2$, $0 < \alpha < 1$ and the norms in these spaces are defined as in [see 1, 2]:

$$\|p\|_D^{(l)} = \sum_{k=0}^{l} \sup_D \left| \frac{\partial^k p(x,t)}{\partial x^k} \right|, \ \ \|q\|_T^{(l)} = \sum_{k=0}^{l} \sup_{[0,T]} \left| \frac{\partial^k q(x,t)}{\partial t^k} \right|,$$

$$p_t = \frac{\partial p(x,t)}{\partial t}, \ p_x = \frac{\partial p(x,t)}{\partial x}, \ p_{xx} = \frac{\partial^2 p(x,t)}{\partial x^2}, q' = \frac{dq(t)}{dt}.$$

We consider the following inverse problem on determining a pair of functions $\{f(t), u(x,t)\}$:

$$u_t - u_{xx} = f(t)g(x), \ \ (x,t) \in D, \tag{1}$$

$$u(x,0) = \varphi(x), \ \ x \in [0,a], \tag{2}$$

$$u_x(0,t) = \psi_0(t), \ \ u_x(\gamma(t),t) = \psi_1(t), \ \ t \in [0,T], \tag{3}$$

$$\int_0^c u(x,t)dx = h(t), \quad t \in [0,T]\,. \tag{4}$$

where c is a constant such that $0 < c < a$, $g(x), \varphi(x,t), \psi_0(t), \psi_1(t), h(t), \gamma(t)$ are given functions.

"Direct" problems for parabolic eqautions in non-cylinderical domains are considered in the works [2,3,4].

The coefficent inverse problem for a parabolic equations in domains with moving boudaries were studied in the papers [5,6].
Problem (1)-(4) belongs to the class of Hadamardr ill-posed problems. Therefore, this problem should be treated proceeding from the general concepts of the theory of ill-posed problems.

For A.N.Tikhonov correct problems, the exictence of a solution is a pair assumed and justifed by the physical meaning of the problem under consideration.

Desipte the fact that the proof of the exictence of a solution to ill-posed problems requires some additional conditions for the input data, but from the point of clew of constructing algorithms for the exact or appoximate fiuding of a solution of the problem, it is ceratialy of fractical interest.

We take the following assumations for the data of problem (1)-(4):

1.1. $g(x) \in C^{\alpha}\left([0,b]\right)$, $\int_0^c g(x)dx = g_0 \neq 0$, (without breaking the generality, for the prostate we take $g_0 = 1$)

1.2. $\varphi(x) \in C^{2+\alpha}\left[0,a\right]$, $\int_0^c \varphi(x)dx = h(0)$;

1.3. $\psi_0(t), \psi_1(t) \in C^{1+\alpha}\left[0,T\right]; \varphi'(0) = \psi_0(0);\ \varphi'(a) = \psi_1(a)$

1.4. $h(t) \in C^{1+\alpha}\left[0,T\right],\ t \in \left[0,T\right].$

1.5. $\gamma(t) \in C^{1+\alpha}\left[0,T\right],\ \gamma'(t) > 0,\ t \in \left[0,T\right], 0 < a = \gamma\left(0\right) \leq \gamma\left(t\right) \leq \gamma\left(T\right) = b < +\infty.$

Definition 1. The a pair of functions $\{f(t), u(x,t)\}$ is called the classical solutions of problem (1)-(4) :

1. $f(t) \in C^{\alpha}\left[0,T\right]$;
2. $u(x,t) \in C^{2+\alpha,1+\alpha/2}\left(D\right) \bigcap C^{1+\alpha,(1+\alpha)/2}\left(\bar{D}\right)$;
3. the conditiouns (1)-(4) hold for these functions.

Lemma 1. Let conditions 1.1-1.5 be satisfied. If problem (1)-(4) has a classical solution $\{f(t), u(x,t)\}$ in the sense definition 1, then these functions are solution and takes (1), (2), (3) and

$$f(t) = h'\left(t\right) - u_x(c,t) + \psi_0(t), \quad t \in \left[0,T\right], \tag{5}$$

and the solution to problem , (1), (2), (3), (5) is the solution to problem (1)-(4).

Proof. Let is assume that the pair of functions $\{f(t),\ u(x,t)\}$ is a solution to problem (1)-(4) in the sense of definition 1. If we integrate equations (1) in the interval $(0,c)$ with respect to the variable $x$we get:

$$\int_0^c u_t dx - \int_0^c u_{xx} dx = f(t)\int_0^c g\left(x\right)dx.$$

Taking into account the conditions of Lemma1, we obtain

$$h'(t) - u_x(a,t) + u_x(0,t) = f(t)\int_0^c g(x)\,dx, \;\; t \in [0,T],$$

Here the reliability of formula (5) is obvious.
Now suppose that the pair of functions$\{f(t),\ u(x,t)\}$ is the classical solution of problem (1), (2), (3) and (5) we take into account formula.

Denoting $\theta(t) = \int_0^c u(x,t)dx - h(t)$,taking into accoupt formula (5) and the conditions of Lemma 1, from the last inequality we obtain

$$\theta'(t) = 0, \;\; \theta(0) = \int_0^c \varphi(x)dx - h(0) = 0.$$

It is clear that the only solution to the problem for ther unkown $\theta(t) = 0$. From here $\int_0^c u(x,t)dx = h(t),\ t \in [0,T]$ we get.

The lemma 1 is proved.

We construct a function $\tilde{\varphi}(x) \in C^{2+\alpha}[0,b]$, such that it satisfies $\tilde{\varphi}(x) \equiv \varphi(x)$, $x \in [0,a]$.

Under the conditions imposed on the input data, the function

$$F(x,t) = \tilde{\varphi}(x) + \frac{2\gamma(t)x - x^2}{2\gamma(t)}\left[\psi_0(t) - \psi_0(0)\right] + \frac{x^2}{2\gamma(t)}\left[\psi_1(t) - \psi_1(0)\right] \quad (6)$$

satisfies:

$$F(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\bar{D}\right), F(x,0) = \tilde{\varphi}(x),\ F_x(0,t) = \psi_0(t),\ F_x(\gamma(t),t) = \psi_1(t).$$

In relations (1)- (3), taking the substitution $u(x,t) = w(x,t) + F(x,t)$ we obtain:

$$\begin{gathered} w_t - w_{xx} = \Phi(x,t), (x,t) \in D, \\ w(x,0) = 0, \quad x \in [0,a], \\ w_x(0,t) = w_x(\gamma(t),t) = 0, \quad t \in [0,T], \end{gathered} \quad (7)$$

where $\Phi(x,t) = f(t)g(x) + F_{xx}(x,t) - F_t(x,t)$.
We extend the function $\Phi(x,t)$ as follows

$$\tilde{\Phi}(x,t) = \begin{cases} \Phi(0,t), & -\infty < x < 0, \quad 0 \le t \le T, \\ \Phi(x,t), & 0 \le x \le \gamma(t), \qquad 0 \le t \le T, \\ \Phi(\gamma(t),t), & \gamma(t) < x < +\infty, \quad 0 \le t \le T, \end{cases} \quad (8)$$

Then, the extend $\Phi(x,t)$ is bounded conditions and uniformly Holder continuous for each compact subset of D_T.

It can be shown that the potential

$$\begin{gathered} y(x,t) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} G(x-\xi, t-\tau)\,\tilde{\Phi}(\xi,\tau)\,d\xi d\tau = \\ = \int\limits_0^t \int\limits_{-\infty}^{+\infty} G(x-\xi,t-\tau)\left[f(\tau)\tilde{g}(\xi) + \tilde{F}_{\xi\xi}(\xi,\tau) - \tilde{F}_\tau(\xi,\tau)\right] d\xi d\tau, \end{gathered} \quad (9)$$

(where the function $G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$, $t > 0$ is the fundamental solution of equation $y_t - y_{xx} = 0$) has the folowing properties [2, capter 19]:

2.1. $y(x,t) \in C^{2,1}\left\{(-\infty,+\infty)\times(0,T]\right\} \bigcap C^{1,0}\left\{(-\infty,+\infty)\times[0,T]\right\};$

2.2 $y_t - y_{xx} = \tilde{\Phi}(x,t),\ (x,t) \in (-\infty,+\infty)\times(0,T]$

2.3 $|y(x,t)| \le m_1 \left\|\tilde{\Phi}\right\|_D^{(0)} \cdot t,\ \ (x,t) \in (-\infty,+\infty)\times[0,T];$

2.4 $|y_x(x,t)| \le m_2 \left\|\tilde{\Phi}\right\|_D^{(0)} \cdot t^{1/2},\ \ (x,t) \in (-\infty,+\infty)\times[0,T].$

Solution of the problem of determining $w(x,t)$ from relations (7) in the from

$$w(x,t) = z(x,t) + y(x,t)$$

where $y(x,t)$ is a function (9), $z(x,t)$ is the solution to the following problem

$$z_t - z_{xx} = 0,\ \ (x,t) \in D,$$

$$z(x,0) = 0,\ \ x \in [0,a], \tag{10}$$

$$z_x(0,t) = -y_x(0,t),\ \ v_x(\gamma((t),t)) = -y_x(\gamma((t),t)),\ \ t \in [0,T].$$

The solution to problem (10) can be represented as [2, capter 14]:

$$z(x,t) = -2\int_0^t G(x,t-\tau)\rho_1(\tau)\,d\tau + 2\int_0^t G(x-\gamma(\tau),t-\tau)\rho_2(\tau)\,d\tau, \tag{11}$$

where $\rho_1(t)$ and $\rho_2(t)$ are the solution of the systems of integral equations

$$\begin{aligned} -y_x(0,t) &= \rho_1(t) + 2\int_0^t G_x(-\gamma(\tau),t-\tau)\rho_2(\tau)\,d\tau, \\ -y_x(\gamma(t),t) &= \rho_2(t) - 2\int_0^t G_x(\gamma(t)-\gamma(\tau),t-\tau)\rho_1(\tau)\,d\tau. \end{aligned} \tag{12}$$

Note that under the accepted conditions for the initial data $\rho_i(t) \in C[0,T]$, $i = 1,2$. Given the above, you can write

$$u(x,t) = F(x,t) + y(x,t) - 2\int_0^t G(x,t-\tau)\rho_1(\tau)d\tau + 2\int_0^t G(x-\gamma(\tau),t-\tau)\rho_2(\tau)d\tau, \tag{13}$$

Thus, the inverse problem of determining $\{f(t),u(x,t)\}$ from (1)-(3),(5) is reduced
To a system of integral equations (13),(5).

Definition 2. $\{f(t),u(x,t)\}$functions called a generalized solution of the inverse problem (1)-(4) if:

1) $f(t) \in C[0,T];$
2) $u(x,t) \in C^{1,0}\left(\bar{D}\right);$
3) these functions satisfy the suystem of integral equations (13),(5).

Theorem 1. Let the initial data of problem (1)-(4) satisfy conditions 1.1-1.5 .

Then there exists a $T_1 \in (0,T]$ such that in the region $\overline{D}_1 = [0,\gamma(t)]\times[0,T_1]$ the inverse problem has a solution in the sense of the definition 2.

Proof. First we show that for eact given $f(t) \in C([0,T])$ the function $u(x,t)$found from (13) belongs to $C^{1,0}\left(\overline{D}\right)$.
Indeed, under the conditions of Theorem 1, the function $u(x,t)$ from (13) and the function $u_x(x,t)$from

$$u_x(x,t) = F_x(x,t) + y_x(x,t) - 2\int_0^t G_x(x,t-\tau)\rho_1(\tau)d\tau + \\ +2\int_0^t G_x(x-\gamma(\tau),t-\tau)\rho_2(\tau)d\tau. \tag{14}$$

are continuous is a $\overline{D}$ for each giver $f(t) \in C([0,T])$.
Now let us show the existence of the function$f(t) \in C([0,T])$. Denote $Q = C([0,T])$. Equation (5) will be written in operator from:

$$M[f(t)] = f(t), \;\; M : Q \to Q$$

$$M[f(t)] = h'(t) + \psi_0(t) - F_x(c,t) - \\ -\int_0^t \int_{-\infty}^{+\infty} G_x(c-\xi,t-\tau)\left[f(\tau)\tilde{g}(\xi) + F_{\xi\xi}(\xi,\tau) - F_\tau(\xi,\tau)\right]d\xi d\tau - \\ -2\int_0^t G_x(c,t-\tau)\rho_1(\tau)d\tau + 2\int_0^t G_x(c-\gamma(\tau),t-\tau)\rho_2(\tau)d\tau. \tag{15}$$

Let $Q' = \{f|f(t) \in C[0,T],\ |f(t)| \le f_0,\ t \in [0,T]$, where f_0 is a constant number$\}$ It is clear that $M[Q'] \subset Q$.

We show the uniform boundedness and equicontinuty of the set $M[Q']$.
First, we note inequalities that will be useful to us later [1,2].

$$G(x,t-\tau), G(x-\gamma(\tau),t-\tau) \le m_3(t-\tau)^{-1/2}, \tag{16}$$

$$\left|G_x(c,t-\tau)\right|, \left|G_x(c-\gamma(\tau),t-\tau)\right| \le m_4(t-\tau)^{-1/2}, \tag{17}$$

$$|G_x(x-\gamma(t),t_1-\tau) - G_x(x-\gamma(t),t_2-\tau)| \le \\ m_5(t_1-t_2)^{(1+\alpha)/2}(t_2-\tau)^{-3/2}\exp\left(-\frac{(x-\xi)^2}{4(t_2-\tau)}\right), \tag{18}$$

From properties 2.3 and 2.4 for the functions $y(x,t)$and $y_x(x,t)$ we have:

$$|y(x,t)| \le m_1 \left\|\tilde{\Phi}\right\|_D^{(0)} t \le m_1\left[f_0 + \|F\|_D^{(2,1)}\right]t, (x,t) \in (-\infty,+\infty)\times[0,T], \tag{19}$$

$$|y_x(x,t)| \le m_2 \left\|\tilde{\Phi}\right\|_D^{(0)} t^{1/2} \le m_2\left[f_0 + \|F\|_D^{(2,1)}\right]t, (x,t) \in (-\infty,+\infty)\times[0,T], \tag{20}$$

Taking into account inequalities (16) and (20), from (12) we have

$$|\rho_1(t)| \le m_6 \left[f_0 + \|F\|_D^{(2,1)} \right] T^{1/2} + m_7 \|\rho_2\|_T^{(0)} T^{1/2}$$

$$|\rho_2(t)| \le m_8 \left[f_0 + \|F\|_D^{(2,1)} \right] T^{1/2} + m_9 \|\rho_1\|_T^{(0)} T^{1/2}$$

The last inequalities are true for every $t \in [0, T]$. Hence, they must also hold for the maximum values of the left-hand sides

$$\|\rho_1\|_T^{(0)} \le m_6 \left[f_0 + \|F\|_D^{(2,1)} \right] T^{1/2} + m_7 \|\rho_2\|_T^{(0)} T^{1/2}$$

$$\|\rho_2\|_T^{(0)} \le m_8 \left[f_0 + \|F\|_D^{(2,1)} \right] T^{1/2} + m_9 \|\rho_1\|_T^{(0)} T^{1/2}.$$

Summing up the last inequalities we obtain

$$\|\rho_1\|_T^{(0)} + \|\rho_2\|_T^{(0)} \le m_{10} \left[f_0 + \|F\|_D^{(2,1)} \right] T^{1/2} + m_{11} \left[\|\rho_1\|_T^{(0)} + \|\rho_2\|_T^{(0)} \right] T^{1/2}.$$

Let say $T_1 \in (0, T]$ is a number such that $m_{11} T_1^{1/2} < 1$.

Then from the last inequality we have

$$\|\rho_1\|_T^{(0)} + \|\rho_2\|_T^{(0)} \le m_{12} \left[f_0 + \|F\|_D^{(2,1)} \right] {T_1}^{1/2}, \tag{21}$$

Taking in (14) $x = c$ and taking into account the inequalities (17),(20),(21) we get

$$|u_x(c,t)| \le m_{13} \left[f_0 + \|F\|_D^{(2,1)} \right] {T_1}^{1/2}. \tag{22}$$

Thus, for any $f(t) \in Q'$

$$M[f(t)] = |h'(t)| + |\psi_0(t)| + m_{13} \left[f_0 + \|F\|_D^{(2,1)} \right] {T_1}^{1/2},$$

that, is the set $M[Q']$ is uniformly bounded.

Now we will show the equcontinuity of the set $M[Q']$: we write the formula (15) for different values of $t_1, t_2 \in (0, T)(t_1 > t_2$ for clarity) and estimate their difference

$$|M[f(t_1) - M[f(t_2)] \le |h'(t_1) - h'(t_2)| + |\psi_0(t_1) - \psi_0(t_2)| + |u_x(c,t) - u_x(c,t_2)| \le$$

$$\le |h'(t_1) - h'(t_2)| + |\psi_0(t_1) - \psi_0(t_2)| + |F_x(c,t_1) - F_x(c,t_2)| +$$

$$+ |y_x(c,t_1) - y_x(c,t_2)| + 2\int_0^{t_2} |G_x(c, t_1 - \tau) - G_x(c, t_2 - \tau)||\rho_1(\tau)| d\tau +$$

$$+2\int_{t_2}^{t_1} |G_x(c, t_1 - \tau)| \cdot |\rho_1(\tau)| d\tau + 2\int_0^{t_2} |G_x(c - \gamma(\tau), t - \tau) -$$

$$-G_x(c - \gamma(\tau), t_2 - \tau||\rho_2(\tau)| d\tau + 2\int_{t_2}^{t_1} |G_x(c - \gamma(\tau), t - \tau)| \cdot |\rho_2(\tau)| d\tau.$$

Taking into account conditions 1.3 and 1.4 of theorem 1, the properties of functions (6) and (9), we can write

$$|h'(t_1) - h'(t_2)| \le m_{14}(t_1 - t_2),$$

$$|\psi_0(t_1) - \psi_0(t_2)| \le m_{15}(t_1 - t_2),$$

$$|F_x(c, t_1) - F_x(c, t_2)| \le m_{16}(t_1 - t_2),$$

$$|y_x(c, t_1) - y_x(c, t_2)| \le m_{17}(t_1 - t_2).$$

Using inequalities (14) and (18) we obtain

$$\int_{t_2}^{t_1} |G_x(c, t - \tau)| \cdot |\rho_1(\tau)| d\tau \le m_{18} \|\rho_1\|_T^{(0)} (t_1 - t_2)^{1/2},$$

$$\int_{t_2}^{t_1} |G_x(c - \gamma(\tau), t - \tau)| \cdot |\rho_2(\tau)| d\tau \le m_{19} \|\rho_2\|_T^{(0)} (t_1 - t_2)^{1/2},$$

$$\int_{0}^{t_1} |G_x(c, t_1 - \tau) - G_x(c, t_2 - \tau)| \cdot |\rho_1(\tau)| d\tau \le$$

$$\le m_{20} \|\rho_1\|_T^{(0)} (t_1 - t_2)^{(1+\alpha)/2} \int_0^{t_2} (t_2 - \tau)^{-3/2} \exp(-c^2/4(t_2 - \tau)) d\tau$$

$$\int_{0}^{t_2} |G_x(c - \gamma(\tau) t_1 - \tau) - G_x(c - \gamma(\tau) t_2 - \tau)| \cdot |\rho_2(\tau)| d\tau \le$$

$$\le m_{21} \|\rho_2\|_T^{(0)} (t_1 - t_2)^{(1+\alpha)/2} \int_0^{t_2} (t_2 - \tau)^{-3/2} \exp(-(c - \gamma(\tau))^2/(4(t_2 - \tau)) d\tau.$$

Generalizing the above estimates, we conclude that $M[Q']$ is an equicontentious set.

Thus , by frgels theorem, the set $M[Q']$ is compact in Q [7]. In this case, according to the Shounder theorem, the operator $M[f(t)]$ has at least one fixed point, in other words, ther operator equation $M[f(t)] = f(t)$ has solution $f(t) \in Q = C(0, T])$.

Theorem 1 is proved.

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