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Hardy-Littlewood-Stein-Weiss inequality in the generalized Morrey spaces

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Abstract. We consider generalized weighted Morrey spaces $\mathcal{M}^{p,\omega,|\cdot|^{\gamma}}(\mathbb{R}^n)$ and a general function $\omega(x,r)$ defining the Morrey-type norm. We prove the Riesz potential I_α is bounded from the weighted Morrey space $\mathcal{M}^{p,\omega_1,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\omega_2,|\cdot|^{\mu}}(\mathbb{R}^n)$, $1<p<\frac{n}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$, $\alpha p-n<\gamma<n(p-1)$, $\mu=\frac{q\gamma}{p}$.

Key Words and Phrases: Maximal operator, fraction maximal operator, Riesz potential, generalized Morrey space, Hardy-Littlewood-Stein-Weiss type estimate, BMO space.

2020 Mathematics Subject Classifications: Primary 42B20, 42B25, 42B35

1. Introduction

Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [17]), they are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$. In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ play an important role. Later, Morrey spaces found important applications to Navier-Stokes ([28]) and Schrödinger ([20], [22], [23], [26], [27]) equations, elliptic problems with discontinuous coefficients ([3], [5]), and potential theory ([1]). An exposition of the Morrey spaces can be found in the books [6] and [15].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [2], [16], [18], [19]. In [9] there was proved the boundedness of the maximal operator, singular integral operator and the potential operators in generalized variable exponent Morrey spaces.

The results on the boundedness of potential operators and classical Calderon-Zygmund singular operators go back to [1] and [21], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [4].

Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces was proved in H.G.Hardy and J.E.Littlewood [13] in the one-dimensional case and to E.M.Stein and G.Weiss [25] in the case $n>1$.

One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|dy, \;\; 0 \le \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ and is intimately related to the Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 3 we treat Riesz potentials.

The main results are given in Theorems 3, 3, 3, 3.

2. Preliminaries on Lebesgue and Morrey spaces

$L_p \equiv L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions f with finite norm

$$\|f\|_{L_p} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}, \;\; 1 \le p < \infty$$

and also $WL_p(\mathbb{R}^n)$, the weak L_p space defined as the set of all measurable functions f on $\mathbb{R}^n$ with the following finite norms

$$\|f\|_{WL_p} = \sup_{r>0} r \left|\{x \in \mathbb{R}^n \; : \; |f(x)| > r\}\right|^{1/p}, \;\; 1 \le p < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Let $L_{p,\omega}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ with finite norm

$$\|f\|_{L_{p,\omega}} = \|f\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p}, \quad 1 \le p < \infty.$$

and for $p = \infty$ the space $L_{\infty,\omega}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

Definition 1. *The weight function* ω *belongs to the class* $A_p(\mathbb{R}^n)$ *for* $1 \le p < \infty$ *, if the following statement*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \omega(y)dy \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \omega^{-\frac{1}{p-1}}(y)dy \right)^{p-1}$$

is finite and ω *belongs to* $A_1(\mathbb{R}^n)$ *, if there exists a positive constant* C *such that for any* $x \in \mathbb{R}^n$ *and* $r > 0$

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} \omega(y)dy \le C \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{\omega(y)}.$$

The following theorem was proved in [25].

Theorem 1. *Let* $0 < \alpha < n$ *,* $1 < p < \frac{n}{\alpha}$ *,* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *,* $\alpha p - n < \gamma < n(p-1)$ *,* $\mu = \frac{q\gamma}{p}$ *. Then the operator* I^α *is bounded from* $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{q,|\cdot|^\mu}(\mathbb{R}^n)$ *.*

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy,$$

where $f_{B(x,t)}(x) = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

Definition 2. *We define the* $BMO(\mathbb{R}^n)$ *space as the set of all locally integrable functions* f *with finite norm*

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - C| dy,$$

where $f_{B(x,t)}(x) = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

Definition 3. *We define the* $BMO_{p,\omega}(\mathbb{R}^n)$ *(* $1 \le p < \infty$ *) space as the set of all locally integrable functions* f *with finite norm*

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}}.$$

Theorem 2. *[14, Theorem 4.4] Let* $1 \le p < \infty$ *and* ω *be a Lebesgue measurable function. If* $\omega \in A_p(\mathbb{R}^n)$ *, then the norms* $\|\cdot\|_{BMO_{p,\omega}}$ *and* $\|\cdot\|_{BMO}$ *are mutually equivalent.*

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 4. *Let* $1 \le p < \infty$ *. The generalized Morrey space* $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ *and generalized weighted Morrey space* $\mathcal{M}^{p,\omega,|\cdot|^\gamma}(\mathbb{R}^n)$ *are defined by the norms*

$$\|f\|_{\mathcal{M}^{p,\omega}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(x,r)} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{\mathcal{M}^{p,\omega,|\cdot|^\gamma}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(x,r)} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,r))}.$$

Everywhere in the sequel we assume that

$$\inf_{x\in\mathbb{R}^n, r>0} \omega(x,r) > 0 \tag{1}$$

which makes the space $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ nontrivial. Note that when p is constant, in the case of $w(x,r) \equiv const > 0$, we have the space $L^\infty(\mathbb{R}^n)$.

3. Riesz potential operator in the spaces $\mathcal{M}^{p,\omega,|\cdot|^\gamma}(\mathbb{R}^n)$

Theorem 1. *Let* $0 < \alpha < n$ *,* $1 < p < \frac{n}{\alpha}$ *,* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *,* $\alpha p - n < \gamma < n(p-1)$ *,* $\mu = \frac{q\gamma}{p}$ *. Then*

$$\|I^\alpha f\|_{L_{q,|\cdot|^\mu}(B(x,t))} \le C t^{\frac{n}{q}+\frac{\gamma}{p}} \int_t^\infty s^{-\frac{n}{q}-\frac{\gamma}{p}-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} ds, \quad t > 0 \tag{1}$$

where C *does not depend on* f *,* x *and* t*.*

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\mathbb{R}^n \setminus B(x,2t)}(y), \quad t > 0, \tag{2}$$

and have

$$I^\alpha f(x) = I^\alpha f_1(x) + I^\alpha f_2(x).$$

By Theorem 2 we obtain

$$\|I^\alpha f_1\|_{L_{q,|\cdot|^\mu}(B(x,t))} \le \|I^\alpha f_1\|_{L_{q,|\cdot|^\mu}(\mathbb{R}^n)} \le C\|f_1\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} = C\|f\|_{L_{p,|\cdot|^\gamma}(B(x,2t))}.$$

Then

$$\|I^\alpha f_1\|_{L_{q,|\cdot|^\mu}(B(x,t))} \le C\|f\|_{L_{p,|\cdot|^\gamma}(B(x,2t))},$$

where the constant C is independent of f .

Taking into account that

$$\|f\|_{L_{p,|\cdot|^\gamma}(B(x,2t))} \le C t^{\frac{n}{q}+\frac{\gamma}{p}} \int_t^\infty s^{-\frac{n}{q}-\frac{\gamma}{p}-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} ds,$$

we get

$$\|I^\alpha f_1\|_{L_{q,|\cdot|^\mu}(B(x,t))} \le C t^{\frac{n}{q}+\frac{\gamma}{p}} \int_t^\infty s^{-\frac{n}{q}-\frac{\gamma}{p}-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} ds. \tag{3}$$

When $|x-z| \le t$, $|z-y| \ge 2t$, we have $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|I^\alpha f_2(x)| \le \int_{\mathbb{R}^n \setminus B(x,2t)} |z-y|^{\alpha-n} |f(y)| dy \le C \int_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{\alpha-n} |f(y)| dy.$$

We choose $\beta > \frac{n}{q}$ and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{\alpha-n} |f(y)| dy \\
&\quad = \beta \int_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{\alpha-n+\beta} |f(y)| \left(\int_{|x-y|}^\infty s^{-\beta-1} ds \right) dy \\
&\quad = \beta \int_{2t}^\infty s^{-\beta-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \le |x-y| \le s\}} |x-y|^{\alpha-n+\beta} |f(y)| dy \right) ds \\
&\quad \le C \int_{2t}^\infty s^{-\beta-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} \| |x-y|^{\alpha-n+\beta} \|_{L_{p'(\cdot),|\cdot|^{\gamma/(1-p)}}(B(x,s))} ds \\
&\quad \le C \int_{2t}^\infty s^{\alpha-\frac{n}{p}-\frac{\gamma}{p}-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} ds
\end{aligned}$$

Hence

$$\|I^\alpha f_2\|_{L_{q,|\cdot|^\mu}(B(x,t))} \le C \int_{2t}^\infty s^{-\frac{n}{q}-\frac{\gamma}{p}-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} ds \|\chi_{B(x,t)}\|_{L_{q,|\cdot|^\mu}(\mathbb{R}^n)}.$$

Therefore

$$\|I^\alpha f_2\|_{L_{q,|\cdot|^\mu}(B(x,t))} \le C t^{\frac{n}{q}+\frac{\gamma}{p}} \int_{2t}^\infty s^{-\frac{n}{q}-\frac{\gamma}{p}-1} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,s))} ds \tag{4}$$

which together with (3) yields (1). □

Theorem 2. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $\alpha p - n < \gamma < n(p-1)$, $\mu = \frac{q\gamma}{p}$ *and the functions* $\omega_1(x,r)$ *and* $\omega_2(x,r)$ *fulfill the condition*

$$\int_r^\infty t^{\alpha-\frac{\gamma}{p}} \omega_1(x,t) \frac{dt}{t} \le C\, r^{-\frac{\gamma}{p}} \omega_2(x,r). \tag{5}$$

Then the operators M^α *and* I^α *are bounded from* $\mathcal{M}^{p,\omega_1(\cdot),|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\omega_2(\cdot),|\cdot|^\mu}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}^{p,\omega_1,|\cdot|^\gamma}(\mathbb{R}^n)$. As usual, when estimating the norm

$$\|I^\alpha f\|_{\mathcal{M}^{q,\omega_2,|\cdot|^\mu}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, t>0} \frac{t^{-\frac{n}{q}}}{\omega_2(x,t)} \|I^\alpha f \chi_{B(x,t)}\|_{L_{q,|\cdot|^\mu}(\mathbb{R}^n)}. \tag{6}$$

We estimate $\|I^\alpha f\chi_{B(x,t)}\|_{L_{q,|\cdot|^\mu}(\mathbb{R}^n)}$ in (6) by means of Theorem 3 and obtain

$$\begin{aligned}&\|I^\alpha f\|_{\mathcal{M}^{q,\omega_2,|\cdot|^\mu}(\mathbb{R}^n)}\\ &\leq C\sup_{x\in\mathbb{R}^n,\,t>0}\frac{t^{\frac{\gamma}{p}}}{\omega_2(x,t)}\int_t^\infty r^{-\frac{n}{q}-\frac{\gamma}{p}-1}\|f\|_{L_{p,|\cdot|^\gamma}(B(x,r))}dr\\ &\leq C\|f\|_{\mathcal{M}^{p,\omega_1,|\cdot|^\gamma}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\,t>0}\frac{t^{\frac{\gamma}{p}}}{\omega_2(x,t)}\int_t^\infty\frac{r^{\alpha-\frac{\gamma}{p}}\omega_1(x,r)}{r}dr.\end{aligned}$$

It remains to make use of condition (5). □

Theorem 3. *Let* $0<\alpha<n$ *,* $1<p<\frac{n}{\alpha}$ *,* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ *,* γ *,* μ *satisfy condition*

$$0\leq\gamma<\frac{n}{p'},\;\;\mu=\frac{\gamma}{p},\tag{7}$$

and let $\omega(x,t)$ *satisfy condition*

$$r^\alpha\omega(x,r)\leq C,\tag{8}$$

Then the operators M^α *is bounded from* $\mathcal{M}^{p,\omega(\cdot),|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{\infty,|\cdot|^\mu}(\mathbb{R}^n)$ *.*

Proof. Let $x\in\mathbb{R}^n$ and $r>0$. By the Hölder inequality we get successively

$$\begin{aligned}&r^{\alpha-n}\int_{B(x,r)}|f(y)|dy\\ &\leq Cr^{\alpha-n}r^{\frac{n}{p}}\omega(x,r)r^{-\frac{n}{p}}\omega^{-1}(x,r)\|f\|_{L_{p,|\cdot|^\gamma}(B(x,r))}\|\chi_{B(x,r)}\|_{L_{p',|\cdot|^{\gamma/(1-p)}}}\\ &\leq Cr^\alpha\omega(x,r)|x|^{-\frac{\gamma}{p}}\|f\|_{\mathcal{M}^{p,\omega(\cdot),|\cdot|^\gamma}}\leq C|x|^{-\frac{\gamma}{p}}\|f\|_{\mathcal{M}^{p,\omega(\cdot),|\cdot|^\gamma}}.\end{aligned}$$

□

Theorem 4. *Let* $0<\alpha<n$ *,* $1<p<\frac{n}{\alpha}$ *,* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ *,* $\alpha p-n<\gamma<n(p-1)$ *,* γ *,* μ *satisfy condition* (7)*,* $0<\alpha<n$ *and let* $\omega(x,t)$ *satisfy condition* (8)*.*

Then the operator I^α *is bounded from* $\mathcal{M}^{p,\omega(\cdot),|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $BMO(\mathbb{R}^n)$ *.*

Proof. Let $f\in\mathcal{M}^{p,\omega(\cdot),|\cdot|^\gamma}(\mathbb{R}^n)$. In [1] was proved

$$M^\sharp(I^\alpha f)(x)\leq CM^\alpha f(x).\tag{9}$$

The proof Theorem 3, by the Theorem 3 and inequality (9). □

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