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Fundamental Solution of Cauchy and Goursat Type Problems in Lines with Characteristic and Non-Characteristic Pieces for Third Order Hyperbolic Equations

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Abstract. In the paper, the existence and uniqueness of the solution for boundedness and summability type rather weak restrictions on the coefficients of a third order hyperbolic equation of a Cauchy and Goursat type problem were proved and an integral representation of the solution for a non-homogeneous problem was found.

Key Words and Phrases: third order hyperbolic equation, θ -fundamental solution, $W_p^{(2,1)}(G)$ space, Cauchy and Goursat problem.

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1. Introduction

In the paper, for boundedness and summability type rather weak restriction on the coefficients of a third order hyperbolic equation of a Cauchy and Goursat type problem we introduce the notion of θ -fundamental solution that generalizes the notion of the Riemann function in a natural way for the case third-orderer hyperbolic equations with non-smooth coefficients and allows to find integral representation of the solution of a non-homogeneous problem, is introduced. The concept of a θ -fundamental solution was first introduced in the general case in [1] by S.S.Akhiev. Furthermore, sufficient conditions under which this problem is well-defined together with its adjoint system and there exists a unique θ -fundamental solution are found. This problem includes as a particular case the Cauchy problem and the Goursat problem [2,3].

2. Problem statement

On a rectangular domain

$$G = G_1 \times G_2,\ G_1 = (x_0, x_1),\ G_2 = (y_0, y_1)$$

of the plane XOY we consider a continuous line Γ, that is located on $\overline{G}$, and connects the points (x_0, y_1) and (x_1, y_0), and satisfies the conditions with all straightlines of the form

$$x=\alpha=const \text{ and } y=\beta=const\left(\alpha\in\overline{G}_1,\ \beta\in\overline{G}_2\right)$$

the line Γ has a unique intersection point with the possible exception of finite number of straight-lines $x=\alpha_k,\ k=1,2,...,N_1,$ and $y=\beta_k,\ k=1,2,...,N_2,$ $\left(\alpha_k\in\overline{G}_1,\ \beta_k\in\overline{G}_2\right)$, with which Γ can interest along some sections. We call any such a line a monotone line. We will consider monotone lines that can be determined as combination

$$\left\{(x,S_\Gamma(x))\left|\dot{}x\in\overline{G}_{1^{\text{‘}}}\right.\right\}\quad U\quad\left\{(\nu_\Gamma(y),y)\left|\dot{}y\in\overline{G}_{2^{\text{‘}}}\right.\right\}$$

of graphs of some pair of monotonically increasing functions $y=S_\Gamma(x)$ and $x=\nu(y)$ continuous on $\overline{G}_1$ *and* $\overline{G}_2$, with possible exception of finitely many points

$$x=\alpha_k\in\overline{G}_2,\ k=1,2,...,N_1 \text{ and } y=\beta_k\in\overline{G}_2,\ k=1,2,...,N_2,$$

where they can have first kind discontinuities (at discontinuity points their values are considered to be equal to one of the limits at this point from the right or left).

This time the function $x=\nu_\Gamma(y)$ can be considered as a generalized inverse for the function $y=S_\Gamma(x)$ and vice versa. Such monotone lines, in particular, can be built up as combination of finitely many pieces of straightlines parallel to coordinate axes.

Let $W_p^{(2,1)}(G)$, $1\le p\le\infty$ be a space of all $u\in L_p(G)$ having S.L.Sobolev generalized derivatives

$$D_x^iD_y^ju\in L_p(G),\ i=0,1,2;\ j=0,1,$$

where $D_z=\frac{\partial}{\partial z}$ .The space $W_p^{(2,1)}(G)$ is Banach in the norm

$$||u||_{W_p^{(2,1)}(G)}=\sum_{\substack{0\le i\le 2\\0\le j\le 1}}||D_x^iD_y^ju||_{L_p(G)}\ .$$

Let us consider the equation

$$(Lu)(x,y)=D_x^2D_yu(x,y)+a_{2,0}(x,y)D_x^2u(x,y)+$$
$$+a_{1,1}(x,y)D_xD_yu(x,y)+a_{1,0}(x,y)D_xu(x,y)+$$
$$+a_{0,1}(x,y)D_y(x,y)+a_{0,0}(x,y)u(x,y)=\varphi^0(x,y),\ (x,y)\in G, \tag{1}$$

where $\varphi^0\in L_p(G)$

Let $a_{i,j}(x,y)$ $(i=0,1,2;\ j=0,1)$ be measurable on G, $0_{i,0}\in L_p(G)$ $(i=0,1)$ and there exist such functions $a_{i,1}^0\in L_p(G_1),\ i=0,1;$ $a_{2,0}^0\in L_p(G_2)$ that $|0_{i,1}(x,y)|\le a_{i,1}^0(x)$, $|0_{2,0}(x,y)|\le a_{2,0}^0(y)$.
Under the imposed conditions the operator L of the equation (1) acts from $W_p^{(2,1)}(G)$ *to* $L_p(G)$ and is bounded.

For the equation (1) on some monotone line Γ we give the conditions:

$$(L_{2,0}u)(x)\equiv D_x^2u(x,y)\left|_{y=S_\Gamma(x)}\right.=\varphi_{2,0}(x),\ x\in G_1$$

$$(L_{1,1}u)(y) \equiv D_x D_y u(x,y)\left|_{x=\nu_\Gamma(y)}\right. = \varphi_{1,1}(y),\ y \in G_2$$
$$(L_{0,1}u)(y) \equiv D_y u(x,y)|_{x=\nu(y)} = \varphi_{0,1}(y)$$
$$L_{1,0}u \equiv D_x u(x,y)\Big|_{\substack{x=x_0\\y=y_1}} = \varphi_{1,0}\ , L_{0,0}u \equiv u(x,y)\Big|_{\substack{x=x_0\\y=y_1}} = \varphi_{0,0}; \tag{2}$$

where
$$varphi_{2,0} \in L_p(G_1),\ \varphi_{i,1} \in L_p(G_2),\ \varphi_{i,0} \in R,\ (i = 0,1)$$
are the given elements.

The operations $L_{2,0},\ L_{i_1,1},\ L_{i_1,0}\ (i=0,1)$ of taking traces are continuous from $W_p^{(2,1)}(G_1)$ to $L_p(G_1), L_p(G_2), R,$ respectively. Therefore, the operator $\hat{L} = (L_{0,0}, L_{1,0}, L_{0,1}, L_{1,1}, L_{2,0}, L)$ of the problem (1), (2) acts from $W_p^{(2,1)}(G)$ to $E_p^{(2,1)} = R \times R \times L_p(G_2) \times L_p(G_2) \times L_p(G_1) \times\ L_p(G)$ and is bounded.
We can also write the problem (1), (2) in the form of the operator equation

$$\hat{L}u = \overset{\wedge}{\varphi}, \tag{3}$$

where
$$\hat{\varphi} = \left(\varphi_{0,0}, \varphi_{1,0}, \varphi_{0,1}, \varphi_{1,1}, \varphi_{2,0}, \varphi^0\right) \in E_p^{(2,1)}.$$

As the solution of the problem (1), (2) we will call the function $u \in W_p^{(2,1)}(G)$ for which equality (1) is fulfilled almost everywhere on G, the first second and third equalities from (2), almost everywhere on G1 and G2, and also the fourth and fifth equality from (2), in the usual sense.

We call the problem (1), (2) a Cauchy or Goursat type problem. If $\Gamma = \left\{(x,\varphi(x))\left|x \in \overline{G}_1\right.\right\}$, then the problem (1), (2) is equivalent to the classic Cauchy problem, where $\varphi(x)$ is a continuously differentiable function on $\overline{G}_1$ $\varphi^1(x) < 0,\ \varphi(x_0) = y_1,\ \varphi(x_1) = y_0$ (is this case the line Γ is non-characteristic).
If

$$\Gamma = \{(x,y)\,|y = S_\Gamma(x) = y_0,\ x \in [x_0,x_1]\}\, U\, \{(x,y)\,|x = \nu_\Gamma(y) = x_0,\ y \in [y_0,y_1]\}$$

or

$$\Gamma = \{(x,y)\,|y = S_\Gamma(x) = y_1,\ x \in [x_0,x_1]\}\, U\, \{(x,y)\,|x = \nu_\Gamma(y) = x_1, y \in [y_0,y_1]\}$$

the problem (1), (2) is equivalent to the classic form Goursat problem (in this case Γ is a characteristic line). In this sense, problem (1), (2) includes as a particular case the Cauchy and Goursat problems.

If the problem (1), (2) for any $\hat{\varphi} \in E_p^{(2,1)}$ has a unique solution $u \in W_p^{(2,1)}(G)$ and this time
$$\|u\|_{W_p^{(2,1)}(G)} \le M\,\|\hat{\varphi}\|_{E_p^{(2,1)}}\,,$$
where
$$\|\hat{\varphi}\|_{\substack{(2,1)\\p}} = |\varphi_{0,0}|_R + |\varphi_{1,0}|_R + \|\varphi_{0,1}\|_{L_p(G_2)} + \|\varphi_{1,1}\|_{L_p(G_2)} + \|\varphi_{2,0}\|_{L_p(G_1)} + \left\|\varphi^0\right\|_{L_p(G)}.$$

M- is a positive constant independent of $\hat{\varphi}$, we will say that the problem (1), (2) is everywhere well-defined. Note that well-defined solvability of the problem (1), (2) is equivalent to the fact that its operator $\hat{L}$ has a bounded inverse $\hat{L}^{-1}$ determined on $E_p^{(2,1)}$.

3. Constracting an adjoint operator

For the operator $\hat{L}$ of the problem (1), (2) the explicit form of the conjugated operator $\hat{L}^*$ was found. We take some functional $\hat{f} \in E_q^{(2,1)}$, $\frac{1}{p}+\frac{1}{q}=1$ bounded on $E_q^{(2,1)}$.

Then $\hat{f}$ has the form

$$\hat{f}=\left(f_{0,0}, f_{1,0},\ f_{0,1}\left(y\right),\ f_{1,1}\left(y\right), f_{2,0}\left(x\right),\ f\left(x,y\right)\right),$$

where

$$f_{0,0}\in R,\ f_{1,0}\in R,\ f_{0,1}\in L_q\left(G_2\right),\ f_{1,1}\in L_q\left(G_2\right), f_{2,0}\in L_q\left(G_1\right), f\in L_q\left(G\right)$$

and by the definition we have

$$\hat{f}\left(\hat{L}u\right)=u\left(x_0,y_1\right)f_{0,0}+D_xu\left(x_0,y_1\right)f_{1,0}+\int_{G_2}D_yu\left(\nu\left(y\right),y\right)f_{0,1}\left(y\right)dy+$$
$$+\int_{G\ 2}D_xD_yu\left(\nu\left(y\right),y\right)f_{1,1}\left(y\right)dy+\int_{G_1}D_x^2u\left(x,S\left(x\right)\right)f_{2,0}\left(x\right)dx+$$
$$+\iint_G\left(Lu\right)\left(x,y\right)f\left(x,y\right)dxdy \tag{4}$$

Taking into account the expression of the operator L from (1), we reduce the right hand side of the equality (4) to the form

$$\hat{f}\left(\hat{L}u\right)=\left(V_{0,0}\hat{f}\right)u\left(x_1,y_1\right)+\left(V_{1,0}\hat{f}\right)D_xu\left(x_1,y_1\right)+$$
$$+\int_{G_2}\left(V_{0,1}\hat{f}\right)D_yu\left(x_1,y\right)dy+\int_{G_2}\left(V_{1,1}\hat{f}\right)\left(x_1,y\right)D_xD_yu\left(x_1,y\right)dy+$$
$$+\int_{G_1}\left(V_{2,0}\hat{f}\right)D_x^2u\left(x,y_1\right)dx+\iint_G\left(V\hat{f}\right)D_x^2D_yu\left(x,y\right)dxdy, \tag{5}$$

where

$$\left(V\hat{f}\right)\left(\tau,\xi\right)\equiv f\left(\tau,\xi\right)-\theta\left(\xi-S_\Gamma\left(\tau\right)\right)f_{2,0}\left(\tau\right)-\theta\left(\tau-\nu_\Gamma\left(\xi\right)\right)f_{1,1}\left(\xi\right)+$$
$$+\int_{x_0}^{\tau}\theta\left(\tau_1-\nu_\Gamma\left(\xi\right)\right)d\tau_1\cdot f_{0,1}+$$
$$+\iint_G\left[a_{1,0}\left(x,y\right)-a_{0,0}\left(x,y\right)\left(\tau-x\right)\right]\theta\left(\tau-x\right)\theta\left(\xi-y\right)f\left(x,y\right)dxdy-$$

$$-\int_{G_1}\left[a_{1,1}(x,\xi)\,\theta(\tau-x)-a_{0,1}(x,\xi)(\tau-x)\,\theta(\tau-x)\right]f(x,\xi)\,dx-$$

$$-\int_{G_2}a_{2,0}(\tau,y)\,\theta(\xi-y)\,f(\tau,y)\,dy;$$

$$\left(V_{2,0}\hat{f}\right)(\tau)\equiv f_{2,0}(\tau)-f_{1,0}+(\tau-x_0)\cdot f_{0,0}+\int_{G_2}f(\tau,y)\,a_{2,0}(\tau,y)\,dy-$$

$$-\iint_G a_{1,0}(x,y)\,f(x,y)\,\theta(\tau-x)\,dxdy+$$

$$+\iint_G a_{0,0}(x,y)\,f(x,y)\,\theta(\tau-x)(\tau-x)\,dxdy;$$

$$\left(V_{1,1}\hat{f}\right)(\xi)\equiv f_{1,1}(\xi)-(x_1-\nu_\Gamma(\xi))\,f_{0,1}-\iint_G a_{1,0}(x,y)\,f(x,y)\,\theta(\xi-y)\,dydx+$$

$$+\int_{G_1}a_{1,1}(x,\xi)\,f(x,\xi)\,dx-\int_{G_1}a_{0,1}(x,\xi)(x_1-x)\,f(x,\xi)\,dx+$$

$$+\iint_G a_{0,0}(x,y)\,f(x,y)(x_1-x)\,\theta(\xi-y)\,dxdy;$$

$$\left(V_{0,1}\hat{f}\right)(\xi)\equiv f_{0,1}(\xi)+\int_{G_1}a_{0,1}(x,\xi)\,f(x,\xi)\,dx-\iint_G a_{0,0}(x,y)\,f(x,y)\,\theta(\xi-y)\,dxdy$$

$$V_{1,0}\hat{f}\equiv f_{1,0}-(x_1-x_0)\,f_{0,0}+\iint_G a_{1,0}(x,y)\,f(x,y)\,dydx-$$

$$-\iint_G a_{0,0}(x,y)\,f(x,y)(x_1-x)\,dxdy$$

$$V_{0,0}\hat{f}\equiv f_{0,0}+\iint_G a_{0,0}(x,y)\,f(x,y)\,dydx \tag{6}$$

Comparing the equality (4) and (5) and considering the general form of linear bounded functionals on $W_p^{(2,1)}(G)$ (see $[1,2,3]$) we obtain that the operator $\hat{L}$, has an adjoint $\hat{L}^+\to\hat{L}^*$ that acts in $E_q^{(2,1)}$, bounded and is of the form $\hat{L}^*=(V_{0,0},V_{1,0},V_{0,1},V_{1,1},V_{2,0},V)$, where the operators are determined by means of the equality (6). Hence it follows that the conjugate equation

$$\hat{L}^*\hat{f}=\hat{\psi} \tag{7}$$

can be written also in the form of the equivalent system of equations

$$\begin{cases} V_{0,0}\hat{f}=\psi_{0,0};\\ V_{1,0}\hat{f}=\psi_{1,0};\\ \left(V_{0,1}\hat{f}\right)(y)=\psi_{0,1}(y)\,,\;\; y\in G_2;\\ \left(V_{1,1}\hat{f}\right)(y)=\psi_{1,1}(y)\,,\;\; y\in G_2;\\ \left(V_{2,0}\hat{f}\right)(x)=\psi_{2,0}(x)\,,\;\; x\in G_1;\\ \left(V\hat{f}\right)(x,y)=\psi(x,y)\,,\;\; (x,y)\in G, \end{cases} \tag{8}$$

where $\hat{\psi} = (\psi_{0,0}, \psi_{1,0}, \psi_{0,1}, \psi_{1,1}, \psi_{2,0}, \psi) \in E_q^{(2,1)}$ is a given ,

$$\hat{f} = (f_{0,0}, f_{1,0}, f_{0,1}, f_{1,1}, f_{2,0}, f) \in E_q^{(2,1)}$$

are the desired elements.

4. Existence and uniqueness of the solution of problem (1), (2) and its adjoint system

The system (6) by its form at first sight may seem to be complicated, below we will show that this system can be reduced to such an equivalent form in which the equation f is an independent equation. For that, we will study system (8) separately in the domains G^+ and G^- , where

$$G^+ = \{(x,y) \in G | y > S_\Gamma(x), \, x \in G_1\}, \quad G^- = G | G^+$$

Note that if $\hat{f} = (f_{0,0}, f_{1,0}, f_{0,1}, f_{1,1}, f_{2,0}, f)$ is some solution of (8), then from (6) it follows that the components $f_{2,0}, f_{1,1}, f_{0,1}, f_{1,0}, f_{0,0}$ of this solution can be expressed by means of its first component f as follows:

$$\begin{cases} f_{0,0} = \psi_{0,0} - \iint_G a_{0,0}(x,y) f(x,y)\, dxdy, \\ f_{1,0} = \psi_{1,0} + (x_1 - x_0)\, \psi_{0,0} - \\ - \iint_G (x - x_0)\, a_{0,0}(x,y) f(x,y)\, dxdy - \iint_G a_{1,0}(x,y) f(x,y)\, dxdy, \\ f_{0,1}(\xi) = \psi_{0,1} - \int_{G_1} a_{0,1}(x,\xi) f(x,\xi)\, dx + \iint_G a_{0,0}(x,y) f(x,y)\, \theta(\xi - y)\, dxdy, \\ f_{1,1}(\xi) = \psi_{1,1} + (x_1 - \nu_\Gamma(\xi))\, \psi_{0,1} - \int_{G_1} a_{0,1} f(x,\xi)\, (x - \nu_\Gamma(\xi))\, dx + \\ + \iint_G (x_1 - \nu_\Gamma(\xi))\, a_{0,0} f(x,y) \cdot \theta(\xi - y)\, dxdy + \\ + \iint_G a_{1,0} f(x,y)\, \theta(\xi - y)\, dxdy - \int_{G_1} a_{1,1}(x,\xi) f(x,\xi)\, dx, \\ f_{2,0}(\xi) = \psi_{2,0} + \psi_{1,0} + (x_1 - \tau_0)\, \psi_{0,0} + \int_\tau^{x_1} \int_{G_2} a_{0,0}(x,y) f(x,y)\, (\tau - x)\, dxdy - \\ - \int_\tau^{x_1} \int_{G_2} a_{1,0}(x,y) f(x,y)\, \theta(\tau - x)\, dxdy - \int_{G_2} a_{2,0}(\tau,y) f(\tau,y)\, dy\,. \end{cases} \tag{9}$$

If $(\tau, \xi) \in G^+$, then

$$\theta(\xi - S_\Gamma(\tau)) = \theta((\tau - \nu_\Gamma(\xi)) = 1 \;\; u \;\; \int_{x_0}^{\tau} \theta((\tau_1 - \nu_\Gamma(\xi))\, d\tau_1 = \tau - \nu_\Gamma(\xi)\,.$$

Then, using (9), the first equation of the system (8) can be reduced to the form

$$\begin{aligned} \left(A_{(x_1,y_1)} f\right) = f(\tau,\xi) + \int_\tau^{x_1} \int_\xi^{y_1} [a_{1,0}(x,y) + a_{0,0}(x,y)\, (x - \tau)]\, f(x,y)\, dxdy + \\ + \int_\tau^{x_1} [a_{1,1}(x,\xi) + (x - \tau)\, a_{0,1}(x,\xi)]\, f(x,\xi)\, dx + \\ + \int_\xi^{y_1} a_{2,0}(\tau,y) f(\tau,y)\, dy = \psi_+(\tau,\xi) \;\; (x,\xi) \in G^+ \end{aligned} \tag{10}$$

But if $(\tau, \xi) \in G^-$ then $\theta(\xi - S_\Gamma(\tau) = \theta(\tau - \nu_\Gamma(\xi)) = 0$, and then the first equation of the system (8) has the following form:

$$\begin{aligned} \left(A_{(x_0,y_0)} f(\tau,\xi)\right) = f(\tau,\xi) + \int_{x_0}^{\tau} \int_{y_0}^{\xi} [a_{1,0}(x,y) + a_{0,0}(x,y)\, (x - \tau)]\, f(x,y)\, dxdy + \\ + \int_{x_0}^{\tau} [a_{1,1}(x,\xi) + a_{0,1}(x,\xi)\, (x - \tau)]\, f(x,\xi)\, dx + \\ + \int_{y_0}^{\xi} a_{2,0}(\tau,y) f(\tau,y)\, dy = \psi(\tau,\xi)\,, (\tau,\xi) \in G^- \end{aligned} \tag{11}$$

Thus we showed that the solution of the system (8) is equivalent to the solution of the pair of independent Volterra integral equations (10) and (11).

The operators $A(x_1, y_1)\ and\, A(x_0, y_0)$ are bounded operators acting in the spaces $L_q(G^+)$ and $L_q(G^-)$, respectively. Furthermore, $A(x_1, y_1)\ u\, A(x_0, y_0)$ are Volterra operators in negative and positive directions of the variables, $\tau, \xi,$ respectively. Using this we can show that each of these equations (10) and (11) has a unigue solution $f_+ \in L_q(G^+)\ \ and\ \ f_- \in L_q(G^-)$.

This shows that the solution of the system (8) is equivalent to the solution of the pair of independent equations (10) and (11), with unique solutions $f_+\, and\ f_-$. The inverse is also valid in the sense that if $f_+ \in L_q(G^+)\ and\ f_- \in L_q(G^-)$ are the solutions of equations (10) and (11), then the function $f(\tau,\xi) = f_\pm(\tau,\xi), (\tau,\xi) \in G^\pm$ together with the equalities (9) determines some solution $\hat{f} = (f_{0,0}, f_{1,0}, f_{0,1}, f_{1,1}, f_{2,0}, f)$ of the system (8) from $E_q^{(2,1)}$. Hence it follows that the system for any $\hat{\psi} \in E_q^{(2,1)}$ has a unique solution $\hat{f} \in E_q^{(2,1)}$. Using the Banach theorem, we obtain that $\hat{L}^*$ has a bounded inverse $\left(\hat{L}^*\right)^{-1}$ acting in $E_q^{(2,1)}$. Since $\left(\hat{L}^*\right)^{-1}$ is bounded, then $\hat{L}$ has a bounded inverse $\hat{L}^{-1}$ determined in ${}_p^{(2,1)}$.
Thus, we prove the following theorem:

Theorem 1. *The problem* (1) *and* (2) *and its conjugate system* (8)*, unconditionally are everywhere well-defined.*

The θ – fundamental solution of the problem (1), (2) is the function

$$\hat{f}(x,y) = (f_{0,0}(x,y),\, ,\ f_{1,0}(x,y), f_{0,1}(.,x,y), \\ f_{1,1}(.,x,y), f_{2,0}(.,x,y),\ f(.,.,x,y),\,) \in H_q^{(2,1)}$$

That for each fixed point $x \in \bar{G}$ is the solution of the system of equations

$$\begin{cases} \left(V\hat{f}\right)(\tau,\xi) = -(\tau - x)\,\theta(\tau - x)\,\theta(\xi - y)\ , \\ \left(V_{2,0}\hat{f}\right)(\tau) = -(\tau - x)\,\theta(\tau - x)\ , \\ \left(V_{1,1}\hat{f}\right)(\xi) = (x_1 - x)\,\theta(\xi - y) \\ \left(V_{0,1}\hat{f}\right)(\xi) = -\theta(\xi - y)\ , \\ V_{1,0}\hat{f} = -(x_1 - x)\ , \\ V_{0,0}f = 1\,. \end{cases} \tag{12}$$

The system (12) for each $x \in \bar{G}$ is a special case of the system (8) for

$$\begin{cases} \psi(\tau,\xi) = -(\tau - x)\,\theta(\tau - x)\,\theta(\xi - y)\ , \\ \psi_{2,0}(\tau) = -(\tau - x)\,\theta(\tau - x)\ , \\ \psi_{1,1}(\xi) = (x_1 - x)\,\theta(\xi - y) \\ \psi_{0,1}(\xi) = -\theta(\xi - y)\ , \\ \psi_{0,1} = -(x_1 - x)\ , \\ \psi_{0,0} = 1\,. \end{cases} \tag{13}$$

Therefore the existence and uniqueness of the θ – fundamental solution $\hat{f} \in H_q^{(2,1)}$ of problem (1), (2) follows from theorem 1.
Then we prove the following theorem:

5. Existence and uniqueness of θ fundamental solution and representation of the solution

Theorem 2. *The Cauchy and Goursat type problem* (1), (2)*, unconditionally is everywhere well-defined and has a unique θ – fundamental solution $\hat{f}(x,y) \in H_q^{(2,1)}$, and its solution $u \in W_p^{(2,1)}(G)$, $1 \le p \le \infty$, is represented in the form*

$$u(x,y) = \iint_G f(\tau,\xi,\, x,y)\,\varphi(\tau,\xi)\,d\tau d\xi + \int_{G_1} f_{2,0}(\tau,x,y)\,\varphi_{2,0}(\tau)\,d\tau +$$
$$+ \int_{G_2} f_{1,1}(\xi,\, x,y)\,\varphi_{1,1}(\xi)\,d\xi + \int_{G_2} f_{0,1}(\xi,\, x,y)\,\varphi_{0,1}(\xi)\,d\xi +$$
$$+ f_{1,0}(x)\,\varphi_{1,0} + f_{0,0}(x)\,\varphi_{0,0}. \tag{14}$$

Proof. By theorem 1 the operator $\hat{L}^*$ has a bounded inverse on $H_q^{(2,1)}$. Therefore, the operator $\hat{L}$ has a bounded inverse determined in $H_p^{(2,1)}$. This in particular means that the problem (1), (2) for a small $\hat{\varphi} \in H_p$ has a unique solution $u \in W_p^{(2,1)}(G)$ and this time

$$\|u\|_{W_p^{(2,1)}(G)} \le M\,\|\hat{\varphi}\|_{Hp},\ M = const\ .$$

The validity of the equality (14) follows directly from identity (5) allowing for (12) and representation of functions $u \in W_p^{(2,1)}(G)$.

The authors of [4,5,6,7] also dealt with the application of the θ fundamental solution to various problems

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