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Two Fold Expansion Formula For a Non Self Adjoint Boundary Value Problem

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Abstract. We investigate the Sturm-Liouville operator with a nonlinear spectral parameter in boundary condition in the space $L_2(0,\infty)$. Unlike other studies, the condition is non self-adjoint. For the problem, the scattering data is defined, the resolvent operator is constructed and in terms of scattering data the two-fold expansion formula is obtained by using Titchmarsh method.

Key Words and Phrases: Expansion Formula; Eigenfunction; Scattering Data; Resolvent Operator.

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1. Introduction

In this paper, we consider on the half line $[0,\infty)$ the differential equation

$$\ell(y) \equiv -y'' + V(x)y = \rho^2 y, \tag{1}$$

with nonlinear dependence on the spectral parameter in the boundary condition

$$U(y) \equiv y'(0) + \left(\beta_0 + \beta_1\rho + \beta_2\rho^2\right) y(0) = 0. \tag{2}$$

Here ρ is a spectral parameter, $V(x)$ is real valued function, $(1+x)|V(x) \in L_1(0,\infty)$ and $\beta_j (j = 0,1,2)$ are real numbers.

The Sturm-Liouville equations with a nonlinear spectral parameter in the boundary condition arise in various problems of mathematical physics. The application to the heat conduction problem as a special case of this problem is given in [1]. The physical application of expansions problem for differential equation on the half line $[0,\infty)$ with the boundary condition depending on the spectral parameter is considered in the works of T.Regge [2],[3]. The Regge problem on the half line has been studied in [4]- [6]. Spectral analysis for the boundary value problem when the spectral parameter appearing in the boundary condition is examined in [7],[8].

In this paper, we study the spectral analysis of the boundary value problem (1)-(2). By using the Jost solution of equation (1) and Titchmarsh's method in [9],[10], we construct the resolvent operator and we obtain two-fold expansion formula according to the scattering data. For the classical Sturm-Liouville operator, a similar problem has been studied completely (see[9]-[12] and the references therein).

Expansion formulas according to the eigenfuctions for the equation (1) with different types of boundary conditions are obtained in [13]- [16] . The scattering theory for boundary value problem (1)-(2) are studied in [14]. In this case, the boundary value problem (1)-(2) is not selfadjoint and it may have a complex eigenvalue (see [12] - [17]). For this reason, the scattering data of problem (1)-(2) is differently defined.

By $\mathscr{D}_\rho$, let us indicate the set of functions that satisfy the following condition:

1.) The functions $y(x), y'(x)$ are absolute continuous on each the interval $[0, b]$ ($\forall b > 0$),

2.) $\ell(y) \in L_2(0, \infty)$,

3.) (2) is provided for each fixed ρ.

Let L_ρ is operator with domain $\mathscr{D}_\rho = \mathscr{D}(L_\rho)$ such that for $L_\rho y = \ell(y)$. If ρ runs through the set of all point of the ρ-plane , then we obtain a family of non-selfadjoint singular operators L_ρ depending on the parameter ρ (see [18]).

$f(x, \rho)$ is the solution of the Eq. (1) possessing the asymptotics $f(x, \rho) \to e^{i\rho x}$ as $x \to \infty$ and for any ρ, $(Im\rho \geq 0)$ the solution $f(x, \rho)$ can be represented in the form

$$f(x, \rho) = e^{i\rho x} + \int_x^\infty L(x, t) e^{i\rho t} dt. \tag{3}$$

The kernel $L(x, t)$ satisfies the inequality

$$|L(x, t)| \leq \frac{1}{2} \sigma(\frac{x+t}{2}) \exp\left\{ \sigma_1(x) - \sigma_1(\frac{x+t}{2}) \right\},$$

where

$$\sigma(x) = \int_x^\infty |q(t)| dt, \quad \sigma_1(x) = \int_x^\infty \sigma(t) dt.$$

The solution $f(x, \rho)$ is an analytic function of ρ in the upper half plane and is continuous on the real line. The following estimates are valid in the half plane $Im\rho \geq 0$:

$$|f(x, \rho)| \leq exp\left\{ -Im\rho x + \sigma_1(x) \right\},$$

$$\left| f(x, \rho) - e^{i\rho x} \right| \leq \left\{ \sigma_1(x) - \sigma_1(x + \frac{1}{|\rho|}) \right\} exp\left\{ -Im\rho x + \sigma_1(x) \right\},$$

$$\left| f'(x, \rho) - i\rho e^{i\rho x} \right| \leq \sigma(x) exp\left\{ -Im\rho x + \sigma_1(x) \right\}.$$

For real $\rho \neq 0$ the functions $f(x, \rho)$ and $f(x, -\rho)$ form a fundamental solution system of equation (1) and their Wronskian is

$$W\left\{ f(x, \rho), f(x, -\rho) \right\} = f'(x, \rho) f(x, -\rho) - f(x, \rho) f'(x, -\rho) = 2i\rho.$$

Similarly, we denote by $\widehat{f}(x, \rho)$ the solution of equation (1) in the half plane $Im\rho > 0$ possessing the asymptotics;

$$\widehat{f}(x,\rho) = e^{-i\rho x}\left(1+\bar{o}(1)\right), \quad \widehat{f}'(x,\rho) = e^{-i\rho x}\left(i\rho+\bar{o}(1)\right),$$

as $\rho \to \infty$,

$$\widehat{f}(x,\rho) = e^{-i\rho x}\left(1+O\left(\frac{1}{|\rho|}\right)\right), \quad \widehat{f}'(x,\rho) = -i\rho e^{i\rho x}\left(1+O\left(\frac{1}{|\rho|}\right)\right),$$

as $|\rho| \to \infty$. The existence and properties of these solutions are studied in ([12], p.299).

Let $\varphi(x,\rho)$ be the special solution of the equation (1) under the initial-value conditions

$$\varphi(0,\rho) = 1, \quad \varphi'(0,\rho) = -\left(\beta_0+\beta_1\rho+\beta_2\rho^2\right).$$

Analogously to the Lemma 1 in [14] is like that

$$\frac{2i\rho\varphi(x,\rho)}{F(\rho)} = f(x,-\rho) - S(\rho)f(x,\rho),$$

for any real number $\rho \neq 0$. Thus the identity is valid:

$$S(\rho) = \frac{f'(0,-\rho)+\left(\beta_0+\beta_1\rho+\beta_2\rho^2\right)f(0,-\rho)}{f'(0,\rho)+\left(\beta_0+\beta_1\rho+\beta_2\rho^2\right)f(0,\rho)}.$$

Denote

$$F_1(\rho) \equiv f'(0,-\rho)+\left(\beta_0+\beta_1\rho+\beta_2\rho^2\right)f(0,-\rho),$$

$$F(\rho) \equiv f'(0,\rho)+\left(\beta_0+\beta_1\rho+\beta_2\rho^2\right)f(0,\rho).$$

We shall define scattering function $S(\rho)$ for the problem (1)-(2). The scattering function $S(\rho)$ determines the asymptotics for $x \to \infty$ of normalized eigenfunctions of the operator L_ρ.

For the solution $\widehat{f}(x,\rho)$, we have

$$\frac{2i\rho\varphi(xi\rho)}{F(\rho)} = \widehat{f}(x,\rho) - \frac{F(\rho)}{\widehat{F}(\rho)}f(x,\rho),$$

where

$$\widehat{F}(\rho) = \widehat{f}'(0,\rho)+\left(\beta_0+\beta_1\rho+\beta_2\rho^2\right)\widehat{f}(0,\rho).$$

The roots of the equation $F(\rho)=0$ in the half plane $Im\rho>0$ form a finite set of complex numbers and non pure imaginary . The multiplicity m_j of a root ρ_j $(j=1,2,...,n)$ of the equation $F(\rho)=0$ is called the multiplicity of the singular value ρ_j.

From the relation $S(\rho)$, it is obtained

$$S(\rho) = 1+O(\frac{1}{\rho}), \quad \text{as } |\rho| \to \infty.$$

Denote

$$f_j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\widehat{F}(\rho)}{F(\rho)} - 1 \right] e^{i\rho x} d\rho = i \underset{\rho=\rho_j}{Res} \frac{\widehat{F}(\rho)}{F(\rho)} e^{i\rho x},$$

where $j = 1, 2, ..., n$.

We shall call the polynomial

$$P_j(x) = e^{-i\rho_j x} f_j(x), \quad j = 1, 2, ..., n.$$

The functions $f_j(x)$ $j = 1, 2, ..., n$ is the characteristic of the operator L_ρ on the discrete spectrum.

The set of values $\{S(\rho), \rho_j, P_j(x)\}$ $(j = 1, 2, ..., n)$ is called the scattering data of the boundary value problem (1)- (2). This set provides a complete description of the infinite behavior of all the eigenfunctions of the problem (1)-(2).

2. The Construction of Resolvent Operator

It is possible to construct the resolvent operator by using the above results. Assume that ρ is not a spectrum point of operator L_ρ. Then there exists resolvent operator $R_\rho = (L - \rho I)^{-1}$. Let's find the expression of the operator R_ρ.

Theorem 2.1. For $Im\rho \geq 0$ and $F(\rho) \neq 0$, *all numbers* ρ *belong to the resolvent set of the operator* R_ρ. *The resolvent* R_ρ *is the integral operator*

$$R_\rho g = \int_0^\infty K(x, t, \rho) g(t) dt, \tag{4}$$

with the kernel,

$$K(x, t, \rho) = -\frac{1}{F(\rho)} \begin{cases} f(x, \rho)\varphi(t, \rho), & 0 \leq t \leq x, \\ \varphi(x, \rho) f(t, \rho), & x \leq t \leq \infty. \end{cases} \tag{5}$$

Moreover

$$|K(x, t, \rho)| \leq \frac{c(x)}{|F(\rho)|} exp \left\{ Im\rho |x - t| \right\}, \tag{6}$$

where

$$c(x) = c \exp \left\{ x\sigma(0) + \sigma_1(0) \right\},$$

$c > 0$ *is an arbitrary constant.*

Proof. Let $g(x) \in \mathscr{D}_\rho$ and assume that it is a finite function at infinity. To construct the resolvent operator, we need to solve the boundary value problem

$$-y'' + q(x)y = \rho^2 y + g(x), \tag{7}$$

$$y'(0) + \left(\beta_0 + \beta_1\rho + \beta_2\rho^2\right) y(0) = 0. \tag{8}$$

By using Lagrange method to the properties of solutions of equation (1), we find (4). Here $K(x,t,\rho)$ has the form (5). Then, from the explicit expression of functions $f(x,\rho)$ and $\varphi(x,\rho)$ for $K(x,t,\rho)$, we get (6). Theorem is proved. □

Lemma 2.2. *Let $g(x)$ be continuously differentiable finite function. Then, the following is valid:*

$$\int_0^\infty K(x,t,\rho)g(t)dt = -\frac{g(x)}{\rho^2} + \frac{1}{\rho^2}K_1(x,\rho), \tag{9}$$

where

$$K_1(x,\rho) = \int_0^\infty K(x,t,\rho)(t)\left\{-g''(t) + q(t)g(t)\right\}dt.$$

Proof. The below equality is valid:

$$-K''(x,t,\rho) + q(x)K(x,t,\rho) - \rho^2 K(x,t,\rho) = \delta(x-t),$$

Here $\delta(x)$ is Dirac-delta function. By multiplying both sides of last equation by $g(x)$ and integrating from zero to infinity, we get

$$-\int_0^\infty K''(x,t,\rho)g(t)dt + \int_0^\infty q(t)K(x,t,\rho)g(t)dt - \rho^2\int_0^\infty K(x,t,\rho)g(t)dt = \int_0^\infty \delta(x-t)g(t)dt.$$

From this equation, (9) is obtained. Lemma is proved. □

3. The Expansion Formula According To The Eigenfunctions

Let us Γ_R denotes the circle of radius R and center is zero which contour is positive oriented. Let us $\Gamma_{R,\varepsilon}^{(1)}$ denotes contour be half arc of Γ_R that doesn't include points z satisfying the conditions $Imz > \varepsilon$ and let $\Gamma_{R,\varepsilon}^{(2)}$ be half arc that does not include $Imz < -\varepsilon$ points of Γ_R and we define $\Gamma_{R,\varepsilon} \equiv \Gamma_{R,\varepsilon}^{(1)} \cup \Gamma_{R,\varepsilon}^{(2)}$, it is clear that $\Gamma_{R,\varepsilon}$ is positive oriented. $\Gamma_{R,\varepsilon}^{(3)}$ denotes negative oriented curve formed with $Imz = \pm\varepsilon$ lines and be arcs including points z satisfying the conditions $|Imz| < \varepsilon$ and let's represent these domains by $\mathscr{D}$ and $\bar{\mathscr{D}}$, respectively. It is clear that $\Gamma_{R,\varepsilon} = \Gamma_R \cup \Gamma_{R,\varepsilon}^{(3)}$. Then, we can use the property of the integration

$$\int_{\Gamma_{R,\varepsilon}} = \int_{\Gamma_R} + \int_{\Gamma_{R,\varepsilon}^{(3)}} . \tag{10}$$

Let us call

$$\phi(x,\rho) = \int_0^\infty K(x,t,\rho)g(t)dt.$$

It follows that, we have

$$\phi(x,\rho) = -\frac{g(x)}{\rho^2} + \frac{1}{\rho^2}\int_0^\infty K(x,t,\rho)\tilde{g(t)}dt.$$

Now multiplying both sides of the equality by $\frac{1}{2\pi i}\rho$ and integrating over ρ the contour $\Gamma_{R,\varepsilon}$, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}} \rho\phi(x,\rho)d\rho = -\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}} \frac{g(x)}{\rho}d\rho + \frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}} \frac{1}{\rho}\left\{\int_0^\infty K(x,t,\rho)\tilde{g}(t)dt\right\}d\rho, \qquad (11)$$

where

$$\tilde{g}(t) = -g''(t) + q(t)g(t).$$

Using the residue calculus, we get

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}} \rho\phi(x,\rho)d\rho = \sum_{\mathscr{D}} Res\left[\rho\phi(x,\rho)\right] + \sum_{\bar{\mathscr{D}}} Res\left[\rho\phi(x,\rho)\right]. \qquad (12)$$

According to the equation (10), we get

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}} \rho\phi(x,\rho)d\rho = \frac{1}{2\pi i}\int_{\Gamma_R} \rho\phi(x,\rho)d\rho + \frac{1}{2\pi i}\int_{\Gamma^{(3)}_{R,\varepsilon}} \rho\phi(x,\rho)d\rho. \qquad (13)$$

Using formula (3), let us calculate the integral on the right hand side of the last equality:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_R} \rho\phi(x,\rho)d\rho &= -\frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(x)}{\rho}d\rho + \frac{1}{2\pi i}\int_{\Gamma_R} \frac{1}{\rho}\left\{\int_0^\infty K(x,t,\rho)\tilde{g}(t)dt\right\}d\rho \\ &= -g(x) + \frac{1}{2\pi i}\int_{\Gamma_R} O(\frac{1}{\rho^2})d\rho \\ &= -g(x), \qquad (14)\end{aligned}$$

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \frac{1}{2\pi i}\int_{\Gamma^{(3)}_{R,\varepsilon}} \rho\phi(x,\rho)d\rho = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \rho\left[\phi(x,\rho+i0) - \phi(x,\rho-i0)\right]d\rho. \qquad (15)$$

Taking into consideration(14) and (15) into (13), we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \rho\phi(x,\rho)d\rho = -g(x) + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \rho\left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho.$$

Using this equation in (12), we obtain

$$\begin{aligned} g(x) &= -\sum_{\mathscr{D}} Res\left[\rho\phi(x,\rho)\right] + \sum_{\bar{\mathscr{D}}} Res\left[\rho\phi(x,\rho)\right] \\ &\quad + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \rho\left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho. \end{aligned} \tag{16}$$

Now multiplying both sides of equality (3) by $\frac{1}{2\pi i}$ and integrating over ρ the contour $\Gamma_{R,\varepsilon}$, we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \phi(x,\rho)d\rho = -\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \frac{g(x)}{\rho^2}d\rho + \frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \left\{\int\limits_{0}^{\infty} \frac{1}{\rho^2}K(x,t,\rho)\tilde{g}(t)dt\right\}d\rho.$$

By similar calculations, we easily see that

$$-\sum_{\mathscr{D}} Res[\phi(x,\rho)] - \sum_{\bar{\mathscr{D}}} Res[\phi(x,\rho)] + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho = 0.$$

Therefore, we get the expansion formula with respect to eigenfunctions as

$$\begin{aligned} g(x) &= -\sum_{\mathscr{D}} Res[\phi(x,\rho)] - \sum_{\tilde{\mathscr{D}}} Res[\phi(x,\rho)] \\ &\quad + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho, \end{aligned} \tag{17}$$

$$0 = -\sum_{\mathscr{D}} Res[\phi(x,\rho)] - \sum_{\bar{\mathscr{D}}} Res[\phi(x,\rho)] + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho. \tag{18}$$

Now, let's convert formulas (17) and (18). Let $\Psi(x,\rho)$ be the solution of the Eq. (1) satisfying the initial conditions

$$\psi(0,\rho)=0, \quad \psi'(0,\rho)=1.$$

It is clear that

$$\begin{aligned} W\{\varphi(x,\rho),\psi(x,\rho)\} &= \varphi'(0,\rho)\psi(0,\rho)-\varphi(0,\rho)\psi'(0,\rho) \\ &= -1, \end{aligned}$$

and

$$f(x,\rho)=f(0,\rho)\varphi(x,\rho)+F(\rho).$$

Then, according to (5), we have

$$K(x,t,\rho)=-\frac{1}{F(\rho)}f(0,\rho)\varphi(x,\rho)\varphi(t,\rho)+\begin{cases}\psi(x,\rho)\varphi(t,\rho), & x\le t,\\ \varphi(x,\rho)\psi(t,\rho), & t\le x.\end{cases} \tag{19}$$

Therefore, we get

$$\phi(x,\rho)=-\frac{1}{F(\rho)}f(0,\rho)\varphi(x,\rho)\int_0^\infty \varphi(t,\rho)g(t)dt+\psi(x,\rho)\int_0^x \varphi(t,\rho)g(t)dt$$
$$+\varphi(x,\rho)\int_x^\infty \psi(t,\rho)g(t)dt.$$

By defininition of functions $\varphi(x,\rho)$ and $\psi(x,\rho)$, it is clear that

$$\Psi(x,t,\rho)=\begin{cases}\psi(x,\rho)\varphi(t,\rho), & 0\le t\le x,\\ \varphi(x,\rho)\psi(t,\rho), & x\le t\le\infty.\end{cases}$$

and we have

$$Res\left\{\psi(x,\rho)\int_0^x \varphi(t,\rho)g(t)dt+\varphi(x,\rho)\int_x^\infty \psi(t,\rho)g(t)dt=0\right\}.$$

From this, it follows that

$$\underset{\mathscr{D}}{Res}[\rho\phi(x,\rho)]=-\underset{\mathscr{D}}{Res}\frac{f(0,\rho)}{F(\rho)}\rho\varphi(x,\rho)\int_0^\infty \varphi(t,\rho)g(t)dt.$$

Let

$$\varphi(g,\rho)=\int_0^\infty \varphi(t,\rho)g(t)dt.$$

Then, the estimate holds

$$\underset{D}{Res}[\rho\phi(x,\rho)]=-\underset{D}{Res}\frac{f(0,\rho)}{F(\rho)}\rho\varphi(x,\rho)\varphi(g,\rho).$$

We assume that $Im\rho \geq 0$, each $\rho_k \in \mathscr{D}$ is zeros of equation $F(\rho) = 0$ with multiplicity m_k, $(k = 1, 2, ..., n)$.

Let's denote $M_k(\rho) = \frac{(\rho-\rho_k)^{m_k}}{(m_k-1)!}\frac{f(0,\rho)}{F(\rho)}$. Then for $\rho_k \in \mathscr{D}$ by the residue formula, we have

$$\underset{\mathscr{D}}{Res}[\rho\phi(x,\rho)] = -\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho M_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}|_{\rho=\rho_k}. \tag{20}$$

Further, we can obtain an analogous formula for $Im\rho \leq 0$, each $\bar{\rho_k} \in \bar{\mathscr{D}}$ in the following form:

$$\underset{\bar{\mathscr{D}}}{Res}[\rho\phi(x,\rho)] = -\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho \bar{M}_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}|_{\rho=\bar{\rho_k}}, \tag{21}$$

where

$$\bar{M}_k(\rho) = \frac{(\rho-\bar{\rho_k})^{m_k-1}}{(m_k-1)!}\frac{f(0,\rho)}{F(\rho)}.$$

We denote by $\bar{\rho_k} \in \bar{\mathscr{D}}$ the roots of the equation $F_1(\rho) = 0$ with multiplicity m_k $(k = 0, 2, ..., n.)$

Since,

$$\begin{aligned}\phi(x,\rho+i0)-\phi(x,\rho-i0) &= -\frac{2i\rho}{F_1(\rho)F(\rho)}\varphi(x,\rho)\int_0^\infty \varphi(t,\rho)g(t)dt \\ &= -\frac{2i\rho}{F_1(\rho)F(\rho)}\varphi(x,\rho)\varphi(f,\rho),\end{aligned} \tag{22}$$

we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\rho\left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\rho^2}{F_1(\rho)F(\rho)}\varphi(x,\rho)\varphi(f,\rho)d\rho.$$

Substituting (20), (21) and (22) in (17), (18) respectively, we obtain two-fold expansion formulas in the following form

$$\begin{aligned}f(x) &= -\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho M_k(\rho)\varphi(x,\rho)\varphi(f,\rho)\right\}|_{\rho=\rho_k} \\ &- \sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho \bar{M}_k(\rho)\varphi(x,\rho)\varphi(f,\rho)\right\}|_{\rho=\bar{\rho_k}} \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho^2}{F_1(\rho)F(\rho)}\varphi(x,\rho)\varphi(f,\rho)d\rho.\end{aligned}$$

$$
\begin{aligned}
0 &= -\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho M_k(\rho)\varphi(x,\rho)\varphi(f,\rho)\right\}|_{\rho=\rho_k} \\
&- \sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho \bar{M}_k(\rho)\varphi(x,\rho)\varphi(f,\rho)\right\}|_{\rho=\bar{\rho_k}} \\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho^2}{F_1(\rho)F(\rho)}\varphi(x,\rho)\varphi(f,\rho)d\rho.
\end{aligned}
$$

for $\rho_k \in \mathscr{D}$, $\bar{\rho_k} \in \bar{\mathscr{D}}$. It can be shown that the integrals on the right hand side converge in the metric of the space $L_2(0,\infty)$.

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