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A problem on Determining the Unknown Coefficient and Free Term of a Linearized Benney-Luke Equation with not Self-Adjoint Boundary Conditions

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Abstract. We study an inverse problem for a linearized Benney-Luke equation with not self-adjoint boundary conditions. At first the initial problem is reduced to the equivalent problem (in a certain sense) for which a theorem on the existence and uniqueness is proved. Furthermore, based on these facts the existence and uniqueness of the classic solution of the initial problem is proved.

Key Words and Phrases: inverse problem, Benney-Luke equation, existence, uniqueness, classic solution.

1. Introduction

Many problems of mathematical physics, continuum mechanics are boundary value problem reduced to integration of a differential equation or a system of partial equations for the given boundary and initial conditions. Many problems of gas dynamics, theory of elasticity, theory of plates and shells are reduced to consideration of higher order partial differential equations [1]. Fourth order differential equations (see: [2, 3]) are of great interest in terms of applications. Benny-Luke type partial differential equations have applications in mathematical physics (see [3]).

The problems in which together with the solution of this or another differential equation it is required also to determine coefficients of the equation itself or the right-hand side of the equation, in mathematics or in mathematical modeling are called inverse problems. Theory of inverse problems for differential equations is a dynamically developing section of modern science. Recently, inverse problems have arisen in various fields of human activity, such as seismology, exploration of minerals, biology, medicine, quality control of industrial products, etc. and this lists them among current problems of modern mathematics. Various inverse problems were studied for separate types of partial differential equations in a number of works. Here we note first of all the works of A.N. Tikhonov [4], M.M. Lavrentev [5,6], V.K. Ivanov [7] and their followers. You can read more about this in the monograph of A.M. Denisov [8].

Theory of inverse problems for fourth order differential equations are still under-researched. The works [9-12] have been devoted to boundary value problems for Benny-Luke equation.

The goal of this paper is to prove the existence and uniqueness of the solution to the inverse boundary value problem for a Benny-Luke equation with not sel-adjoint boundary conditions.

2. Statement of the problem and its reduction to an equivalent problem

Let $D_T = \{(x,t): \ 0 \le x \le 1, \ 0 \le t \le T\}$. In the rectangle D_T we consider the following inverse boundary value problem: find the triple $\{u(x,t), a(t), b(t)\}$ of the function $u(x,t)$, $a(t)$, $b(t)$ satisfying the equation [3]

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) = a(t)u(x,t) + b(t)g(x,t) + f(x,t) \ , \quad (1)$$

with nonlocal initial conditions

$$u(x,0) = \varphi(x) + \int_0^T p_1(t)u(x,t)\,dt\,,\ u_t(x,0) = \psi(x) + \int_0^T p_2(t)u(x,t)\,dt \ (0 \le x \le 1), \quad (2)$$

with not self-adjoint boundary conditions

$$u(1,t) = 0, u_x(0,t) = u_x(1,t) \ , \ u_{xx}(1,t) = 0, \ u_{xxx}(0,t) = u_{xxx}(1,t) \ (0 \le t \le T) \quad (3)$$

and with additional conditions

$$u(0,t) = h_1(t) \qquad (0 \le t \le 1), \quad (4)$$

$$u\left(\frac{1}{2}, t\right) = h_2(t) \qquad (0 \le x \le 1) \quad (5)$$

where $\alpha > 0$, $\beta > 0$ are fixed numbers, $f(x,t)$, $g(x,t)$, $\varphi(x)$, $\psi(x)$, $p_1(t)$,$p_2(t)$, $h_1(t)$, $h_2(t)$ and the given functions.

Denote

$$\tilde{C}^{4,2}(D_T) = \{u(x,t): \ u(x,t) \in C^2(D_T)\ , u_{ttx}(x,t), \\ u_{ttxx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\}\,.$$

Definition. *Under the classic solution of the inverse boundary value problem* (1)-*(5) we understand the triple* $\{u(x,t), a(t), b(t)\}$ *of the functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$, $b(t) \in C[0,T]$ *satisfying equation* (1) *and conditions* (2)-(5) *in the usual sense.*

Along with the inverse boundary value problem (1)-(5) we consider the following auxiliary inverse boundary value problem: it is required to determine the triple $\{u(x,t), a(t), b(t)\}$ of the functions $u(x,t) \in \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$, $b(t)(t) \in C[0,T]$, from the relations (1)-(5) ,

$$h_1''(t) - u_{xx}(0,t) + \alpha u_{xxxx}(0,t) - \beta u_{xxtt}(0,t) =$$

$$= a(t)h_1(t) + b(t)g(0,t) + f(0,t) \ (0 \le t = T), \tag{6}$$

$$h_2''(t) - u_{xx}\left(\frac{1}{2}, t\right) + \alpha u_{xxxx}\left(\frac{1}{2}, t\right) - \beta u_{xxtt}\left(\frac{1}{2}, t\right) =$$

$$= a(t)h_2(t) + b(t)g\left(\frac{1}{2}, t\right) + f\left(\frac{1}{2}, t\right) \quad (0 \le t = T), \tag{7}$$

The following theorem is proved in the same way [13]

Theorem 1. *Let* $\varphi(x),\ \psi(x) \in C[0,1]$, $P_1(t), P_2(t) \in C[0,T]$, $f(x,t),\ g(x,t) \in C(D_T)$,

$$h_i(t) \in C^2[0,T] \ (i = 1,2),\ h(t) \equiv h_1(t)\, g(\frac{1}{2}, t) - h_2(t)g(0,t) \ne 0 \ (0 \le t \le T)$$

$f(x,t), g(x,t) \in C(D_T)$*and the following matching condition be fulfilled*

$$\varphi(0) = h(0) - \int_0^T P_1(t)h_1(t)dt,\ \ \varphi(0) = h_1'(0) - \int_0^T P_2(t)h_2(t)dt,$$

$$\varphi\left(\frac{1}{2}\right) = h_2(0) - \int_0^T P_1(t)h_1(t)dt,\ \ \psi\left(\frac{1}{2}\right) = h_2'(0) - \int_0^T P_2(t)h_2(t)dt$$

Then the following statements are valid:

A. Each classic solution $\{u(x,t), a(t), b(t)\}$ *of the problem* (1)-*(5) is also the solution of the problem* (1)-(3),(6), *(7);*

B. Each solution $\{u(x,t), a(t), b(t)\}$ *of the problem* (1)-(3), *(6)*,(7) *such that*

$$\left(\|p_1(t)\|_{C[0,T]} + T\,\|p_2(t)\|_{C[0,T]} + \|a(t)\|_{C[0,T]}\right) T^2 < 1,$$

is the solution of (1)-*(5).*

3. Solvability of the inverse boundary value problem.

It is known [14] that the sequences of functions

$$X_0(x) = 2(1-x),\ \ X_{2k-1}(x) = 4(1-x)\cos\lambda_k x,\ \ X_{2k}(x) = 4\sin\lambda_k x \ (k = 1,2,...), \tag{8}$$

$$Y_0(x) = 1,\ \ Y_{2k-1}(x) = \cos\lambda_k x,\ \ Y_{2k}(x) = x\sin\lambda_k x \ (k = 1,2,...) \tag{9}$$

form a biorthogonal system, and the system 8) forms a Riesz basis in $L_2(0,1)$, where $\lambda_k = 2k\pi\,(k = 1,2,...)$. Then the arbitrary function $\vartheta(x) \in L_2(0,1)$ expands in biorthogonal series:

$$\vartheta(x) = \vartheta_0\, X_0(x) + \sum_{k=1}^{\infty} \vartheta_{2k-1}\, X_{2k-1}(x) + \sum_{k=1}^{\infty} \vartheta_{2k}\, X_{2k}(x),$$

where

$$\vartheta_0 = \int_0^1 \vartheta_0 \, Y_0\,(x)dx\,, \;\; \vartheta_{2k-1} = \int_0^1 \vartheta_{2k-1}\, Y_{2k-1}\,(x)dx\,, \;\; \vartheta_{2k-1} = \int_0^1 \vartheta_{2k-1}\, Y_{2k-1}\,(x)dx\,.$$

It is known [15] that if

$$\vartheta(x) \in C^{2i-1}[0,1], \quad \vartheta^{(2i)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0, \;\; \vartheta^{(2s+1)}(0) = \vartheta^{(2s)}(1) \;\; (s = \overline{0, i-1}\;\;),$$

then

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i} \vartheta_{2k-1}\;\right)^2 \le \frac{1}{2}\left\| \vartheta^{(2i)}(x) \right\|_{L_2(0,1)}^2,$$

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i} \vartheta_{2k}\;\right)^2 \le \frac{1}{2}\left\| \vartheta^{(2i)}(x)x + 2i\vartheta^{(2i-1)}(x) \right\|_{L_2(0,1)}^2. \tag{10}$$

Under the assumptions

$$\vartheta(x) \in C^{2i}[0,1], \quad \vartheta^{(2i+1)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0, \;\; \vartheta^{(2s-1)}(0) = \vartheta^{(2s-1)}(1) \;\; (i \ge 1, \; s = \overline{0, i}\;\;),$$

the validity of the following estimations [15] is estabilished:

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i+1} \vartheta_{2k-1}\;\right)^2 \le \frac{1}{2}\left\| \vartheta^{(2i+1)}(x) \right\|_{L_2(0,1)}^2,$$

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i+1} \vartheta_{2k}\;\right)^2 \le \frac{1}{2}\left\| \vartheta^{(2i+1)}(x)x + (2i+1)\vartheta^{(2i)}(x) \right\|_{L_2(0,1)}^2. \tag{11}$$

To study the problem (1)-(3), (6),(7) we consider the following space.

Denote by $B_{2,T}^5$ [15] the totality of all functions $u\,(x,t)$ of the form

$$u\,(x,t) = \sum_{k=0}^{\infty} u_k\,(t)\,X_k\,(x)\,,$$

considered on a D_T, for which all the functions $u_k\,(t) \in C[0,T]$ and

$$J_T(u) \equiv \|u_0(t)\|_{C[0,T]} +$$

$$+ \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \, \|u_{2k-1}(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \, \|u_{2k}(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}},$$

where the functions $X_k\,(x)$ $\;(k = 0, 1, 2, ...)$ were determined by the relations (8).

On this set we calculate the norm as follows: $\|u(x,t)\|_{B_{2,T}^5} = J_{"}(u)$.

Let E_T^5 denote a space of such vector-functions $\{u(x,t), a(t)\}$ that $u\,(x,t) \in B_{2,T}^5$, $a\,(t) \in C[0,T]$. Let's supply this space with the norm

$$\|z\|_{E_T^5} = \|u(x,t)\|_{B_{2,T}^5} + \|a(t)\|_{C[0,T]} \,.$$

Obviously, $B_{2,T}^5$ and E_T^5 are Banach spaces.

Since the system (8) forms the Riesz basis $L_2\,(0,1)$, and the system (8) and (9) forms a biorthogonal in $L_2\,(0,1)$ system of functions, then we will look for the first component $u\,(x,t)$ of the solution $\{u(x,t), a(t)\}$ of the problem (1)-(3), (6) in the form

$$u\,(x,t) = u_0\,(t)\,X_0\,(x) + \sum_{k=1}^{\infty} u_{2k-1}\,(t)\,X_{2k-1}\,(x) + \sum_{k=1}^{\infty} u_{2k}\,(t)\,X_{2k}\,(x)\,, \qquad (12)$$

where

$$u_0(t) = \int_0^1 u(x,t)Y_0(x)dx,$$

$$u_{2k-1}(t) = \int_0^1 u(x,t)Y_{2k-1}(x)dx,\ u_{2k}(t) = \int_0^1 u(x,t)Y_{2k}(x)dx\ (k=1,2,...)\,, \qquad (13)$$

is the solution of the following problem:

$$u_0''(t) = F_0\,(t;u,a,b) \quad (0 \le t \le T), \qquad (14)$$

$$u_{2k-1}''(t) + \beta_k^2 u_{2k-1}(t) = \frac{1}{1+\beta\lambda_k^2} F_{2k-1}\,(t;u,a,b) \quad (0 \le t \le T,\ k=1,2,...), \qquad (15)$$

$$u_{2k}''\,(t) + \beta_k^2 u_{2k}\,(t) = \frac{1}{1+\beta\lambda_k^2} F_{2k}\,(t;u,a,b) +$$

$$+\frac{2\lambda_k(1+2\alpha\lambda_k^2)}{1+\beta\lambda_k^2} u_{2k-1}(t) + \frac{2\beta\lambda_k}{1+\beta\lambda_k^2}\, u_{2k-1}''(t)\,(0 \le t \le T,\ \ k=1,2,...), \qquad (16)$$

$$u_k(0) = \varphi_k + \int_0^T p_1(t)u_k(t)\,dt,\quad u_k'(0) = \psi_k + \int_0^T p_2(t)u_k(t)\,dt \quad (k=0,1,2,...)\,, \quad (17)$$

moreover,

$$\beta_k^2 = \frac{\lambda_k^2(1+\alpha\lambda_k^2)}{1+\beta\lambda_k^2}, F_k(t;u,a) = a(t)u_k(t) + b(t)g_k(t) + f_k(t),$$

$$f_k(t) = \int_0^1 f(x,t)Y_k(x)dx,\ g_k(t) = \int_0^1 g(x,t)Y_k(x)dx,$$

$$\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, \quad \psi_k = \int_0^1 \psi(x)Y_k(x)dx \ \ (k=0,1,...).$$

Solving the problem (14)-(17) we fined:

$$u_0(t) = \varphi_0 + \int_0^T p_1(t)u_0(t)\,dt + \left(\psi_0 + \int_0^T p_2(t)u_0(t)\,dt\right) t + \int_0^t (t-\tau)F_0(\tau; u, a, b)d\tau, \quad (18)$$

$$u_{2k-1}(t) = \left(\varphi_{2k-1} + \int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t +$$

$$+\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t +$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau; u, a, b)\sin\beta_k(t-\tau)d\tau, \quad (19)$$

$$u_{2k}(t) = \left(\varphi_{2k} + \int_0^T p_1(t)u_{2k}(t)\,dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\psi_{2k} + \int_0^T p_2(t)u_{2k}(t)\,dt\right)\sin\beta_k t +$$

$$+\frac{\lambda_k(1+2\alpha\lambda_k^2+\alpha\beta\lambda_k^4)}{(1+\beta\lambda_k^2)^3}\left[t\left(\varphi_{2k-1} + \int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t +\right.$$

$$+\left(\frac{1}{\beta_k}\sin\beta_k t - t\cos\beta_k t\right)\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T p_2(t)u_{2k-1}(t)\,dt\right) +$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k}(\tau; u, a, b)\sin\beta_k(t-\tau)d\tau +$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi; u, a, b)\sin\beta_k(t-\xi)\,d\xi\right)\sin\beta_k(t-\tau)\,d\tau\right] +$$

$$+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau; u, a, b)\sin\lambda_k(t-\tau)d\tau. \quad (20)$$

After substituting the expression $u_k(t)(k = 0, 1, ...)$ in (12), for determining the component $u(x,t)$of the solution of the problem (1)-(3), (6) ,(7) we obtain:

$$u(x,t) = \left(\varphi_0 + \int_0^T p_1(t)u_0(t)\,dt + \left(\psi_0 + \int_0^T p_2(t)u_0(t)\,dt\right) t +\right.$$

$$\left.+\int_0^t (t-\tau)F_0(\tau; u, a, b)d\tau\right) X_0(x) +$$

$$+\sum_{k=1}^{\infty}\left\{\left(\varphi_{2k-1} + \int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t +\right.$$

$$+\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t +$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau; u, a, b)\sin\beta_k(t-\tau)d\tau\right\} X_{2k-1}(x) +$$

$$+\sum_{k=1}^{\infty}\left\{\left(\varphi_{2k}+\int_0^T p_1(t)u_{2k}(t)\,dt\right)\cos\beta_k t+\right.$$

$$+\frac{1}{\beta_k}\left(\psi_{2k}+\int_0^T p_2(t)u_{2k}(t)\,dt\right)\sin\beta_k t+$$

$$+\frac{\lambda_k(1+2\alpha\lambda_k^2+\alpha\beta\lambda_k^4)}{(1+\beta\lambda_k^2)^3}\left[t\left(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+\right.$$

$$+\left(\frac{1}{\beta_k}\sin\beta_k t-t\cos\beta_k t\right)\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)+$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k}(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau+$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi;u,a,b)\sin\beta_k\,(t-\xi)\,d\xi\right)\sin\beta_k\,(t-\tau)\,d\tau\right]+$$

$$\left.+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,a,b)\sin\lambda_k(t-\tau)d\tau\right\}X_{2k}(x). \tag{21}$$

Now from (6) and (7) , allowing for (12), we have:

$$a(t)=[h(t)]^{-1}\left\{g\left(\frac{1}{2},t\right)(h_1''(t)-f(0,t))-g(0,t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)+\right.$$

$$\left.+2\sum_{k=1}^{\infty}\left(2g\left(\frac{1}{2},t\right)-(-1)^k g(0,t)\right)((\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u_{2k-1}''(t))\right\}, \tag{22}$$

$$b(t)=[h(t)]^{-1}\left\{h_1(t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)-h_2(t)(h_1''(t)-f(0,t))+\right.$$

$$\left.+2\sum_{k=1}^{\infty}(h_1(t)\cdot(-1)^k-2h_2(t))((\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u_{2k-1}''(t))\right\}. \tag{23}$$

Further, from (15), allowing for (19), we obtain:

$$(\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u_{2k-1}''(t)=F_{2k-1}(t;u,a,b)-u_{2k-1}''(t)=$$

$$=\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a,b)-\beta_k^2u_{2k-1}(t)=$$

$$=\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}t;u,a,b)-\beta_k^2\left(\left(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t+\right.$$

$$+\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+$$

$$+\ \frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}t;u,a,b)\sin\beta_k(t-\tau)d\tau\Bigg). \qquad (24)$$

For obtaining an equation for the second and third component $a(t)$ and $b(t)$ of the solution $\{u(x,t),a(t),b(t)\}$ of the problem (1)–(3), (6), (7), we substitute the expression (24) in (22) and (23):

$$a(t)=[h(t)]^{-1}\left\{g\left(\frac{1}{2},t\right)(h_1''(t)-f(0,t))-g(0,t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)+\right.$$

$$+2\sum_{k=1}^{\infty}\left(2g\left(\frac{1}{2},t\right)-(-1)^k g(0,t)\right)\left[\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a,b)-\right.$$

$$-\beta_k^2\left(\left(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t+\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+\right.$$

$$\left.\left.\left.+\ \frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau\right)\right)\right]\Bigg\}, \qquad (25)$$

$$b(t)=[h(t)]^{-1}\left\{h_1(t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)-h_2(t)(h_1''(t)-f(0,t))+\right.$$

$$+2\sum_{k=1}^{\infty}(h_1(t)\cdot(-1)^k-2h_2(t))\left[\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a,b)-\right.$$

$$-\beta_k^2\left(\left(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t+\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+\right.$$

$$\left.\left.\left.+\ \frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau\right)\right)\right]\Bigg\}. \qquad (26)$$

Thus, the solution of the problem (1)-(3), (6),(7) is reduced to the solution of the system (21), (25), (26) with respect to the unknown functions $u(x,t)$,$a(t)$ and $b(t)$.

The following lemma is very important for studying the uniqueness of the solution to the problem (1)-(3), (6),(7)

Lemma 1. *If* $\{u(x,t),a(t),a(t)\}$ *is any solution of the problem* (1)-(3)*, (6), then the functions* $u_k(t)$ $(k=0,1,2,\ldots)$*, determined by the relation (12), satisfy on* $[0,T]$ *the denumerable system (17), (18) and (19).*

Obviously, if $u_k(t)=\int_0^1 u(x,t)Y_k(x)dx$ $(k=0,1,...)$ is the solution of the system (18), (19) and (20), then the triple $\{u(x,t),a(t)\}$ of the functions $u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$,$a(t)$ and $b(t)$ is the solution of the system (21), (25),(25

From Lemma 1 we have the following corollary

Corollary 1. *Let the system (20), (24),*(26) *have a unique solution. Then the problem* (1)-(3)*, (6),*(7) *may have at most one solution, i.e. if the problem* (1)-(3)*, (6) ,*(7) *has a solution, this solution is a unique solution.*

Now in the space E_T^5 we consider the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\}\ ,$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)=\sum_{k=0}^{\infty}\tilde{u}_k(t)X_k(x),\Phi_2(u,a)=\tilde{a}(t),$$

$\tilde{u}_0(t),\ \tilde{u}_{2k-1}(t),\ \tilde{u}_{2k}(t)$, $\tilde{a}(t)$ and $\tilde{b}(t)$ are equal to the right side of (18), (19) , (20), (25) and (26), respectively.

It is easy to see that

$$1+\beta\lambda_k^2>\beta\lambda_k^2,\quad \frac{1}{1+\beta\lambda_k^2}<\frac{1}{\beta\lambda_k^2},$$

$$\sqrt{\frac{\alpha}{1+\beta}}\,\lambda_k\le\beta_k\le\sqrt{\frac{1+\alpha}{\beta}}\,\lambda_k,\quad \sqrt{\frac{\beta}{1+\alpha}}\,\frac{1}{\lambda_k}\le\frac{1}{\beta_k}\le\sqrt{\frac{1+\beta}{\alpha}}\,\frac{1}{\lambda_k},$$

Taking into account this relation, we find:

$$\|\tilde{u}_0\,(t)\|_{C[0,T]}\le|\varphi_0|+T\,|\psi_0|+T\sqrt{T}\left(\int_0^T|f_0\,(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+$$

$$+\left(\|p_1(t)\|_{C[0,T]}+T\,\|p_2(t)\|_{C[0,T]}\right)T\,\|u_0(t)\|_{C[0,T]}+T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}+$$

$$+T\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T|g_0\,(\tau)|^2\,d\tau\right)^{\frac{1}{2}},\tag{27}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le 3\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+3\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right)+$$

$$+\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{3}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,|)^2d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,g_{2k-1}\,(\tau)\,|)^2d\tau\right)^{\frac{1}{2}}\right],\tag{28}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le 4\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k}|)^2\right)^{\frac{1}{2}}+4\,\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k}\,|)^2\right)^{\frac{1}{2}}+$$

$$+\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{3}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,g_{2k}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}\right]+$$

$$,+\frac{4(1+2\alpha+\alpha\beta)}{\beta^3}\left[T\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}+\right.$$

$$+\frac{4}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(T\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T^2\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]+$$

$$+4T\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{6}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,g_{2k-1}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}\right],\qquad(29)$$

$$\|\tilde{a}(t)\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|g\left(\frac{1}{2},t\right)(h_1''(t)-f(0,t))-g(0,t)\left(h_2''(t)-f\left((\frac{1}{2},t)\right)\right)\right\|_{C[0,T]}+\right.$$

$$+2\left\|2\left|g\left(\frac{1}{2},t\right)\right|+|g(0,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\left\{\frac{1+\alpha}{\beta}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right)+\right.\right.$$

$$+T\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,g_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}\right]\Bigg]+\left(\sum_{k=1}^{\infty}(\lambda_k^2\,\|f_{2k-1}\,(t)\|_{C[0,T]}\,\,|\right)^{\frac{1}{2}}+$$

$$+\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|b(t)\|_{C[0,T]}\,\left(\sum_{k=1}^{\infty}(\lambda_k^2\,\|g_{2k-1}\,(t)\|_{C[0,T]}\,\,|\right)^{\frac{1}{2}}\Bigg\}\Bigg\}\ , \tag{30}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h_1(t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)-h_2(t)(h_1'(t)-f(0,t))\right\|_{C[0,T]}+\right.$$

$$+2\,\||h_1(t)|+2\,|h_2(t)|\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times\left\{\frac{1+\alpha}{\beta}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right)+\right.\right.$$

$$+T\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,g_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}\right]\Bigg]+\left(\sum_{k=1}^{\infty}(\lambda_k^2\,\|f_{2k-1}\,(t)\|_{C[0,T]}\,\,|\right)^{\frac{1}{2}}+$$

$$+\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|b(t)\|_{C[0,T]}\,\left(\sum_{k=1}^{\infty}(\lambda_k^2\,\|g_{2k-1}\,(t)\|_{C[0,T]}\,\,|\right)^{\frac{1}{2}}\Bigg\}\Bigg\}\ . \tag{31}$$

Assume that the data of the problem (1)-(3), (6),(7) satisfy the following condition:

$$1.\alpha>0,\beta>0,p_i(t)\in C[0,T](i=1,2)\ .$$

$$2.\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi(1)=0,\varphi'(0)=\varphi'(1),$$

$$\varphi''(1)=0,\varphi'''(0)=\varphi'''(1),\ \varphi^{(4)}(1)=0.$$

$$3.\psi(x)\in C^3[0,1],\ \psi^{(4)}(x)\in L_2(0,1),\ \psi(1)=0,\ \psi'(0)=\psi'(1),\ \psi''(1)=0,\psi'''(0)=\psi'''(1).$$

4. $f(x,t), f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T),\ f(1,t)=0,\ \ f_x(0,t)=f_x(1,t)\ \ (0 \le t \le T).$

5. $g(x,t), g_x(x,t) \in C(D_T),\ g_{xx}(x,t) \in L_2(D_T),\ g(1,t)=0,\ \ g_x(0,t)=g_x(1,t)\,(0 \le t \le T).$

6. $h_i(t) \in C^2[0,T]\,(i=1,2),\ h(t) \equiv h_1(t) g\left(\frac{1}{2},t\right) - h_2(t) g(0,t) \ne 0\ (0 \le t \le T).$

Then from (27)- (31) we find:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} \le A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} +$$
$$+C_1(T)\|u(x,t)\|_{B^5_{2,T}} + D_1(T)\|b(t)\|_{C[0,T]}\,, \tag{32}$$
$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} +$$
$$+C_2(T))\|u(x,t)\|_{B^5_{2,T}} + D_2(T)\|b(t)\|_{C[0,T]}\ \ , \tag{33}$$
$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le A_3(T) + B_3(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} +$$
$$+C_3(T))\|u(x,t)\|_{B^5_{2,T}} + D_3(T)\|b(t)\|_{C[0,T]}\ \ , \tag{34}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\|f(x,t)\|_{L_2(D_T)} + \sqrt{2}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} +$$
$$+\sqrt{\frac{2(1+\beta)}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)} + \sqrt{\frac{2T(1+\beta)}{\alpha}}\|f_{xx}(x,t)\|_{L_2(D_T)} + \frac{3}{\sqrt{2}}\left\|\varphi^{(5)}(x)+4\varphi^{(3)}(x)\right\|_{L_2(0,1)} +$$
$$+\frac{4}{\sqrt{2}}\sqrt{\frac{1+\beta}{\alpha}}\left\|\psi^{(4)}(x)+3\psi^{(3)}(x)\right\|_{L_2(0,1)} + \frac{4}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\|f_{xx}(x,t)+2f_x(x,t)\|_{L_2(D_T)} +$$
$$+\frac{4(1+2\alpha+\alpha\beta)}{\beta^3}\left(\frac{T}{\sqrt{2}}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{2\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)} +\right.$$
$$\left.+\frac{T}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\|f_{xx}(x,t)\|_{L_2(D_T)}\right) + \frac{6}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\|f_{xx}(x,t)\|_{L_2(D_T)}\,,$$
$$B_1(T) = T^2 + \frac{11T}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(1+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}T\right),$$
$$C_1(T) = 4\left(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}\right)T\,,$$
$$D_1(T) = T\sqrt{T}\|g(x,t)\|_{L_2(D_T)} + \frac{4}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\|g_{xx}(x,t)+2g_x(x,t)\|_{L_2(D_T)} +$$
$$+\sqrt{\frac{2(1+\beta)}{\alpha}}\|g_{xx}(x,t)\|_{L_2(D_T)} + \frac{T}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\|g_{xx}(x,t)\|_{L_2(D_T)} +$$

$$+\frac{6}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|g_{xx}(x,t)\|_{L_2(D_T)},$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|g\left(\frac{1}{2},t\right)(h_1''(t)-f(0,t))-g(0,t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)\right\|_{C[0,T]}+\right.$$

$$+2\left\|2\left|g\left(\frac{1}{2},t\right)\right|+|g(0,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\left\{\frac{1+\alpha}{\beta}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{1+\beta}{\alpha}}\,\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.\right.$$

$$\left.\left.\left.+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right]+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}\right\},$$

$$B_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|2\left|g\left(\frac{1}{2},t\right)\right|+|g(0,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\frac{1+\alpha}{\beta^2}\sqrt{\frac{1+\beta}{\alpha}}T+1\right),$$

$$C_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|2\left|g\left(\frac{1}{2},t\right)\right|+|g(0,t)|\right\|_{C[0,T]}\times$$

$$\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)T,$$

$$D_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|2\left|g\left(\frac{1}{2},t\right)\right|+|g(0,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times\left(\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right),$$

$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h_1(t)\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)-h_2(t)(h_1'(t)-f(0,t))\right\|_{C[0,T]}+\right.$$

$$+2\,\||h_1(t)|+2\,|h_2(t)|\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{1+\beta}{\alpha}}\,\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.\right.$$

$$\left.\left.\left.+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right]+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}\right\},$$

$$B_3(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\||h_1(t)|+2\,|h_2(t)|\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\frac{1+\alpha}{\beta^2}\sqrt{\frac{1+\beta}{\alpha}}T+1\right),$$

$$C_3(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\||h_1(t)|+2\,|h_2(t)|\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right)T,$$

$$D_3(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\||h_1(t)|+2\,|h_2(t)|\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times\left(\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right).$$

From the inequalities (32)- (34) we conclude:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^5}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^5}+$$

$$+C(T)\,\|u(x,t)\|_{B_{2,T}^5}+D(T)\,\|b(t)\|_{C[0,T]}\,, \tag{35}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),\qquad B(T)=B_1(T)+B_2(T)+B_3(T)$$

$$C(T)=C_1(T)+C_2(T)+C_3(T),\qquad D(T)=D_1(T)+D_2(T)+D_3(T)$$

So, we proved the following theorem

Theorem 2. *Let the conditions 1-6 and*

$$(A(T)+2)(B(T)A(T)+2)+C(T)+D(T))<1\;. \tag{36}$$

be fulfilled.

Then the problem (1)-(3), *(6)*,(7) *has in the sphere* $K=K_R(\|\,z\,\|_{E_T^5}\leq R=A(T)+2)$ *from* E_T^5 *a unique solution..*

Proof. In the space E_T^5 we consider the equation

$$z=\Phi z, \tag{37}$$

where $z=\{u,a\}$, the components $\Phi_i(u,a,b)\,(i=1,2)$ of the operator $\Phi(u,a,b)$ are determined by the right hand side of the equation (21), (25), (26), respectively.

Let us consider the operator $\Phi(u,a)$ in the sphere $K=K_R$ from E_T^5. Similar to (35) we obtain that for any z, z_1, $z_2\in K_R$ the following estimations are valid:

$$\|\Phi z\|_{E_t^5}\leq A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^5}+C(T)\,\|u(x,t)\|_{B_{2,T}^5}+D(T)\,\|b(t)\|_{C[0,T]}\leq$$

$$\leq A(T)+B(T)(A(T+2)^2+C(T)(A(T)+2)+D(T)(A(T)+2)=$$

$$=A(T)+(A(t)+2)(B(T)(A(T)+2)+C(T)+D(T)\,), \tag{38}$$

$$\|\Phi z_1-\Phi z_2\|_{E_t^5}\leq B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^5}\right)+$$

$$+C(T)\,\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^5}+D(T)\,\|b_1(t)-b_2(t)\|_{C[0,T]}\,. \tag{39}$$

Then, allowing for (36), from the estimates (38), (39) it follows that the operator Φ acts in the sphere $K = K_R$ and is compressive. Therefore, in the sphere $K = K_R$ the operator Φ has a unique fixed point $\{u, a\}$, that is the solution of the equation (37), i.e. it is a unique solution of the system (21), (25),(26) in the sphere $K = K_R$.

The function $u(x,t)$, as an element of the space $B^5_{2,T}$, has continuous derivatives $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ in D_T.

Similar to [10], we can how that $u_t(x,t)$, $u_{tt}(x,t)$, $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to verify that equation (1) and conditions (2), (3) and (6),(7) are satisfied in the usual sense. So, $\{u(x,t), a(t), b(t)\}$ is the solution of the problem (1)-(3), (6), (7), and by the corollary of Lemma 1, it is unique. The theorem is proved.

By means of Theorem 1 we prove the following theorem

Theorem 3. *Let all the conditions of Theorem 2* and the matching conditions

$$\varphi(0) = h(0) - \int_0^T P_1(t)h_1(t)dt, \;\; \varphi(0) = h_1'(0) - \int_0^T P_2(t)h_2(t)dt,$$

$$\varphi\left(\frac{1}{2}\right) = h_2(0) - \int_0^T P_1(t)h_1(t)dt, \;\; \psi\left(\frac{1}{2}\right) = h_2'(0) - \int_0^T P_2(t)h_2(t)dt$$

and

$$\Big(\|p_1(t)\|_{C[0,T]} + T\,\|p_2(t)\|_{C[0,T]} + A(T) + 2\Big)\,T^2 < 1$$

be fullfield.

Then the problem (1)-(5) has in the sphere $K = K_R(\|\, z\,\|_{E^5_T} \le R = A(T) + 2)$ *from* E^5_T *a unique classic solution.*

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