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On a scattering problem for Sturm-Liouville equation with a rational function of spectral parameter in boundary condition

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Abstract. We consider the Sturm-Liouville equation on the half line $[0,\infty)$ with a rational function of spectral parameter in the boundary condition and investigate the corresponding scattering problem. Scattering data is obtained and its properties are examined.

Key Words and Phrases: Sturm-Liouville equation; scattering data; spectral parameter.

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1. Introduction

Consider the boundary value problem generated by the differential equation and boundary condition:

$$-v''(x) + \{q(x) - \lambda^2\}v(x) = 0, \quad (0 \le x < \infty) \tag{1}$$

$$v'(0) - f(\lambda)\, v(0) = 0, \tag{2}$$

where λ is a spectral parameter, $q(x)$ is real valued function with the condition

$$\int_0^\infty (1+x)\,|q(x)|\,dx < \infty \tag{3}$$

and

$$f(\lambda) = \frac{b_0 + b_1\lambda^2 + b_2\lambda^4}{a_0 + a_1\lambda^2 + a_2\lambda^4}$$

for $\alpha_i,\ \beta_j \in \mathbb{R}\ \ (i,j = 0,1,2)$

$$a_1b_0 - b_0a_1 \ge 0, \qquad a_2b_1 - b_1a_2 \ge 0, \qquad a_2b_0 - b_2a_0 = 0. \tag{4}$$

Spectral problems often appear in mathematics, mechanics, physics and other branches of natural sciences. Direct scattering problem deals with the determination of spectral characteristics for given boundary value problem when $q(x)$ is known. In the case that boundary condition didn't contain spectral parameter, scattering problem for equation (1) was solved by Marchenko [1] and Levitan [2]. In spectral theory, problems with spectral parameters in equations and boundary conditions are extremely important. Sturm-Liouville problems with spectral parameter-dependent boundary conditions arise in studies of heat conduction problems and vibrating string problems. Fulton and Pruess showed a kind of heat conduction problems in [3]. Many examples of spectral problems which arise in mechanical engineering and contain eigen parameter in the boundary conditions were presented in the book [4]. Spectral problem for second order differential operator pencil on the axis was studied in [5]. Problems with boundary conditions depending on spectral parameter were examined in finite interval by several authors [6, 7, 8, 9, 10, 11, 12] and on the half line by [13, 14, 15, 16, 17, 18, 19].

The aim of this paper is to present scattering problem for Sturm-Liouville operator involving fourth order spectral parameter in the boundary condition and investigate the properties of scattering data. More precisely, we will extend the Marchenko method to a more general situtaion in which the boundary condition contains a rational function of spectral parameter.

The remaining paper is organized as follows: In section 2, the required results for boundary value problem (1)-(3) are provided. In Section 3, scattering problem is presented and properties of scattering data are investigated.

2. Preliminaries

This section provides results from the work [1].

As known from [1], if the condition (3) holds, the equation (1) has a unique solution $e(\lambda, x)$ which satisfies the asymptotic behavior, for $Im\lambda \geq 0$,

$$\lim_{x\to+\infty} e^{-i\lambda x} e\left(\lambda, x\right) = 1.$$

This is called Jost solution and can be expressed by

$$e\left(\lambda, x\right) = e^{i\lambda x} + \int\limits_{x}^{\infty} K\left(x, t\right) e^{i\lambda t} dt, \tag{5}$$

where the kernel function $K(x,t)$ satisfies the inequality

$$|K(x,t)| \leq \frac{1}{2}\Omega\left(\frac{x+t}{2}\right) exp\left\{\Omega_1\left(x\right) - \Omega_1\left(\frac{x+t}{2}\right)\right\}$$

and the functions $\Omega(x)$ and $\Omega_1(x)$ have the following notations:

$$\Omega(x) \equiv \int_x^\infty |q(t)|\,dt, \quad \Omega_1(x) \equiv \int_x^\infty \Omega(x)\,dt.$$

Also,

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)\,dt. \tag{6}$$

Moreover, $e(\lambda, x)$ is an analytic function of λ in the upper half plane ($Im\lambda > 0$) and is continuous on the real line. For $Im\lambda \geq 0$, the following estimates hold:

$$|e(\lambda, x)| \leq exp\left\{-Im\lambda x + \Omega_1(x)\right\}, \tag{7}$$

$$\left|e(\lambda, x) - e^{i\lambda x}\right| \leq \left\{\Omega_1(x) - \Omega_1\left(x + \frac{1}{|\lambda|}\right)\right\} exp\left\{-Im\lambda x + \Omega_1(x)\right\} \tag{8}$$

and

$$\left|e'(\lambda, x) - i\lambda e^{i\lambda x}\right| \leq \Omega(x)\, exp\left\{-Im\lambda x + \Omega_1(x)\right\}. \tag{9}$$

The functions $e(\lambda, x)$ and $e(-\lambda, x)$ form a fundamental system of solutions of equation (1) for real $\lambda \neq 0$, and their Wronskian is equal to $2i\lambda$:

$$W\left\{e(\lambda, x), e(-\lambda, x)\right\} = e'(\lambda, x)\, e(-\lambda, x) - e(\lambda, x)\, e'(-\lambda, x) = 2i\lambda.$$

Denote by $\sigma(\lambda, x)$ the solution of (1) satisfying the conditions

$$\sigma(\lambda, 0) = a_0 + a_1\lambda^2 + a_2\lambda^4, \quad \sigma'(\lambda, 0) = b_0 + b_1\lambda^2 + b_2\lambda^4.$$

It is evident that the solution $\sigma(\lambda, x)$ satisfies the boundary condition (2).

3. Scattering data

The direct scattering problem consists of the determination the scattering data when $q(x)$ is known. This section concerns the scattering problem for the boundary value problem (1)-(3) and therefore, we shall obtain scattering data for (1)-(3) and analyze its properties.

Lemma 3.1. *The following identity holds:*

$$\frac{2i\lambda\sigma(\lambda, x)}{P(\lambda)} = \overline{e(\lambda, x)} - S(\lambda)\, e(\lambda, x) \tag{10}$$

for all real $\lambda \neq 0$, and

$$S(\lambda)=\frac{P(-\lambda)}{P(\lambda)}, \tag{11}$$

$$S(-\lambda)=\overline{S(\lambda)}, \quad |S(\lambda)|=1, \tag{12}$$

where

$$P(\lambda)=\left(a_0+a_1\lambda^2+a_2\lambda^4\right)e'(\lambda,0)-(b_0+b_1\lambda^2+b_2\lambda^4)e(\lambda,0). \tag{13}$$

Proof. $e(\lambda,x)$ and $e(-\lambda,x)$ constitute the fundamental solution system of the equation (1) for all real $\lambda\neq 0$. Therefore, we can represent the function $\sigma(\lambda,x)$ as a linear combination of these functions:

$$\sigma(x,\lambda)=\zeta(\lambda)e(\lambda,x)+\eta(\lambda)e(-\lambda,x). \tag{14}$$

The coefficient functions $\zeta(\lambda)$ and $\eta(\lambda)$ are obtained by taking account of the following equalities

$$\begin{aligned}
&W[e(\lambda,x),\sigma(\lambda,x)]=\eta(\lambda)2i\lambda\\
&=\left(a_0+a_1\lambda^2+a_2\lambda^4\right)e'(\lambda,0)-(b_0+b_1\lambda^2+b_2\lambda^4)e(\lambda,0)\\
&W[e(-\lambda,x),\sigma(\lambda,x)]=-\zeta(\lambda)2i\lambda\\
&=\left(a_0+a_1\lambda^2+a_2\lambda^4\right)e'(-\lambda,0)-(b_0+b_1\lambda^2+b_2\lambda^4)e(-\lambda,0),
\end{aligned}$$

and substituted in (14). Hence, we deduce that

$$\sigma(\lambda,x)=-\frac{P(-\lambda)}{2i\lambda}e(\lambda,x)+\frac{P(\lambda)}{2i\lambda}e(-\lambda,x), \tag{15}$$

where the function $P(\lambda)$ is defined by (13).

Let us show $P(\lambda)\neq 0$ for all real $\lambda\neq 0$. Suppose the contrary, then there exists $\mu_0\in\mathbb{R}$, $\mu_0\neq 0$, such that

$$e'(\mu_0,0)=f(\mu_0)e(\mu_0,0).$$

On the other side, we have $W[e(\mu_0,0),e(-\mu_0,0)]=2i\mu_0$. Hence

$$|e(\lambda_0,0)|^2\left(f(\mu_0)-\overline{f(\mu_0)}\right)=2i\mu_0$$

and it follows $0=2i\mu_0$. This is a contradiction since $\mu_0\neq 0$. Thus, by dividing equality (15) by $\frac{1}{2i\lambda}P(\lambda)$, we find (10) where $S(\lambda)$ is defined with (11).

In order to complete proof, we shall show that $S(\lambda)$ satisfies the conditions in (12). For real $\lambda\neq 0$, $\overline{P(\lambda)}=P(-\lambda)$ and hence, it follows that $\overline{S(\lambda)}=S(-\lambda)$. Also,

$$|S(\lambda)|=\left|\frac{P(-\lambda)}{P(\lambda)}\right|=\left|\frac{\overline{P(\lambda)}}{P(\lambda)}\right|=1$$

holds for all real $\lambda\neq 0$, and so it completes the proof. □

The function $S(\lambda)$ defined by (11) is called the scattering function of the boundary value problem (1)-(3).

Let us examine properties of $P(\lambda)$ on the upper half-plane.

Lemma 3.2. *The function $P(\lambda)$ may have only a finite number of zeros on the upper half plane. All zeros are simple and lie on the imaginary axis.*

Proof. By the proof of Lemma (3.1), we have $P(\lambda) \neq 0$ for all real $\lambda \neq 0$, and the point $\lambda = 0$ is the possible zero of $P(\lambda)$. Since $e(\lambda, 0)$ and $e'(\lambda, 0)$ are analytic in the upper plane $Im\lambda > 0$, it follows from the expression $P(\lambda)$ that it has the same property. Hence, we have that the zeros of $P(\lambda)$ are at most countable. Now, we show that the set of zeros is bounded. Assume the contrary, i.e. let the set of zeros be unbounded. Then, there exists the numbers λ_k such that $P(\lambda_k) = 0$ for $Im\lambda_k > 0$ and $|\lambda_k| \to \infty$, and we have

$$e'(\lambda_k, 0) = f(\lambda_k) e(\lambda_k, 0).$$

With the inequality (9)

$$|f(\lambda_k) e(\lambda_k, 0) - i\lambda_k| \leq \Omega(0) \, exp\{\Omega_1(0)\}.$$

Hence

$$|\lambda_k| \leq |f(\lambda_k) e(\lambda_k, 0)| + \Omega(0) \, exp\{\Omega_1(0)\}.$$

Since the equality (8) and $\lim_{k\to+\infty} e(\lambda_k, 0) = 1$, the right side of the equality has a finite limit. We arrived the contradiction. This shows that the set $\{\lambda_k\}$ is bounded. Thus, the set of zeros of $P(\lambda)$ is bounded and form at most countable set having zero the only possible limit point.

Next, we shall present that zeros of $P(\lambda)$ lie on the imaginary axis. Suppose that λ_1 and λ_2 are zeros of $P(\lambda)$. Hence, they satisfy the equation (1):

$$-e''(\lambda_1, x) + q(x) e(\lambda_1, x) = {\lambda_1}^2 e(\lambda_1, x) \tag{16}$$

$$-\overline{e''(\lambda_2, x)} + q(x)\overline{e(\lambda_2, x)} = \overline{\lambda_2}^2 \overline{e(\lambda_2, x)}. \tag{17}$$

We multiply (16) by $\overline{e(\lambda_2, x)}$ and (17) by $e(\lambda_1, x)$, subtract the second from the first, and finally integrate this result according to x over $(0, \infty)$. As a result, we find

$$\left({\lambda_1}^2 - \overline{\lambda_2}^2\right) \int_0^\infty e(\lambda_1, x) \overline{e(\lambda_2, x)} dx - W\left[e(\lambda_1, x), \overline{e(\lambda_2, x)}\right]\Big|_{x=0} = 0. \tag{18}$$

On the other side, we have

$$W\left[e(\lambda_1, x), \overline{e(\lambda_2, x)}\right]\Big|_{x=0} = \left(f(\lambda_1) - \overline{f(\lambda_2)}\right) e(\lambda_1, 0) \overline{e(\lambda_2, 0)}$$

since λ_j $(j=1,2)$ is a zero to $P(\lambda)$. Letting $\lambda_1=\lambda_2=\lambda$, $a(\lambda):=a_0+a_1\lambda^2+a_2\lambda^4$ and substituting in (18), the following is obtained

$$(\lambda^2-\overline{\lambda}^2)\left[\frac{|e(\lambda,0))|^2}{|a(\lambda)|^2}\left\{(a_1b_0-a_0b_1)+(a_2b_1-a_1b_2)|\lambda|^4\right\}+\int_0^\infty |e(\lambda,x)|^2\,dx\right]=0.$$

Because of the the condition (4), the expression in the parentheses is positive and it implies $\lambda^2=\overline{\lambda}^2$. This shows that λ is pure imaginary, i.e. $\lambda=i\mu$, where $\mu\geq 0$.

Now, we shall prove that there are only finitely many zeros. Let δ denote the infimum of the distances between two neighboring zeros of $P(\lambda)$ and show $\delta>0$. Suppose the contrary and let $\{i\lambda_k\}$ and $\left\{i\widehat{\lambda}_k\right\}$ be two sequences of zeros of the function $P(\lambda)$ such that

$$\lim_{k\to\infty}\left(\widehat{\lambda}_k-\lambda_k\right)=0,\quad 0<\lambda_k\leq\widehat{\lambda}_k,\quad max_k\widehat{\lambda}_k<M.$$

Then, it follows from the estimate (8) that, for A large enough, the inequality

$$e(i\lambda,x)>\frac{1}{2}e^{-\lambda x}$$

holds uniformly with respect to $x\in[A,\infty)$ and $\lambda\in[0,\infty)$. Thus, we get

$$\int_A^\infty e\left(i\widehat{\lambda}_k,x\right),\overline{e(i\lambda_k,x)}dx>\frac{1}{4}\frac{e^{-A(\widehat{\lambda}_k+\lambda_k)}}{\left(\widehat{\lambda}_k+\lambda_k\right)}>\frac{e^{-2AM}}{8M}. \tag{19}$$

On the other side, the equality (18) yields

$$0=\int_0^\infty e\left(i\widehat{\lambda}_k,x\right)\overline{e(i\lambda_k,x)}dx+\frac{e\left(i\widehat{\lambda}_k,0\right)\overline{e(i\lambda_k,0)}}{a(i\widehat{\lambda}_k)\overline{a(i\lambda_k)}}\left\{(a_1b_0-a_0b_1)+(a_2b_1-a_1b_2)(\widehat{\lambda}_k\lambda_k)^2\right\}$$

$$=\int_0^A e\left(i\widehat{\lambda}_k,x\right)\left[\overline{e(i\lambda_k,x)}-\overline{e\left(i\widehat{\lambda}_k,x\right)}\right]dx+\int_0^A e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\widehat{\lambda}_k,x\right)}dx$$

$$+\int_A^\infty e\left(i\widehat{\lambda}_k,x\right)\overline{e(i\lambda_k,x)}dx+\frac{e\left(i\widehat{\lambda}_k,0\right)\overline{e(i\lambda_k,0)}}{a(i\widehat{\lambda}_k)\overline{a(i\lambda_k)}}\left\{(a_1b_0-a_0b_1)+(a_2b_1-a_1b_2)(\widehat{\lambda}_k\lambda_k)^2\right\}$$

and letting $k\to\infty$, we obtain

$$\lim_{k\to\infty}\int_A^\infty e\left(i\widehat{\lambda}_k,x\right)\overline{e(i\lambda_k,x)}dx\leq 0 \tag{20}$$

since

$$\lim_{k\to\infty}\left[\overline{e\left(i\lambda_k,x\right)}-\overline{e\left(i\widehat{\lambda}_k,x\right)}\right]=0$$

holds uniformly with respect to $x\in[0,A]$. Comparing (19) and (20), we obtain a contradiction. Therefore, it follows $\delta>0$ and so, this shows that the function $P(\lambda)$ has only a finite number of zeros: $i\lambda_k,\quad k=1,\ldots,n$.

To complete the proof, let us show that all zeros of the function $P(\lambda)$ are simple. By the derivation of the equation

$$-e''\left(\lambda,x\right)+q(x)e\left(\lambda,x\right)=\lambda^2e\left(\lambda,x\right) \tag{21}$$

with respect to λ, we obtain

$$-\dot{e}''\left(\lambda,x\right)+q(x)\dot{e}\left(\lambda,x\right)=\lambda^2\dot{e}\left(\lambda,x\right)+2\lambda e\left(\lambda,x\right), \tag{22}$$

where the dot shows differentiation with respect to λ. Multiplying (21) by $\dot{e}\left(\lambda,x\right)$ and (22) by $e\left(\lambda,x\right)$ and subtracting the second from the first and integrating this relation with respect to x over $(0,\infty)$, we get

$$2\lambda\int_0^\infty e^2\left(\lambda,x\right)dx+W\left[e\left(\lambda,x\right),\dot{e}\left(\lambda,x\right)\right]\Big|_{x=0}=0.$$

Let λ be a zero of the function $P(\lambda)$. Using the expression of the function $P(\lambda)$, the following result is obtained that

$$W\left[e\left(\lambda,x\right),\dot{e}\left(\lambda,x\right)\right]\Big|_{x=0}$$
$$=-\frac{\dot{P}(\lambda)e\left(\lambda,0\right)}{a(\lambda)}+\frac{e^2\left(\lambda,0\right)}{a^2(\lambda)}2\lambda\left[(a_1b_0-a_0b_1)+(a_2b_1-a_1b_2)\lambda^4)\right]$$

and hence, we find that

$$\frac{\dot{P}(\lambda)e\left(\lambda,0\right)}{a(\lambda)}=2\lambda\left\{\int_0^\infty e^2\left(\lambda,x\right)dx+\frac{e^2\left(\lambda,0\right)}{a^2(\lambda)}\left[(a_1b_0-a_0b_1)+(a_2b_1-a_1b_2)\lambda^4)\right]\right\}. \tag{23}$$

If we substitute $\lambda=i\mu_k$, $\mu_k>0$, in (23) and multiply $-i$, then we get that the right side of the equality is positive. Therefore, $\dot{P}(i\mu_k)\neq0$, i.e. the zeros of $P(\lambda)$ are simple. The lemma is proved. □

Denote

$$m_k^{-2}\equiv\int_0^\infty\left|e\left(i\lambda_k,x\right)\right|^2dx+\frac{\left|e\left(i\lambda_k,0\right)\right|^2}{\left|a(i\lambda_k)\right|^2}\left[(a_1b_0-a_0b_1)+(a_2b_1-a_1b_2)\lambda_k^4)\right].$$

The numbers $m_k, k=1,\ldots,n$ are called *norming numbers* for the boundary value problem (1)-(3).

Definition 3.3. *The collection* $\{S(\lambda); i\lambda_1, ..., i\lambda_n; m_1, ..., m_n\}$ *is called the scattering data of the boundary value problem (1)-(3).*

With the help of the solution (5), we get

$$P(\lambda) = \left(a_0 + a_1\lambda^2 + a_2\lambda^4\right)\left\{i\lambda - K(0,0) + \int_0^\infty K_x(0,t)\, e^{i\lambda t} dt\right\}$$

$$-(b_0 + b_1\lambda^2 + b_2\lambda^4)\left\{1 + \int_0^\infty K(0,t)\, e^{i\lambda t} dt\right\}$$

$$= \lambda^5[ia_2 + O\left(\frac{1}{\lambda}\right)]$$

if $a_2 \neq 0$ as $|\lambda| \to \infty$. In a similar way, it follows that

$$P(-\lambda) = \left(a_0 + a_1\lambda^2 + a_2\lambda^4\right)\left\{-i\lambda - K(0,0) + \int_0^\infty K_x(0,t)\, e^{-i\lambda t} dt\right\}$$

$$-(b_0 + b_1\lambda^2 + b_2\lambda^4)\left\{1 + \int_0^\infty K(0,t)\, e^{-i\lambda t} dt\right\}$$

$$= \lambda^5[-ia_2 + O\left(\frac{1}{\lambda}\right)].$$

Taking these into account, we conclude that

$$-1 - S(\lambda) = O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty.$$

In the case that $a_2 = 0$, we get

$$P(\lambda) = \lambda^4[-b_2 + O\left(\frac{1}{\lambda}\right)]$$

and

$$P(-\lambda) = \lambda^4[-b_2 + O\left(\frac{1}{\lambda}\right)]$$

as $|\lambda| \to \infty$. Hence, the following result is obtained

$$1 - S(\lambda) = O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty.$$

Let us define

$$S_0(\lambda) = \begin{cases} 1, & a_2 = 0 \\ -1, & a_2 \neq 0. \end{cases}$$

$S_0(\lambda) - S(\lambda) \in L_2(-\infty, \infty)$ and the function

$$F_S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (S_0(\lambda) - S(\lambda)) e^{i\lambda x} d\lambda$$

belongs to the space $L_2(-\infty, \infty)$.

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