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Inverse scattering problem for Sturm-Liouville equation with a rational function of spectral parameter in boundary condition

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Abstract. The inverse scattering problem is analyzed for the Sturm-Liouville equation on the half line $[0,\infty)$ with a rational function of spectral parameter in the boundary condition. The main equation is derived, its solvability is proved and it is presented that the potential is uniquely recovered in terms of the scattering data.

Key Words and Phrases: Sturm-Liouville equation; nonlinear spectral parameter; inverse problem; main equation.
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1. Introduction

The determination of potential from an appropriate set of spectral data is known as inverse problem of scattering theory. Problems with spectral parameters in equations and boundary conditions are extremely important in spectral theory. Sturm Liouville problems appear in studies of heat conduction problems and vibrating string problems when boundary condition contains spectral parameter [1]. Many examples of spectral problems which arise in mechanical engineering and contain eigen parameter in the boundary conditions were presented in the book [2].

In this work let us consider the boundary value problem generated by the differential equation and boundary condition:

$$-v''(x) + \{q(x) - \lambda^2\}v(x) = 0, \quad (0 \le x < \infty) \tag{1}$$

$$v'(0) - f(\lambda)\, v(0) = 0, \tag{2}$$

where λ is a spectral parameter, $q(x)$ is real valued function with the condition

$$\int_0^\infty (1+x)\,|q(x)|\,dx < \infty \tag{3}$$

and

$$f(\lambda) = \frac{b_0 + b_1\lambda^2 + b_2\lambda^4}{a_0 + a_1\lambda^2 + a_2\lambda^4}$$

for $\alpha_i,\ \beta_j \in \mathbb{R}\ \ (i,j = 0,1,2)$

$$a_1b_0 - b_0a_1 \geq 0, \quad a_2b_1 - b_1a_2 \geq 0, \quad a_2b_0 - b_2a_0 = 0. \tag{4}$$

Marchenko [3] and Levitan [4] studied inverse scattering problem of the Sturm-Liouville operator on the half line with a boundary condition at the origin when there is no spectral parameter. Inverse problem for second order differential operator pencil on the axis was studied in [5]. Problems with boundary conditions depending on spectral parameter were examined in finite interval by several authors [6, 7, 8, 9, 10, 11, 12] and on the half line by [13, 14, 15, 16, 17, 18, 19, 20].

The main goal of this paper is to introduce inverse scattering problem for Sturm-Liouville equation involving fourth order spectral parameter in the boundary condition. More precisely, we will extend the Marchenko method to a more general situtaion in which the boundary condition contains a rational function of spectral parameter.

The remaining paper is organized as follows: In section 2, the required results for boundary value problem (1)-(3) are provided. In Section 3, the main equation is derived, which is needed in recovering potential $q(x)$ uniquely. Finally, in Section 4, we present the uniqueness and reconstruction of the potential of equation (1).

2. Preliminaries

This section provides basic tools and results from the work [20] which allow us to achieve this research.

It is well known [3] when the condition (3) holds, the equation (1) has a unique solution $e(\lambda, x)$ which satisfies the asymptotic behavior, for $Im\lambda \geq 0$,

$$\lim_{x\to+\infty} e^{-i\lambda x} e(\lambda, x) = 1.$$

This is called Jost solution and can be expressed by

$$e(\lambda, x) = e^{i\lambda x} + \int_x^\infty K(x,t)\, e^{i\lambda t} dt, \tag{5}$$

where the kernel function $K(x,t)$ satisfies the inequality

$$|K(x,t)| \leq \frac{1}{2}\Omega\left(\frac{x+t}{2}\right) exp\left\{\Omega_1(x) - \Omega_1\left(\frac{x+t}{2}\right)\right\}$$

and the functions $\Omega(x)$ and $\Omega_1(x)$ have the following notations:

$$\Omega(x) \equiv \int_x^{\infty} |q(t)|\, dt, \quad \Omega_1(x) \equiv \int_x^{\infty} \Omega(x)\, dt.$$

Also,

$$K(x,x) = \frac{1}{2}\int_x^{\infty} q(t)\, dt. \tag{6}$$

Denote by $\sigma(\lambda, x)$ the solution of (1) satisfying the conditions

$$\sigma(\lambda, 0) = a_0 + a_1\lambda^2 + a_2\lambda^4, \quad \sigma'(\lambda, 0) = b_0 + b_1\lambda^2 + b_2\lambda^4.$$

Lemma 2.1. *[20] The following identity holds:*

$$\frac{2i\lambda\sigma(\lambda, x)}{P(\lambda)} = \overline{e(\lambda, x)} - S(\lambda)\, e(\lambda, x) \tag{7}$$

for all real $\lambda \neq 0$, and

$$S(\lambda) = \frac{P(-\lambda)}{P(\lambda)}, \tag{8}$$

$$S(-\lambda) = \overline{S(\lambda)}, \quad |S(\lambda)| = 1, \tag{9}$$

where

$$P(\lambda) = \left(a_0 + a_1\lambda^2 + a_2\lambda^4\right) e'(\lambda, 0) - (b_0 + b_1\lambda^2 + b_2\lambda^4)e(\lambda, 0).$$

The function $S(\lambda)$ defined by (8) is called the scattering function of the boundary value problem (1)-(3).

Lemma 2.2. *[20] The function $P(\lambda)$ may have only a finite number of zeros on the upper half plane. All zeros are simple and lie on the imaginary axis.*

The zeros $i\lambda_k$, $(\lambda_k > 0)$, $k = 1, \ldots, n$, of the function $P(\lambda)$ are called the singular values of the boundary value problem (1)-(3).

The numbers m_k, $k = 1, \ldots, n$ are defined with

$$m_k^{-2} \equiv \int_0^{\infty} |e(i\lambda_k, x)|^2\, dx + \frac{|e(i\lambda_k, 0)|^2}{|a(i\lambda_k)|^2}\left[(a_1b_0 - a_0b_1) + (a_2b_1 - a_1b_2)\lambda_k^4)\right] \tag{10}$$

and called *norming numbers* for the boundary value problem (1)-(3).

Definition 2.3. *[20] The collection $\{S(\lambda); i\lambda_1, ..., i\lambda_n; m_1, ..., m_n\}$ is called the scattering data of the boundary value problem (1)-(3).*

With the help of the solution (5), we have

$$S_0(\lambda) - S(\lambda) = O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty,$$

where

$$S_0(\lambda) = \begin{cases} 1, & a_2 = 0 \\ -1, & a_2 \neq 0. \end{cases}$$

$S_0(\lambda) - S(\lambda) \in L_2(-\infty, \infty)$ and the function

$$F_S(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} (S_0(\lambda) - S(\lambda)) e^{i\lambda x} d\lambda$$

belongs to the space $L_2(-\infty, \infty)$.

3. The main equation

In this section, we shall present the main equation in order to discuss inverse scattering problem.

Theorem 3.1. *For every fixed $x \geq 0$, the kernel $K(x,t)$ of the solution (5) satisfies the integral equation*

$$K(x,y) + F(x+y) + \int\limits_{x}^{\infty} K(x,t) F(t+y) dt = 0, \quad y > x \tag{11}$$

where

$$F(x) = \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x} + \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} (S_0(\lambda) - S(\lambda)) e^{i\lambda x} d\lambda. \tag{12}$$

The integral equation (11) is called the *main equation* for the boundary value problem (1)-(3).

Proof. In order to obtain the main equation for the kernel $K(x,t)$ of the solution (5), we use the equality (7) derived in Lemma 2.1. By rewriting the identity (7), we provide the following result

$$\frac{2i\lambda\sigma(\lambda,x)}{P(\lambda)} - e^{-i\lambda x} + S_0(\lambda) e^{i\lambda x} = [S_0(\lambda) - S(\lambda)] e^{i\lambda x} + \int\limits_{x}^{\infty} K(x,t) e^{-i\lambda t} dt$$

$$-\int_x^\infty S_0(\lambda)K(x,t)\,e^{i\lambda t}dt+\int_x^\infty [S_0(\lambda)-S(\lambda)]\,K(x,t)\,e^{i\lambda t}dt.$$

Multiplying both sides of this equality by $\frac{1}{2\pi}e^{i\lambda y}$ and integrating it according to λ over $(-\infty,\infty)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left[\frac{2i\lambda\sigma(\lambda,x)}{P(\lambda)}-e^{-i\lambda x}+S_0(\lambda)e^{i\lambda x}\right]e^{i\lambda y}d\lambda=\frac{1}{2\pi}\int_{-\infty}^\infty [S_0(\lambda)-S(\lambda)]\,e^{i\lambda(x+y)}d\lambda$$
$$+\frac{1}{2\pi}\int_{-\infty}^\infty\int_x^\infty K(x,t)\,e^{-i\lambda(t-y)}dtd\lambda-\frac{1}{2\pi}\int_{-\infty}^\infty\int_x^\infty S_0(\lambda)K(x,t)\,e^{i\lambda(t+y)}dtd\lambda$$
$$+\frac{1}{2\pi}\int_{-\infty}^\infty\int_x^\infty [S_0(\lambda)-S(\lambda)]\,K(x,t)\,e^{i\lambda(t+y)}dtd\lambda. \qquad (13)$$

Here

$$\frac{1}{2\pi}\int_{-\infty}^\infty\int_x^\infty K(x,t)\,e^{-i\lambda(t-y)}dtd\lambda=\int_x^\infty K(x,t)\,\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda(t-y)}d\lambda dt$$
$$=\int_x^\infty K(x,t)\,\delta(t-y)dt=K(x,y)$$

and since $K(x,-y)=0$ for $y>x$, it yields

$$\frac{1}{2\pi}\int_{-\infty}^\infty\int_x^\infty S_0(\lambda)K(x,t)\,e^{i\lambda(t+y)}dtd\lambda=\int_x^\infty S_0(\lambda)K(x,t)\,\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\lambda(t+y)}d\lambda dt$$
$$=S_0(\lambda)\int_x^\infty K(x,t)\,\delta(-t-y)dt=S_0(\lambda)K(x,-y)=0.$$

Therefore, on the right of (13), we obtain

$$K(x,y)+F_s(x+y)+\int_x^\infty K(x,t)F_s(t+y)dt,\quad y>x$$

where

$$F_S(x)=\frac{1}{2\pi}\int_{-\infty}^\infty (S_0(\lambda)-S(\lambda))e^{i\lambda x}d\lambda. \qquad (14)$$

On the other side, using the residue theorem and Jordan's lemma, we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2i\lambda\sigma(\lambda,x)}{P(\lambda)}-e^{-i\lambda x}+S_0(\lambda)e^{i\lambda x}\right]e^{i\lambda y}d\lambda=-\sum_{k=1}^{n}\frac{2i\lambda_k\sigma\left(i\lambda_k,x\right)}{\dot{P}\left(i\lambda_k\right)}e^{-\lambda_k y}.$$

Taking formula (10) into account, we can transform this expression to the form

$$-\sum_{k=1}^{n}\frac{2i\lambda_k\sigma\left(i\lambda_k,x\right)}{\dot{P}\left(i\lambda_k\right)}e^{-\lambda_k y}=-\sum_{k=1}^{n}\frac{2i\lambda_k a(i\lambda_k)}{e\left(i\lambda_k,0\right)\dot{P}\left(i\lambda_k\right)}e\left(i\lambda_k,x\right)e^{-\lambda_k y}$$
$$=-\sum_{k=1}^{n}m_k^2 e\left(i\lambda_k,x\right)e^{-\lambda_k y}=-\sum_{k=1}^{n}m_k^2\left[e^{-\lambda_k(x+y)}+\int_x^{\infty}K(x,t)e^{-\lambda_k(t+y)}dt\right].$$

Substituting this value into the left side of (13), we get the desired integral equation (11) and $F(x)$ is defined by the formula (12). This completes the proof of theorem. □

4. Solvability of the main equation

The inverse scattering problem deals with the recovery $q(x)$ from the scattering data of the boundary value problem (1)-(3). To determine $q(x)$, it is evident from (6) that it is sufficient to know $K(x,t)$. Therefore, suppose that the data $\{S(\lambda);i\lambda_1,...,i\lambda_n;m_1,...,m_n\}$ is given. We construct the equation (11), take kernel $K(x,t)$ as unknown and regard the equation as a Fredholm-type equation for every fixed x.

In this part, we shall investigate the solvability of the main equation and show that the potential $q(x)$ can be recovered uniquely from the scattering data.

Theorem 4.1. *For every fixed $x\geq 0$, the main equation (11) has a unique solution $K(x,.)$ in the space $L_1[x,\infty)$.*

Proof. Assume that the collection $\{S(\lambda);i\lambda_1,...,i\lambda_n;m_1,...,m_n\}$ is the scattering data of the boundary value problem (1)-(3). The function $F(x)$ is written by the formula (12). The transition function $F(x)$ possesses similar properties to the transition function of the problem without the spectral parameter in the boundary condition. Thus, applying Theorem 2.3.1 in [3], the result is obtained that the equation (11) has a unique solution $K(x,y)$. This proves the theorem. □

Corollary 4.2. *The scattering data of the boundary value problem (1)-(3) determines potential $q(x)$ in equation (1) uniquely.*

Proof. The main equation (11) is constructed only on the basis of the scattering data and by Theorem 4.1, it has a unique solution $K(x, y)$ for every $x \geq 0$. Hence, we have the kernel $K(x, y)$ of the solution (5) and find the potential function with

$$q(x) = -2\frac{d}{dx}K(x, x).$$

□

References

[1] C.T. Fulton, S. Pruess. Numerical methods for a singular eigenvalue problem with eigenparameter in the boundary conditions. J. Math. Anal. Appl., 71(2): 431-462, 1979.

[2] L. Collatz. Eigenwertaufgaben mit technischen Anwendungen. Akad. Verlagsgesellschaft Geest & Portig, Leipzig, 1963, Germany.

[3] V.A. Marchenko. Sturm-Liouville operators and applications. Birkhäuser, Basel, 1986.

[4] B.M. Levitan. Inverse Sturm-Liouville problems. VSP, Zeist, 1987. Translated from the Russian by O. Efimov.

[5] R.F. Efendiev. Spectral analysis for one class of second-order indefinite non-self-adjoint differential operator pencil. Applicable Analysis, 90(12): 1837-1849, 2011.

[6] C.T. Fulton. Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 77(3-4): 293-308, 1977.

[7] P.A. Binding, P.J. Browne, B.A. Watson. Sturm Liouville problems with boundary conditions rationally dependent on the eigenparameter, II. Journal of Computational and Applied Mathematics, 148: 147-168, 2002.

[8] V. Pivovarchik. Direct and inverse problem for a damped string. J. Operator Theory, 42(1): 189-220, 1999.

[9] M.V. Chugunova. Inverse spectral problem for the Sturm-Liouville operator with eigenvalue parameter dependent boundary conditions. Oper. Theory Adv. Appl., 123: 187-194, 2001.

[10] A. Chernozhukova, G. Freiling. A uniqueness theorem for the boundary value problems with non-linear dependence on the spectral parameter in the boundary conditions. Inverse Problems in Science and Engineering, 17: 777-785, 2009.

[11] L.I. Mammadova, I.M. Nabiev. Spectral properties of the Sturm–Liouville operator with a spectral parameter quadratically included in the boundary condition. Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 30(2): 237–248, 2020.

[12] Ch.G. Ibadzadeh, L.I. Mammadova, I.M. Nabiev. Inverse problem of spectral analysis for diffusion operator with nonseparated boundary conditions and spectral parameter in boundary condition. Azerbaijan Journal of Mathematics, 9(1): 171-189, 2019.

[13] E.A. Pocheikina-Fedotova. The inverse boundary value problem on the half-axis for a second order equation. Izv. Vyssh. Uchebn. Zaved. Mat., 7: 75-84, 1972.

[14] Kh.R. Mamedov. Uniqueness of the solution to the inverse problem of scattering theory for the Sturm–Liouville operator with a spectral parameter in the boundary condition. Mathematical Notes, 74(1): 142-146, 2003.

[15] Kh.R. Mamedov, N.P. Kosar. Inverse scattering problem for Sturm-Liouville operator with nonlinear dependence on the spectral paramater in the boundary condition. Math. Meth. Appl. Sci., 34: 231-241, 2011.

[16] Kh.R. Mamedov, H. Menken. On the inverse problem of scattering theory for a differential operator of the second order. North-Holland Mathematics Studies, 197: 185-194, 2004.

[17] H. Menken, Kh.R. Mamedov. On the inverse problem of the scatterıng theory for a boundary-value problem. Geometry, Integrability and Quantization, 1: 226–236, 2006.

[18] K.R. Mamedov, F.A. Cetinkaya. Boundary value problem for a Sturm-Liouville operator with piecewise continuous coefficient. Hacettepe Journal of Mathematics and Statistics, 44(4): 867-874, 2015.

[19] A. Çöl. Inverse spectral problem for Sturm-Liouville operator with discontinuous coefficient and cubic polynomials of spectral parameter in boundary condition. Adv Differ Equ., 2015: 132, 2015.

[20] A. Çöl. On a scattering problem for Sturm-Liouville equation with a rational function of spectral parameter in boundary condition. Journal of Contemporary Applied Mathematics, 12(2): 73-82, 2022.

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