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Algebraic Points of any Given Degree on the Affine Curves $y^2 = x(x + 2p)(x + 4p)(x^2 - 8p^2)$

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Abstract. We give the set of algebraic points of any given degree of a family curves :

$$\mathcal{C}_p : y^2 = x(x + 2p)(x + 4p)(x^2 - 8p^2) \quad \text{with } p \equiv 7\,[16]$$

These curves are described by E.V. Flynn and J. Redmond in [4], who showed that the Mordell-Weill group is finite and explained the generators of the torsion group for this family curves.
Key Words and Phrases : Degree of algebraic point; curves; Mordell-Weil Group, Jacobian, Linear system.
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1. Introduction and main result

1.1. Introduction

Let $\mathcal{C}$ be a smooth projective plane curve defined over $\mathbb{Q}$. For all algebraic extension field K of $\mathbb{Q}$, we denote by $C(K)$ the set of K-rational points of $\mathcal{C}$ on K and by $\mathcal{C}^{(d)}(\mathbb{Q})$ the set of algebraic points of degree d over $\mathbb{Q}$. The degree of an algebraic point R is the degree of its field of definition on $\mathbb{Q}$ i.e $deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$. A famous theorem of Fatlings [3] shows that if $\mathcal{C}$ is a smooth projective plane curve defined over K of genus $g \geq 2$, then $\mathcal{C}(K)$ is finite. Fatling's proof is still ineffective in the sense that it does not provide an algorithm for computing $\mathcal{C}(K)$.
Currently for curve $\mathcal{C}$ defined over a numbers field K of genus $g \geq 2$, there is no know algorithm for computing the set $\mathcal{C}(K)$ or for deciding if $\mathcal{C}(K)$ is empty. But there is a bag of strikes that can be used to show that $\mathcal{C}(K)$ is empty, or to determine $\mathcal{C}(K)$ if it is not empty. These include local method, Chabauty method [2], Descent method [5], Mordell-Weil sieves method [1]. These methods often succeed with less than full knowledge of the jacobian of the curve. If it is finite it is not hard to determine $\mathcal{C}(Q)$ and to generalize for all number field K.
The purpose of this note is to determine a parametrization of the set $\mathcal{C}_p^{(l)}(\mathbb{Q})$ on the curve

$$\mathcal{C}_p : y^2 = x(x + 2p)(x + 4p)(x^2 - 8p^2) \quad \text{with } p \equiv 7\,[16]$$

The curve $\mathcal{C}_p$ studied in [4], has rank 0 when $p \equiv 7[16]$, so the Mordell-Weil groupe of the jacobian $\mathcal{J}(\mathbb{Q})$ is finite.
We denote by $\mathcal{J}$ the Jacobian of $\mathcal{C}_p$ and by $j(P)$ the class $[P-\infty]$ of $P-\infty$, i.e. j is the Jacobian folding :

$$\begin{array}{cccc} j & : & \mathcal{C}_p & \longrightarrow & \mathcal{J}(\mathbb{Q}), \\ & & P & \longmapsto & [P-\infty] \end{array}$$

where $\mathcal{J}(\mathbb{Q})$ represents the Mordell-Weil group of rational points of the Jacobian of $\mathcal{C}_p$; this group is finite (cf. [4]).

1.2. Main result

Our main result is the following theorem :

Theorem 1.1. *The set of algebraic points of degree at most l on $\mathbb{Q}$ on the curve $\mathcal{C}_p$ of affine equation $y^2 = x(x+2p)(x+4p)(x^2-8p^2)$ is given by :*

$$\mathcal{F} = \mathcal{F}_1 \bigcup \left(\bigcup_{k\in\{0,\ 1,\ 2\}} \mathcal{F}_2^k \right) \bigcup \left(\bigcup_{\substack{\ell, \jmath \in \{0,1,2\} \\ \ell \neq \jmath}} \mathcal{F}_3^{\ell,\jmath} \right) \bigcup \mathcal{F}_4$$

with :

$$\mathcal{F}_1 = \left\{ \left(x,\ -\frac{\sum_{i=0}^{\frac{l}{2}} a_i x^i}{\sum_{j=0}^{\frac{l-5}{2}} b_j x^j} \right) \middle| \begin{array}{c} a_0 \text{ and } b_0 \text{ not simultaneously zero,} \\ a_{\frac{l}{2}} \neq 0 \text{ if } l \text{ is even, } b_{\frac{l-5}{2}} \neq 0 \text{ if } l \text{ is} \\ \text{is odd and } x \text{ is a solution} \\ \text{of the equation :} \end{array} \right. \\ \left. \left(\sum_{i=0}^{\frac{l}{2}} a_i x^i \right)^2 = \left(\sum_{j=0}^{\frac{l-5}{2}} b_j x^j \right)^2 x(x+2p)(x+4p)(x^2-8p^2) \right\}$$

$$\mathcal{F}_2^k = \left\{ \left(x, -\frac{\sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i\right)}{\sum_{j=0}^{\frac{l-4}{2}} b_j x^j} \right) \middle| \begin{array}{c} a_{\frac{l+1}{2}} \neq 0 \text{ if } l \text{ is even, } b_{\frac{l-4}{2}} \neq 0 \\ \text{if } l \text{ is odd and } x \text{ solution of} \\ \text{the equation :} \end{array} \right. \\ \left(\sum_{i=1}^{\frac{l+1}{2}} a_i \left(\frac{x^i + \delta_k^i}{x^{\frac{1}{2}}} \right) \right)^2 = \left(\sum_{j=0}^{\frac{l-4}{2}} b_j x^j \right)^2 (x+2p)(x+4p)(x^2-8p^2) \\ \text{with} \\ \left. \delta_0 = 0, \quad \delta_1 = -2p \quad \text{and} \quad \delta_2 = -4p \right\}$$

$$\mathcal{F}_3^{\ell,\jmath} = \left\{ \begin{array}{c} \left(x, -\frac{\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i + \omega^i\right)}{\sum_{j=0}^{\frac{l-4}{2}} b_j x^j} \right) \quad \Bigg| \quad \begin{array}{c} a_{\frac{l+2}{2}} \neq 0 \ si\ l\ is\ even,\ b_{\frac{l-3}{2}} \neq 0 \\ if\ l\ is\ odd\ and\ x\ solution\ of \\ the\ equation: \end{array} \\ \left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\frac{x^i + \omega^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x+2p)(x+4p)(x^2-8p^2) \\ with \\ \omega^i = -\frac{1}{2}\left(\delta_\ell^i + \delta_\jmath^i\right)\ and\ \delta_\ell,\ \delta_\jmath \in \{0, -2p, -4p\},\ such\ that\ \delta_\ell \neq \delta_\jmath \\ and\ \delta_0 = 0,\ \delta_1 = -2p \quad and\ \delta_2 = -4p \end{array} \right\}$$

$$\mathcal{F}_4 = \left\{ \begin{array}{c} \left(x, -\frac{\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i + \nu^i\right)}{\sum_{j=0}^{\frac{l-3}{2}} b_j x^j} \right) \quad \Bigg| \quad \begin{array}{c} a_{\frac{l+2}{2}} \neq 0\ if\ l\ is\ even,\ b_{\frac{l-3}{2}} \neq 0 \\ if\ l\ is\ odd\ x\ solution\ of \\ the\ equation: \end{array} \\ \left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\frac{x^i + \nu^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x+2p)(x+4p)(x^2-8p^2) \\ with \\ \nu^i = -\frac{1}{2}\left(\left(2i\sqrt{2}p\right)^i + \left(-2i\sqrt{2}p\right)^i \right) \end{array} \right\}.$$

2. Auxiliary results

For a divisor D on $\mathcal{C}_p$, let $\mathcal{L}(\mathfrak{D})$ denote the $\overline{\mathbb{Q}}$-vector space of rational functions f defined over $\mathbb{Q}$ such that $f = 0$ or $div(f) \geq -D$; $l(D)$ denotes the $\overline{\mathbb{Q}}$-dimension of $\mathcal{L}(D)$.

Lemma 2.1. *We have :* $\mathcal{J}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Proof. See[4]

□

Let P_0, P_1, P_2 and P_∞ be the points of $\mathcal{C}_p$, defined by : $P_0 = [0 : 0 : 1]$, $P_1 = [-2p : 0 : 1]$, $P_2 = [-4p : 0 : 1]$, $P_3 = [2\sqrt{2}p : 0 : 1]$, $P_4 = [-2\sqrt{2}p : 0 : 1]$ and $P_\infty = [0 : 1 : 0]$.
We have the following lemma :

Lemma 2.2. *For curve* $\mathcal{C}_p : y^2 = x(x+2p)(x+4p)(x^2-8p^2)$. *We have :*

(i) $div(x) = 2P_0 - 2P_\infty$,

(ii) $div(y) = P_0 + P_1 + P_2 + P_3 + P_4 - 5P_\infty$.

Proof. Let x, y be the affine coordinates and X, Y, Z the projective coordinates, $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$.

(i) $div(x) = div\left(\frac{X}{Z}\right) = (X = 0) \cdot \mathcal{C}_p - (Z = 0) \cdot \mathcal{C}_p \qquad (**)$.

- For $X = 0$, it follows from $(**)$ that : $Y^2 = 0$ where $Z^3 = 0$.
  This gives the points : $P_0 = [0 : 0 : 1]$ and $P_\infty = [0 : 1 : 0]$ with a the order of the multiplication 2 and 3 respectively.

$$(X = 0) \cdot \mathcal{C}_p = 2P_0 + 3P_\infty \tag{1}$$

- Similarly for $Z = 0$, this implies that : $X^5 = 0$.
  We therefore obtain the point $P_\infty = [0 : 1 : 0]$ with a multiplicitous order equal to 5. Hence

$$(Z = 0) \cdot \mathcal{C}_p = 5P_\infty. \tag{2}$$

  From relations (1) and (2), we deduce that :

$$div(x) = 2P_0 - 2P_\infty.$$

(ii) let's calculate : $div(x - \eta_k)$ for $\eta_k \in \left\{-2p, -4p, 2\sqrt{2}p, -2\sqrt{2}p\right\}$ with $P_k = = [\eta_k : 0 : 1]$. We have

$$div(x - \eta_k) = (X = \eta_k Z) \cdot \mathcal{C}_p - (Z = 0) \cdot \mathcal{C}_p$$

- For $X = \eta_k Z$, it follows that : $Y^2 = 0$ or $Z^3 = 0$. This gives the points $P_k = [\eta_k : 0 : 1]$ and $P_\infty = [0 : 1 : 0]$ with a multiplicative order 2 et 3 respectively. Hence

$$(X = Z) \cdot \mathcal{C}_p = 2P_k + 3P_\infty. \tag{3}$$

- For $Z = 0$, we find the relation (2).
  Thus from relations (2) and (3), we deduce that :

$$div(x - \eta_k) = 2P_k - 2P_\infty.$$

- We have :

$$y^2 = \prod_{k=0}^{4}(x - \eta_k) \Longrightarrow 2div(y) = \sum_{k=0}^{4} div(x - \eta_k)$$

so $div(y) = P_0 + P_1 + P_2 + P_3 + P_4 - 5P_\infty$.

□

Corollary 2.3. *The following results are the consequences of Lemma 2*

- $j(P_0) + j(P_1) + j(P_2) + j(P_3) + j(P_4) = 0,$
- $2j(P_0) = 2j(P_1) = 2j(P_2) = 2j(P_3) = 2j(P_4) = 0$

The following lemma give a basis of $\mathcal{J}(\mathbb{Q})$.

Lemma 2.4. $\mathcal{J}(\mathbb{Q}) = \langle [P_0 - P_\infty],\ [P_1 - P_\infty],\ [P_2 - P_\infty] \rangle$.

Proof. See [4] □

Remark 2.5. *We have* $\mathcal{J}(\mathbb{Q}) = \langle\ j(P_0),\ j(P_1),\ j(P_2)\ \rangle$
For all $x \in \mathcal{J}(\mathbb{Q})$, $x = n_0 j(P_0) + n_1 j(P_1) + n_2 j(P_2)$ *whith* $n_k \in \{0\ ,\ 1\}$ *for all* $k \in \{0\ ,\ 1,\ 2\}$.

Lemma 2.6. *A* $\mathbb{Q}$*-base of* $\mathcal{L}(mP_\infty)$ *is given by :*

$$\mathcal{B}_m = \left\{x^i,\ 0 \le i \le \frac{m}{2}\right\} \bigcup \left\{yx^j,\ 0 \le j \le \frac{m-5}{2}\right\}$$

Proof. It is easy to show that $\mathcal{B}_m$ is a free family, it then remains toshow that card $\mathcal{B}_m = \dim \mathcal{L}(mP_\infty)$. Since the curve is of genus 2 (see [4]), according to the Riemann-Roch theorem, we have $\dim \mathcal{L}(mP_\infty) = m - g + 1 = m - 1$ dès que $m \ge 2g - 1 = 3$. Two cases are possible :

1^{st} **case :** suppose that m is even, then $m = 2h$, we get :
$i \le \frac{m}{2} \Leftrightarrow i \le \frac{2h}{2} = h$ the same $j \le \frac{m-5}{2} \Leftrightarrow j \le \frac{2h-5}{2} \Leftrightarrow j \le h - \frac{5}{2} \Longrightarrow j < h - \frac{4}{2} = h - 2 \Longrightarrow j \leqslant h - 3$. So we have :

$$\mathcal{B}_m = \left\{1,\ x,\ \ldots, x^h\right\} \bigcup \left\{y,\ yx,\ \ldots, yx^{h-3}\right\}.$$

It follows that :

$$\text{card}\, \mathcal{B}_m = h + 1 + h - 3 + 1 = 2h - 1 = m - 1 = \dim \mathcal{L}(mP_\infty).$$

$2^{\mathbf{nd}}$ **case :** suppose that m is odd, then $m = 2h + 1$, we get :
$i \le \frac{m}{2} \Leftrightarrow i \le \frac{2h+1}{2} \Leftrightarrow i \le h + \frac{1}{2} \Longrightarrow i < h + 1 \Longrightarrow i \leqslant h$ the same $j \le \frac{m-5}{2} \Leftrightarrow j \le \frac{2h-4}{2} = h - 2$. Thus we have :

$$\mathcal{B}_m = \left\{1,\ x,\ \ldots, x^h\right\} \bigcup \left\{y,\ yx,\ \ldots, yx^{h-2}\right\}.$$

It follows that :

$$\text{card}\, \mathcal{B}_m = h + 1 + h - 2 + 1 = 2h = m - 1 = \dim \mathcal{L}(mP_\infty).$$

□

3. Proof of the main theorem

Let $R \in \mathcal{C}_p(\overline{\mathbb{Q}})$ with $[\mathbb{Q}(R):\mathbb{Q}] = l$. Consider $R_1, \ldots, R_l$ the Galois conjugates of R and let $t = [R_1 + \ldots + R_l - lP_\infty] \in \mathcal{J}(\mathbb{Q})$. From Remark 1, we have $t = -n_0 j(P_0) - n_1 j(P_1) + n_2 j(P_2)$ whith $n_k \in \{0\ ,\ 1\}$ for all $k \in \{0,\ 1,\ 2\}$ and hence

$$[R_1 + \ldots + R_l - lP_\infty] \;=\; \left[\left(\sum_{k=0}^{2} n_k\right) P_\infty - n_0 P_0 - n_1 P_1 - n_2 P_2\right]$$

whith $n_k \in \{0\ ,\ 1\}$ for all $k \in \{0,\ 1,\ 2\}$. This gives the following formula

$$\left[R_1 + \ldots + R_l + n_0 P_0 + n_1 P_1 + n_2 P_2 - \left(l + \sum_{k=0}^{2} n_k\right) P_\infty\right] \;=\; 0$$

whith $n_{k,\ k\in\{0,\ 1,\ 2\}} \in \{0\ ,\ 1\}$. According to Abel Jacobi's theorem ([**?** , page 156]), there exists a rational function f defined on $\mathbb{Q}$ such that :

$$div(f) \;=\; R_1 + \ldots + R_l + n_0 P_0 + n_1 P_1 + n_2 P_2 - \left(l + \sum_{k=0}^{2} n_k\right) P_\infty \qquad (\star)$$

Four cases are possible :

Case 1: $n_k = 0$**,** $\forall\, i \in \{0,\ 1,\ 2\}$**.**
The formula $(\star)$ becomes : $div(f) = R_1 + \ldots + R_l - lP_\infty$, donc $f \in \mathcal{L}(lP_\infty)$. According to Lemma 4, we have : $f = \sum_{i=0}^{\frac{l}{2}} a_i x^i + \sum_{j=0}^{\frac{l-5}{2}} b_j y x^j$ with b_0 and a_0 not simultaneously all zero (otherwise one of the R_i should be equal to P_0, which would be absurd), $a_{\frac{l}{2}} \neq 0$ if l is even (otherwise one of R_i should be equal to P_∞, which would be absurd) and $b_{\frac{l-5}{2}} \neq 0$ if l is odd (otherwise one of R_i should be equal to P_∞, which would be absurd). At point R_1, we have : $\sum_{i=0}^{\frac{l}{2}} a_i x^i + \sum_{j=0}^{\frac{l-5}{2}} b_j y x^j = 0$, implying that,

$$y \;=\; -\frac{\sum_{i=0}^{\frac{l}{2}} a_i x^i}{\sum_{j=0}^{\frac{l-5}{2}} b_j x^j}.$$

By replacing the value of y in the expression of the equation of the curve, we obtain :

$$\left(-\frac{\sum_{i=0}^{\frac{l}{2}} a_i x^i}{\sum_{j=0}^{\frac{l-5}{2}} b_j x^j} \right)^2 = x(x + 2p)(x + 4p)(x^2 - 8p^2)$$

this equation can be written :

$$\left(\sum_{i=0}^{\frac{l}{2}} a_i x^i \right)^2 = \left(\sum_{j=0}^{\frac{l-5}{2}} b_j x^j \right)^2 x(x + 2p)(x + 4p)(x^2 - 8p^2) \tag{4}$$

Expression (4) is an equation of degree l in x. Indeed, whatever the parity of l, the first member of equation (4) has degree $2 \times \left(\frac{l}{2}\right) = l$ and the second member has degree $2 \times \left(\frac{l-5}{2}\right) + 5 = l$.

This gives a family of points of degree l :

$$\mathcal{F}_1 = \left\{ \left(x, \; -\frac{\sum_{i=0}^{\frac{l}{2}} a_i x^i}{\sum_{j=0}^{\frac{l-5}{2}} b_j x^j} \right) \;\middle|\; \begin{array}{c} a_0 \text{ and } b_0 \text{ not simultaneously zero,} \\ a_{\frac{l}{2}} \neq 0 \text{ if } l \text{ is even, } b_{\frac{l-5}{2}} \neq 0 \text{ if } l \text{ is} \\ \text{is odd and } x \text{ is a solution} \\ \text{of the equation :} \\ \left(\sum_{i=0}^{\frac{l}{2}} a_i x^i \right)^2 = \left(\sum_{j=0}^{\frac{l-5}{2}} b_j x^j \right)^2 x(x + 2p)(x + 4p)(x^2 - 8p^2) \end{array} \right\}$$

Case 2: only one of n_k is non-zero with $k \in \{0, 1, 2\}$ and let us denote $P_k = [\delta_k : 0 : 1]$ with $\delta_0 = 0$, $\delta_1 = -2p$ and $\delta_2 = -4p$.

The formula $(\star)$ becomes : $div(f) = R_1 + \ldots + R_l + P_k - (l + 1)P_\infty$, so $f \in \mathcal{L}((l + 1)P_\infty)$, according to Lemma 4, we have :

$$f = \sum_{i=0}^{\frac{l+1}{2}} a_i x^i + \sum_{j=0}^{\frac{l-4}{2}} b_j y x^j$$

and since $ord_{P_k} f = 1$, so $b_0 = -\sum_{i=1}^{\frac{l+1}{2}} a_i \delta_k^i$ implies that

$$f = \sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i \right) + \sum_{j=0}^{\frac{l-4}{2}} b_j y x^j$$

with $a_{\frac{l+1}{2}} \neq 0$ if l is even (otherwise one of the R_i should be equal P_∞, which would be absurd) and $b_{\frac{l-4}{2}} \neq 0$ if l if l is odd (otherwise one of the R_i should be equal P_∞, which would be absurd).
At the points R_i, we have :

$$\sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i\right) + \sum_{j=0}^{\frac{l-4}{2}} b_j y x^j = 0,$$

which implies that

$$y = -\frac{\sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i\right)}{\sum_{j=0}^{\frac{l-4}{2}} b_j x^j}.$$

By replacing the value of y in the expression of the equation of the curve, we obtain :

$$\left(-\frac{\sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i\right)}{\sum_{j=0}^{\frac{l-4}{2}} b_j x^j} \right)^2 = x(x + 2p)(x + 4p)(x^2 - 8p^2)$$

this equation can be written :

$$\left(\sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i\right) \right)^2 = \left(\sum_{j=0}^{\frac{l-4}{2}} b_j x^j \right)^2 x(x + 2p)(x + 4p)(x^2 - 8p^2)$$

which also corresponds to the equation :

$$\left(\sum_{i=1}^{\frac{l+1}{2}} a_i \left(\frac{x^i + \delta_k^i}{x^{\frac{1}{2}}} \right) \right)^2 = \left(\sum_{j=0}^{\frac{l-4}{2}} b_j x^j \right)^2 (x + 2p)(x + 4p)(x^2 - 8p^2) \qquad (5)$$

Expression (5) is an equation of degree l. Indeed, whatever the parity of l, the first member of (5) is of degree $2\times\left(\frac{l+1}{2} - \frac{1}{2}\right) = l$ and the second member is of degree $2\times\left(\frac{l-4}{2}\right)+4 = l$.

This gives a family of points of degree l:

$$\mathcal{F}_2^k = \left\{ \begin{array}{c} \left(x, -\frac{\sum_{i=1}^{\frac{l+1}{2}} a_i \left(x^i + \delta_k^i\right)}{\sum_{j=0}^{\frac{l-4}{2}} b_j x^j} \right) \quad \Bigg| \quad \begin{array}{c} a_{\frac{l+1}{2}} \neq 0 \text{ if } l \text{ is even, } b_{\frac{l-4}{2}} \neq 0 \\ \text{if } l \text{ is odd and } x \text{ solution of} \\ \text{the equation :} \end{array} \\ \left(\sum_{i=1}^{\frac{l+1}{2}} a_i \left(\frac{x^i + \delta_k^i}{x^{\frac{1}{2}}} \right) \right)^2 = \left(\sum_{j=0}^{\frac{l-4}{2}} b_j x^j \right)^2 (x+2p)(x+4p)(x^2-8p^2) \\ \text{with} \quad \delta_0 = 0, \quad \delta_1 = -2p \quad \text{and} \quad \delta_2 = -4p \end{array} \right\}$$

Case 3: Only two of n_k are non-zero, and let's denote n_ℓ and $n_\jmath$ with $P_\ell = [\delta_\ell : 0 : 1]$ and $P_\jmath = [\delta_\jmath : 0 : 1]$ with $\delta_\ell,\ \delta_\jmath \in \{0,\ -2p,\ -4p\}$, such as $\delta_\ell \neq \delta_\jmath$ and $\ell,\ \jmath \in \{0,\ 1,\ 2\}$ such that $\ell \neq \jmath$. The formula ($\star$) becomes :

$$div(f) = R_1 + \ldots + R_l + P_\ell + P_\jmath - (l+2)P_\infty$$

so $f \in \mathcal{L}((l+2)P_\infty)$, according to Lemma 4, we have :
$f = \sum_{i=0}^{\frac{l+2}{2}} a_i x^i + \sum_{j=0}^{\frac{l-3}{2}} b_j y x^j$ and since $ord_{P_\ell} f = ord_{P_\jmath} f = 1$, results in so :

$$b_0 = -\frac{1}{2} \left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\delta_\ell^i + \delta_\jmath^i\right) \right).$$

By posing : $\omega^i = -\frac{1}{2}\left(\delta_\ell^i + \delta_\jmath^i\right)$, we hence $b_0 = \sum_{i=1}^{\frac{l+2}{2}} a_i \omega^i$ implies that

$$f = \sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i + \omega^i\right) + \sum_{j=0}^{\frac{l-3}{2}} b_j y x^j$$

with $a_{\frac{l+2}{2}} \neq 0$ if l is even (otherwise one of the R_i should be equal P_∞, which would be absurd) and $b_{\frac{l-3}{2}} \neq 0$ si l is odd (otherwise one of the R_i should be equal P_∞, which would be absurd).

At point R_i, we have : $\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i \ + \ \omega^i\right) \ + \ \sum_{j=0}^{\frac{l-3}{2}} b_j y x^j \ = \ 0$, which implies that

$$y \ = \ - \frac{\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i \ + \ \omega^i\right)}{\sum_{j=0}^{\frac{l-3}{2}} b_j x^j}.$$

By replacing the value of y in the expression of the equation of the curve, we obtain :

$$\left(- \frac{\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i \ + \ \omega^i\right)}{\sum_{j=0}^{\frac{l-3}{2}} b_j x^j} \right)^2 \ = \ x(x \ + \ 2p)(x \ + \ 4p)(x^2 \ - \ 8p^2)$$

this equation can be written :

$$\left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i + \omega^i\right) \right)^2 \ = \ \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^j \right)^2 x(x \ + \ 2p)(x \ + \ 4p)(x^2 \ - \ 8p^2)$$

This also corresponds to the equation :

$$\left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\frac{x^i + \omega^i}{x} \right) \right)^2 \ = \ \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x + 2p)(x \ + \ 4p)(x^2 - 8p^2) \qquad (6)$$

Expression (6) is an equation of degree l. Indeed : whatever the parity of l, the first member of equation (6) is of degree $2 \times \left(\frac{l+2}{2} - 1\right) = l$ and the second member is of degree

$2 \times \left(\frac{l-3}{2} - \frac{1}{2}\right) + 4 = l$. This gives a point family of degree l :

$$\mathcal{F}_3^{\ell,\jmath} = \left\{ \begin{array}{c|c} \left(x, -\dfrac{\sum_{i=1}^{\frac{l+2}{2}} a_i\left(x^i+\omega^i\right)}{\sum_{j=0}^{\frac{l-4}{2}} b_j x^j}\right) & \begin{array}{c} a_{\frac{l+2}{2}} \neq 0 \text{ si } l \text{ is even, } b_{\frac{l-3}{2}} \neq 0 \\ \text{if } l \text{ is odd and } x \text{ solution of} \\ \text{the equation :} \end{array} \end{array} \right.$$

$$\left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\frac{x^i+\omega^i}{x}\right)\right)^2 = \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 (x+2p)(x+4p)(x^2-8p^2)$$

with

$$\omega^i = -\frac{1}{2}\left(\delta_\ell^i + \delta_\jmath^i\right) \text{ and } \delta_\ell, \delta_\jmath \in \{0, -2p, -4p\}, \text{ such as } \delta_\ell \neq \delta_\jmath$$

$$\left. \ell,\ \jmath \in \{0,1,2\} \text{ such that } \ell \neq \jmath \text{ and } \delta_0 = 0, \delta_1 = -2p \text{ and } \delta_2 = -4p \right\}$$

Case 4 : none of the n_k are zero. The formula $(\star)$ becomes : $div(f) = R_1 + \ldots + R_l + P_3 + P_4 - (l+2)P_\infty$, so $f \in \mathcal{L}((l+2)P_\infty)$, according to Lemma 4, we have : $f = \sum_{i=0}^{\frac{l+2}{2}} a_i x^i + \sum_{j=0}^{\frac{l-3}{2}} b_j y x^j$, and since $ord_{P_3} f = ord_{P_4} f = 1$ and $ord_{P_3} f = 1$, results in so : $b_0 = -\frac{1}{2}\left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\left(2i\sqrt{2}p\right)^i + \left(-2i\sqrt{2}p\right)^i\right)\right)$.

By positing : $\nu^i = -\frac{1}{2}\left(\left(2i\sqrt{2}p\right)^i + \left(-2i\sqrt{2}p\right)^i\right)$, we hence : $b_0 = \sum_{i=1}^{\frac{l+2}{2}} a_i\left(x^i+\nu^i\right)$, implies that

$$f = \sum_{i=1}^{\frac{l+2}{2}} a_i\left(x^i + \nu^i\right) + \sum_{j=0}^{\frac{l-3}{2}} b_j y x^j$$

with $b_{\frac{l+2}{2}} \neq 0$ if l is even (otherwise one of the R_i should be equal P_∞, which would be absurd) and $b_{\frac{l-4}{2}} \neq 0$ if l is odd (otherwise one of the R_i should be equal P_∞, which would be absurd).

At point R_i, we have : $\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i \ + \ \nu^i\right) \ + \ \sum_{j=0}^{\frac{l-3}{2}} b_j y x^j \ = \ 0$, implying that,

$$y \ = \ - \frac{\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i \ + \ \nu^i\right)}{\sum_{j=0}^{\frac{l-3}{2}} b_j x^j}.$$

By replacing the value of y in the expression of the equation of the curve, we obtain :

$$\left(- \frac{\sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i \ + \ \nu^i\right)}{\sum_{j=0}^{\frac{l-3}{2}} b_j x^j} \right)^2 \ = \ x(x \ + \ 2p)(x \ + \ 4p)(x^2 \ - \ 8p^2)$$

this equation can be written :

$$\left(a\left(x^{\frac{5}{2}} + \mu\right) \ + \ \sum_{i=1}^{\frac{l+2}{2}} a_i \left(x^i + \nu^i\right) \right)^2 \ = \ \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^j \right)^2 x(x \ + \ 2p)(x \ + \ 4p)(x^2 \ - \ 8p^2)$$

which also corresponds tothe equation :

$$\left(\sum_{i=1}^{\frac{l+2}{2}} a_i \left(\frac{x^i + \nu^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x+2p)(x+4p)(x^2 - 8p^2) \qquad (7)$$

Expression (7) is an equation of degree l. Indeed : whatever the parity of l, the first member of equation (7) is of degree $2\left(\frac{l+2}{2} - 1\right) = l$ and the second member is of degree

$2\left(\dfrac{2\times(\frac{l-3}{2})-1}{2}\right)+6=l$. This gives a family of points of degree l:

$$\mathcal{F}_4=\left\{\begin{array}{c|c} \left(x,\ -\dfrac{\sum_{i=1}^{\frac{l+2}{2}} a_i\left(x^i+\nu^i\right)}{\sum_{j=0}^{\frac{l-3}{2}} b_j x^j}\right) & \begin{array}{c} a_{\frac{l+2}{2}}\neq 0 \text{ if } l \text{ is even, } b_{\frac{l-3}{2}}\neq 0 \\ \text{if } l \text{ is odd } x \text{ solution of} \\ \\ \text{the equation :}\end{array} \\ \multicolumn{2}{c}{\left(\sum_{i=1}^{\frac{l+2}{2}} a_i\left(\dfrac{x^i+\nu^i}{x}\right)\right)^2=\left(\sum_{j=0}^{\frac{l-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 (x+2p)(x+4p)(x^2-8p^2)} \\ \multicolumn{2}{c}{\text{with}} \\ \multicolumn{2}{c}{\nu^i=-\dfrac{1}{2}\left(\left(2i\sqrt{2}p\right)^i+\left(-2i\sqrt{2}p\right)^i\right)} \end{array}\right\}$$

□

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