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Eigenvalues of Fredholm type limit integral equations in the space of Bohr almost periodic functions

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Abstract. In present paper it is considered the question on eigenvaluesof Fredholm type limit integral equations in the space of Bohr almost periodic functions. We prove basic results of the theory of ordinary Fredholm integral equations for limit integral equations' case.

Key Words and Phrases: Fredholm integral equation, symmetric kernel, almost periodic function, limit periodic function

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1. Introduction

Let us consider the limit integral equation of Fredholm type in Bohr space of almost periodic functions, introduced in [12]:

$$\phi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\phi(\xi)d\xi \tag{1}$$

We shall use some known facts from the theory of almost periodic functions of Bohr. Recall definition of almost periodic function of H. Bohr [8, p. 41] (see also [1-3]). Let we are given with a continuous real function $f(x)$, given on all real axes $f : \mathbb{R} \to X$, where X is some normed space.

The number $\tau$is called to be ε-almost period, if for every $x \in \mathbb{R}$ the following inequality is satisfied

$$|f(x+\tau) - f(x)| \leq \varepsilon.$$

Let $E \in \mathbb{R}$ be some enumerable subset in $\mathbb{R}$. This subset is called to be relatively dense, if there exists a nimber $L > 0$ such that every interval of the view

$$a < x < a + L,\, a \in \mathbb{R}$$

of the length L contains a point from the subset E.

The function $f(x)$ is said to be almost periodic, if for every$\varepsilon > 0$ the set of ε-almost periods is relatively dense in $\mathbb{R}$.

Definition 1. Let $F_1, F_2, ...$ be a sequence of continuous periodic functions, $F_k : \mathbb{R}^n \to \mathbb{R}$. If $F_k \to F$, uniformly with respect to $x \in \mathbb{R}^n$, then the function F is said to be the almost periodic.

Denote by $\mathbb{R}^\infty$ the set of all sequences of real numbers of a view ($\alpha_1, \alpha_2, ..., \ \alpha_k, ...$). This sets up a linear space in which it is possible introduce the Tichonoff metric as below:

$$d(x, y) = \sum_{n=1}^{\infty} e^{1-n} \left| x_n - y_n \right|,$$

where $x \in \mathbb{R}^\infty$, $y \in \mathbb{R}^\infty$. Every function in finite number of variables of a view $\Phi : \mathbb{R}^m \to \mathbb{R}$ could considered as a function $\Phi' : \mathbb{R}^\infty \to \mathbb{R}$ in denumerable number of variables by taking $\alpha_k = 0$ for all $k > m$.

Definition 2. Let we are given with a sequence of continuous periodic functions $F_k : \mathbb{R}^{m_k} \to \mathbb{R}, \ m_1 < m_2 < \cdots, \ m_k \to \infty$. If $F_k \to F$ uniformly with respect to $x \in \mathbb{R}^\infty$, then the limit of this sequence, that is the function $F : \mathbb{R}^\infty \to \mathbb{R}$ is called a limit periodic function in denumerable number of variables.

Lemma 1. Every almost periodic function is a diagonal function of some limit-periodic function in finite or infinite number of variables.

These notions are generalized to infinite dimensional case. Ther pair of real numbers (τ, η) we call an ε-almost period for the function $f(x, y)$, if for every pair of real numbers (x, y) the following relations are satisfied:

$$|f(x + \tau, y) - f(x, y)| \leq \varepsilon,$$

$$|f(x, y + \eta) - f(x, y)| \leq \varepsilon.$$

Definition 3. A continuous function defined in all plane $(x, y) \in \mathbb{R} \times \mathbb{R}$ is called to be almost periodic, if for every real $\varepsilon > 0$ there exists a number $l = l(\varepsilon) > 0$ such that for every pair $(x, y) \in \mathbb{R} \times \mathbb{R}$ there exists a pair (τ, η) of ε-almost period in the open quadrate$(x, x + l) \times (y, y + l) \in \mathbb{R} \times \mathbb{R}$.

It is best known that almost periodic functions can uniformly approximated in all plane by trigonometric polynomials. The corresponding result is formulated as follows [3, p. 66].

Lemma 2. Every continuous function $f(x, y)$ is almost periodic in $\mathbb{R} \times \mathbb{R}$. For every $\varepsilon > 0$ it is possible to find natural numbers N_ε and L_ε such that the inequality

$$\left| f(x, y) - \sum_{n=1}^{N_\varepsilon} \sum_{r=1}^{L_\varepsilon} a_{n,r} e^{2\pi i (\tau_n x + \mu_r y)} \right| < \varepsilon$$

is satisfied for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

From this result one deduces

Lemma 3. Every almost periodic function $f(x, y)$ is a diagonal function for some limit-periodic function in two system of finite or infinite number of variables.

Let the function $\varphi(x_1, x_2, ..., x_n; y_1, y_2, ..., y_m)$ be a limit periodic function of two system of variables. This means that

$$\varphi(x_1, x_2, ..., x_n; y_1, y_2, ..., y_m) = \lim_{k\to\infty} G_k(x_1, x_2, ..., x_n; y_1, y_2, ..., y_m)$$

uniformly in $\mathbb{R}^{n+m}$, moreover the functions $G_k(x_1, x_2, ..., x_n; y_1, y_2, ..., y_m)$ are periodic. Then taking values of the function G_k on the principle diagonal of the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, we get almost periodic function of two variables.

In infinite dimensional case, the said above is possible reformulate as follows: it is possible to find two system of variables $x_1, ..., x_m, ...$ and $y_1, ..., y_s, ...$, and a sequence of periodic with respect to every variable functions $G_k(x_1, x_2, ...; y_1, y_2, ...)$, $k = 1, 2, ...$ such that

$$G(x_1, x_2, ...; y_1, y_2, ...) = \lim_{k\to\infty} G_k(x_1, x_2, ...; y_1, y_2, ...),$$

uniformly (in the Tichonoff metric introduced above), and moreover,

$$K(x, \xi) = G(x, x, ...; \xi, \xi, ...).$$

In this work we consider the case of symmetric kernel when both system of variables are finite. Then the function

$$G(x_1, ..., x_m, y_1, ..., y_m)$$

will be limit –periodic in $2m$ variables, moreover the periods with respect to pairs of variables x_i and y_i are coinsident.

In the works of H. Bohr [8] there were studied means of a view (1) on the base of theorem of Croneker's theorem on uniform distribution (mod 1) of some curves in multi-dimensional unite cube. Cronecer's theorem [6, p. 345] ([11]) states:

Lemma 4. Let the real numbers $\alpha_1, \alpha_2, ..., \alpha_N$ be linearly independent over the field of rational numbers. Let, further, $I_\gamma(T)$ be the measure of such points $t \in (0, T)$ for which $(\alpha_1 t, \alpha_2 t, ..., \alpha_N t) \in \gamma(mod 1)$. Then

$$\lim_{T\to\infty} \frac{I_\gamma(T)}{T} = \Gamma.$$

We shall use also the following definition [9].

Definition 3. The family F of functions f defined in the subset E *of the metric space* X is called to be equicontinuous in E, if for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$, when $d(x, y) < \delta$, $x \in E$, $y \in E$, $f \in$ F; here d means the distance in X.

Following lemma is given in [6, p. 348] ([7]).

Lemma 5. Let the curve $\gamma(t)$ be uniformly distributed (mod 1) in R^n, D is a closed domain in the unite cube being measurable in Jordan meaning, Φ means a family of complex valued continuous functions defined in D. If Φ is uniformly bounded and equicontinuous family, then the following relation is satisfied uniformly with respect to $f \in \Phi$:

$$\lim_{T\to\infty} T^{-1} \int_0^T f(\{\gamma(t)\})dt = \int_D f dx_1 ... dx_N,$$

where in the left hand side of the equality the integration is taken over such $t \in (0, T)$, for which

$$\gamma(t) = (\{\gamma_1(t)\}, \ldots, \{\gamma_N(t)\}) \in D(mod\ 1).$$

2. Basic auxiliary results

In [12] it was given the general formula for solving of the equation (1) at the points λ at which the Fredholm function $D(\lambda)$ is distinct from zero $D(\lambda) \neq 0$. We have considered only the case of symmetric kernel. Let us consider now the general case of arbitrary kernel being almost periodic in two variables and solve this equation by the method of sequel substitutions. Substitute the expression of the function $\phi(\xi)$ in right hand side of the equation (1). We get

$$\phi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) \left(f(\xi) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(\xi,\eta)\,\phi(\eta)\,d\eta \right) d\xi =$$

$$= f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, f(\xi)\, d\xi + \lambda^2 \lim_{T\to\infty} \frac{1}{T^2} \int_0^T \int_0^T K(x,\xi)\, K(\xi,\eta)\, \phi(\eta)\, d\xi d\eta.$$

Substituting again the expression of the function $\phi(x)$ from the equation (1) in both integrals we get

$$\phi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, f(\xi)\, d\xi + \lambda^2 \lim_{T\to\infty} \frac{1}{T^2} \int_0^T \int_0^T K(x,\xi)\, K(\xi,\eta) \times$$

$$\times f(\eta)\, d\xi d\eta + \lambda^3 \lim_{T\to\infty} \frac{1}{T^3} \int_0^T \int_0^T \int_0^T K(x,\xi)\, K(\xi,\eta)\, K(\eta,\theta)\phi(\theta)\, d\xi d\eta d\theta.$$

And so on. Repeating this procedure, after of n steps we get

$$\phi(x) = \; f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, f(\xi)\, d\xi +$$

$$+\lambda^2 \lim_{T\to\infty} \frac{1}{T^2} \int_0^T \int_0^T K(x,\xi)\, K(\xi,\eta)\, f(\eta)\, d\xi d\eta + \cdots +$$

$$+\lambda^n \lim_{T\to\infty} \frac{1}{T^n} \int_0^T \int_0^T \cdots \int_0^T K(x,\xi) K(x,\xi_1) \cdots K(\xi_{n-2},\xi_{n-1}) f(\xi_{n-1})\, d\xi d\xi_1 \cdots d\xi_{n-1} +$$

$$+\lambda^{n+1} \lim_{T\to\infty} \frac{1}{T^{n+1}} \int_0^T \int_0^T \cdots \int_0^T K(x,\xi) K(x,\xi_1) \cdots K(\xi_{n-1},\xi_n) \phi(\xi_n)\, d\xi d\xi_1 \cdots d\xi_n.$$

Tending the number of steps to infinity we find a series

$$f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, \phi(\xi)\, d\xi \; +$$

$$+\lambda^2 \lim_{T\to\infty} \frac{1}{T^2} \int_0^T \int_0^T K(x,\xi)\, K(\xi,\eta)\, \phi(\eta)\, d\xi d\eta + \cdots .$$

We show that this series is uniformly convergent and its sum is a solution of the equation (1). Denoting

$$u_n(\lambda, x) =$$

$$= \lambda^n \lim_{T\to\infty} \frac{1}{T^n} \int_0^T \int_0^T \cdots \int_0^T K(x,\xi) K(x,\xi_1) \cdots K(\xi_{n-2},\xi_{n-1}) f(\xi_{n-1})\, d\xi d\xi_1 \cdots d\xi_{n-1}$$

we consider the functional series

$$\sum_{n=1}^{\infty} u_n(\lambda, x). \tag{2}$$

Since the functions $f(x)$ and $K(x,\xi)$ are almost periodic, then this function is uniformly bounded. So, there is a constant M such that $|K(x,\xi)| \le M$. We see that the function $u_n(\lambda, x)$ has a bound

$$|u_n(\lambda, x)| \le |\lambda|^n M^{n+1} \le M^{1-n},$$

when $|\lambda| \le M^{-2}$. We can suppose that $M > 1$. So, the function $u_n(\lambda, x)$ uniformly bounded and

$$\left| \sum_{n=1}^{\infty} u_n(\lambda, x) \right| \le \frac{M}{M-1}.$$

Consequently, the series (2) converges uniformly and has a sum being continuous and bounded in every finite segment on real axes. Now we can formulate the following result.

Theorem 1. Let the functions $f(x)$ and $K(x,\xi)$ be almost periodic in Bohr sense, and the function $f(x)$ is not zero identically. Then the integral equation (1) has for every fixed λ, $|\lambda| \le M^{-2}$ a unique solution given by the uniform convergent series

$$\phi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, f(\xi)\, d\xi +$$

$$+\lambda^2 \lim_{T\to\infty} \frac{1}{T^2} \int_0^T \int_0^T K(x,\xi)\, K(\xi,\eta)\, f(\eta)\, d\xi d\eta + \cdots .$$

Proof. Take the function

$$H(\xi) = f(\xi) + \sum_{n=1}^{\infty} u_n(\lambda, \xi).$$

This function is continuous on all real axes and uniformly bounded as it was showed above. Multiplying both sides of this equality by $K(x,\xi)\, f(\xi)$, let us take the mean value. We get

$$\lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, H(\xi)\, d\xi = \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\, f(\xi)\, d\xi +$$

$$+\sum_{n=1}^{\infty}\lambda \lim_{T\to\infty}\frac{1}{T}\int_0^T K\left(x,\xi\right)u_n\left(\lambda,\xi\right)f\left(\xi\right)d\xi=$$

$$=\lambda \lim_{T\to\infty}\frac{1}{T}\int_0^T K\left(x,\xi\right)f\left(\xi\right)d\xi+\lambda^2\lim_{T\to\infty}\frac{1}{T^2}\int_0^T\int_0^T K\left(x,\xi\right)K\left(\xi,\xi_1\right)f\left(\xi_1\right)d\xi d\xi_1+$$

$$+\lambda^{n+1}\lim_{T\to\infty}\frac{1}{T^{n+1}}\int_0^T\int_0^T\cdots\int_0^T K\left(x,\xi\right)K\left(x,\xi_1\right)\cdots K\left(\xi_{n-1},\xi_n\right)f\left(\xi_n\right)d\xi d\xi_1\cdots d\xi_n.$$

The series in the right hand side is equal to $H\left(\xi\right)-\ f\left(\xi\right)$. So, the function $H\left(\xi\right)$ satisfices the equation (1).

To prove that the equation (1) has no other solutions, take any it's solution. Repeating the reasoning spend above we get for this solution the same expansion into series, which shows uniqueness of the solution. Theorem 1 is proven.

It is clear that taking

$$K_1\left(x,y\right)=K\left(x,y\right),\ K_{j+1}\left(x,y\right)=\frac{1}{T}\int_0^T K\left(x,\xi\right)K_j\left(\xi,y\right)d\xi,$$

we get, so called, iterated kernels. In terms of iterated kernels, the formula of the lemma 1 can be written as follows:

$$\phi\left(x\right)=f\left(x\right)+\sum_{n=1}^{\infty}\lambda^n\lim_{T\to\infty}\frac{1}{T}\int_0^T K\left(x,\eta\right)K_{n-1}\left(\eta,\xi\right)f\left(\eta\right)d\eta. \quad (3)$$

3. Hilbert-Smith theory for limit integral equations in the case of symmetric kernels

In [12] it was shown that the limit integral equation (1) in the Bohr space of almost periodic functions equivalent to some family of ordinary integral equations ([10]) in some multidimensional unite cubes, solutions of which defines the solution of limit integral equation by tending to the limit. From the theory of ordinary Fredholm type integral equations it is best known that in the case of non-symmetric kernel the equation may not have eigenvalues. By this reason to introduce and study the corresponding properties of limit integral equations of a view (1), we assume that the kernel is a symmetric. In this case the all of results established for ordinary integral equations have their analogs in the limit integral equations cases. In this paper we shall consider the question on existence of eigenvalues in the case of symmetric kernel.

Theorem 2. If the kernel of the equation (1) is symmetric, then it has an eigenvalue.

For the proof of Theorem 2 we need in several auxiliary lemmas.

Lemma 6. The function $D\left(x,y;\lambda\right)/D(\lambda)$ has the following expansion into power series:

$$\frac{D\left(x,y;\lambda\right)}{D\left(\lambda\right)}=\sum_{n=1}^{\infty}K_n\left(x,y\right)\lambda^n,$$

which converges uniformly in some neighborhood of the origin on complex plane.

Proof. For establishing this relation let us to get the expansion of the logarithmic derivative of the Fredholm function $D(\lambda)$ into power series with respect to λ. In accordance with the theorem 2 of the work [12], if $D(\lambda) \neq 0$ then the equation (1) has a unique solution given by the formula

$$\phi(x) = f(x) + \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\xi) \frac{D(x,\xi;\lambda)}{D(\lambda)} d\xi. \tag{4}$$

Such a value for the function $D(\lambda)$ exists, because $D(0) = 1$. Moreover, it could found such an interval $|\lambda| < \delta$, $\delta < 1$, in which $D(\lambda) \neq 0$. Then the function $1/D(\lambda)$ is an analytic function in the considered in some disk with the center at the origin on complex plane. By this reason this function can expanded into power series as follows

$$\frac{1}{D(\lambda)} = d_0 + d_1\lambda + d_2\lambda^2 + \cdots .$$

Since the function $D(x, \xi; \lambda)$ is also an entire function, then it has also the expansion into power series:

$$D\left(x, y; \lambda\right) = \lambda K\left(x, y\right) + \sum_{n=1}^{\infty} (-1)^n \frac{Q_n(x,y)}{n!} \lambda^{n+1}.$$

Then the function $D\left(x, y; \lambda\right)/D(\lambda)$ also has the expansion into power series. Let us write it as:

$$\frac{D\left(x, y; \lambda\right)}{D\left(\lambda\right)} = \sum_{n=1}^{\infty} q_n\left(x, y\right)\lambda^n, \tag{5}$$

which is convergent in some disc $|\lambda| < \delta$. From the formula [5], we deduce that this function has not a term with λ^0. Then from (4) we have

$$\phi\left(x\right) = f\left(x\right) + \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda^n q_n\left(x, \xi\right) f\left(\xi\right) d\xi =$$

$$= f\left(x\right) + \sum_{n=1}^{\infty} \lambda^n \lim_{T\to\infty} \frac{1}{T} \int_0^T q_n\left(x, \xi\right) f\left(\xi\right) d\xi.$$

From the relation (3), we have

$$\phi\left(x\right) = \ f\left(x\right) + \sum_{n=1}^{\infty} \lambda^n \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\xi\right) K_n\left(x, \xi\right) d\xi.$$

Comparing the two last equalities, we can write

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\xi\right)_n\left(x, \xi\right) d\xi = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\xi\right) q_n\left(x, \xi\right) d\xi.$$

So,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\xi\right)(K_n\left(x,\xi\right)-q_n\left(x,\xi\right))d\xi=0.$$

We can take $f\left(\xi\right)=K_n(x,\xi)-q_n\left(x,\xi\right)$, for every fixed x. Then we get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\xi\right)(K_n\left(x,\xi\right)-q_n(x,\xi))^2d\xi=0.$$

This is a key relation from which we deduce that $K_n(x,\xi)-q_n\left(x,\xi\right)=0$. The reasoning is not trivial and based on basic properties of almost periodic functions, explained above. First of all we note that the coefficients $q_n\left(x,\xi\right)$ of the series (5), and also iterated kernels could expressed as mean values of determinants ([12]), with almost periodic entries. Representing almost periodic functions, using Lemma 3, as a diagonal function of multivariate limit periodic functions, we can express this mean values as Riemann integral taken over unite cubes of corresponding dimensions using Lemma 5, as in the work [12]. Then we get the equality of the type

$$\int_\Delta \left|G_n(\bar{x},\bar{\xi})-H_n(\bar{x},\bar{\xi})\right|^2 d\bar{\xi}d\,\bar{x}=0.$$

Using continuity, from this equality we deduce the identity $G_n(\bar{x},\bar{\xi})=H_n(\bar{x},\bar{\xi})$. From this relation it follows that $K_n(x,\xi)-q_n\left(x,\xi\right)=0$, for all fixed x, identically with respect to ξ. So, we have proved that

$$\frac{D\left(x,y;\lambda\right)}{D\left(\lambda\right)}=\sum_{n=1}^{\infty}K_n\left(x,y\right)\lambda^n.$$

Lemma 6 is proved.

Lemma 7. In some neighborhood of the origin on complex plane the following relation holds true:

$$\frac{D'\left(\lambda\right)}{D\left(\lambda\right)}=\sum_{n=1}^{\infty}U_{n+1}\lambda^n;$$

here

$$U_n=\lim_{T\to\infty}\frac{1}{T}\int_0^T K_n\left(\xi,\xi\right)d\xi.$$

Proof. Take $x=y$ in the lemma 6. Then we have

$$\frac{D\left(x,x;\lambda\right)}{D\left(\lambda\right)}=\sum_{n=1}^{\infty}K_n\left(x,x\right)\lambda^n.$$

Since the series converges uniformly, then we can integrate over x:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{D\left(\xi,\xi;\lambda\right)}{D\left(\lambda\right)}d\xi=\sum_{n=1}^{\infty}\lambda^n.\lim_{T\to\infty}\frac{1}{T}\int_0^T K_n\left(\xi,\xi\right)d\xi.$$

As it was shown in [J-N-A],

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T D(\xi,\xi;\lambda)\,d\xi = -\frac{\lambda dD(\lambda)}{d\lambda}.$$

Dividing by $D(\lambda)$ and using the equality above, we find:

$$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty}\lambda^n.\lim_{T\to\infty}\frac{1}{T}\int_0^T K_{n+1}(\xi,\xi)\,d\xi = -\sum_{n=0}^{\infty}U_{n+1}\lambda^n.$$

Lemma 7 is proved.

For continue our reasoning we must note that from symmetricity of the kernel $K(x,y)$ it follows that the kernel $K_n(x,y)$ is also symmetric. We suffice with the proof of this fact when *n=2*. Really,

$$K_2(x,y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_1(x,\xi)\,K_1(\xi,y)\,d\xi =$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T K_1(y,\xi)\,K_1(\xi,x)\,d\xi = K_2(y,x).$$

The general case can be established as above.

Lemma 8. For any natural n, the kernel $K_n(x,y)$ is not equal to zero identically.

Proof. If we have $K_n(x,y)\equiv 0$ identically, then $K_m(x,y)\equiv 0$ for every $m\ge n$. Since one of the numbers n or *n+1* is an even number of a view *2k*, then we have

$$K_{2k}(x,y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_k(x,\xi)\,K_k(\xi,y)\,d\xi = \lim_{T\to\infty}\frac{1}{T}\int_0^T (K_k(y,\xi))^2 d\xi = 0.$$

Then the reasoning was spend above, during proof or previous lemma, shows that $K_k(x,y)\equiv 0$. So, supposing that n is a least natural number for which $K_n(x,y)\equiv 0$, we arrive at contradiction, because $n>k$ (since $n>1$). The Lemma 8 is proved.

From Lemma 8 it follows that $U_{2n}>0$, so all of coefficients with even indices are positive. By an induction one can establish the result

$$K_{s+t}(x,y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_s(x,\xi)\,K_t(\xi,y)\,d\xi\ .$$

Then for every $n>s$ we have

$$K_{2n}(x,y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_{n+s}(x,\xi)\,K_{n-s}(\xi,y)\,d\xi\ .$$

Lemma 9. For every natural n, the following inequalities are satisfied:

$$\frac{U_{2n+2}}{U_{2n}} \ge \frac{U_{2n}}{U_{2n+2}} \ge \cdots \ge \frac{U_4}{U_2}.$$

Proof. It is enough to show only first inequality, that is the inequality

$$\frac{U_{2n}}{U_{2n-2}} \geq \frac{U_{2n-2}}{U_{2n-4}}.$$

As it was shown above

$$K_{2n}\left(x,y\right) = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_{n+1}\left(x,\xi\right)K_{n-1}\left(\xi,y\right)d\xi \ .$$

Applying the Swartz inequality, we get:

$$\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T K_{2n}\left(x,x\right)dx\right)^2 =$$

$$= \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T K_{n+1}\left(x,y\right)K_{n-1}\left(y,x\right)dydx\right)^2 \leq$$

$$\leq \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T K_{n+1}\left(x,y\right)dydx\right)^2 \times$$

$$\times \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T K_{n-1}\left(y,x\right)dydx\right)^2 \leq$$

$$\leq \left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T K^2_{n+1}\left(x,y\right)dydx\right\} \times$$

$$\times \left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T K^2_{n-1}\left(x,y\right)dydx\right\} =$$

$$= \left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T K_{2n+2}\left(x,y\right)dydx\right\}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T K_{2n-2}\left(x,y\right)dydx\right\} = U_{2n+2}U_{2n-2}.$$

So, we have proved that

$$\left(U_{2n}\right)^2 \leq U_{2n+2}U_{2n-2}.$$

Then

$$\frac{U_{2n}}{U_{2n-2}} \geq \frac{U_{2n-2}}{U_{2n-4}},$$

and first inequality of the Lemma 8. The other relations can be established similarly. Lemma 9 is proved.

Lemma 10. The radius of convergence of the power series

$$\sum_{n=1}^{\infty} U_{2n}\lambda^{2n}$$

does not exceed $\sqrt{\frac{U_2}{U_4}}$.

Proof. It is known that the radius of given power series coincides with the supremum of such positive λ for which the series is convergent. From Lemma 2 we get

$$\left|\frac{U_{2n}}{U_{2n-2}}\lambda^2\right| \geq \frac{U_4}{U_2}|\lambda|^2.$$

If now $\frac{U_4}{U_2}|\lambda|^2 \geq 1$, or $|\lambda| > \sqrt{\frac{U_2}{U_4}}$, the series diverges, by the Ratio test for series [9, p.66] ([4]). Lemma 10 is proved.

Proof of Theorem 2. If the kernel $K(x, y)$ has not eigenvalues, that is the function $D(\lambda)$ has not complex roots, then the logarithmic derivative can be expanded into convergent power series on all complex plane. From Lemma 9 it follows that the series

$$\sum_{n=1}^{\infty} U_{2n}|\lambda|^{2n}$$

diverges when $|\lambda| > \sqrt{\frac{U_2}{U_4}}$. Since this series is a subseries of the series

$$\sum_{n=1}^{\infty} U_{n+1}|\lambda|^{n+1},$$

then the last series diverges for considered values of $|\lambda|$. So, the series

$$\frac{D'(\lambda)}{D(\lambda)} = \sum_{n=0}^{\infty} U_{n+1}\lambda^n$$

diverges also. This contradicts our assumption. So, the proof of Theorem 2 is finished.

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