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Interactions of various shaped bodies in PCR3BP

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Abstract. The dynamical properties of the motion of the infinitesimal body are investigated in the perturbed circular restricted 3-body problem (PCR3BP). Here primary as radiating oblate, secondary as dipole, infinitesimal body varies its mass according to Jeans law, interactions between these bodies, the effects of Coriolis and centrifugal forces are considered. Equations of motions and quasi-Jacobian integral are determined by assuming the above said perturbations. Further the stationary points, regions of motion, Poincaré surfaces of section and periodic orbits are illustrated numerically. Furthermore the stability of the stationary points are examined.

Key Words and Phrases: Radiating Oblate primary; Dipole secondary; Three-body interactions; Coriolis and centrifugal forces.

2010 Mathematics Subject Classifications: Primary 70F05, 70F07, 70F15

1. Introduction

Restricted three-body problem in an interesting and application based problem. Therefore, it is the most studied problem in the celestial mechanics and dynamical astronomy. The difference among the researchers are the perturbations with various types such as: the various shapes of the bodies, circular or elliptical motions of the primaries, various types of the forces acts on the bodies, variable mass of the bodies, interactions between bodies etc.

R.K. Sharma [1] have studied the solutions and their characteristic exponent in the restricted three-body problem where they have considered one primary as oblate and another one as point mass. A. Abdulraheem [2] and E.I. Abouelmagd [3] have investigated the periodic orbits in the restricted three-body problem where they have assumed radiating oblate primary and coriolis as well as centrifugal forces. F. Bouaziz [4] have illustrated the restricted three-body problem where they have considered the variable mass of the infinitesimal body, radiating primaries and the effects of asteroids belt. A.A. Ansari [5] have studied the robes restricted three-body problem where the primary is taken as heterogeneous body, secondary is taken as point mass, the outer layer of the heterogeneous body contains the viscous fluid where the infinitesimal body is moving. A.A. Ansari [6] have investigated the dynamical behaviour of the infinitesimal variable mass body in the restricted three-body problem where the primaries are radiating and moving in the elliptical orbits. A.A. Ansari [7] have illustrated the hills problem in the restricted three-body problem where the primaries triaxial in shapes and are moving in the circular orbits. S.K. Sahdev[8, 9] have investigated the Robes restricted three-body problem where primary is taken as heterogeneous in shape and outer layer of this body contains the viscous fluid in which the infinitesimal body is moving. They have also assumed that the secondary is radiating oblate with modified Newtonian force.

V.V. Radzievsky [10, 11], Y. A. Chernikov [12], A.A. Perezhogin [13], A.L. Kunitsyn [14] have studied the restricted three-body problem when the primaries are having the effects of solar radiation pressure. J. Singh [15, 16] V. Szevehely [17], E. Sarris [18], H. Peng [19], J. Singh [20], A. Narayan [21], A.A. Ansari([22], [23]) have studied the restricted three-body problem where the primaries are moving in the elliptical orbits. J. Singh [24], M.J. Zhang [25], A.A. Ansari ([26, 27], [28], [29], [29]) have investigated the restricted three-body problem where the infinitesimal body varies its mass according to Jeans law.

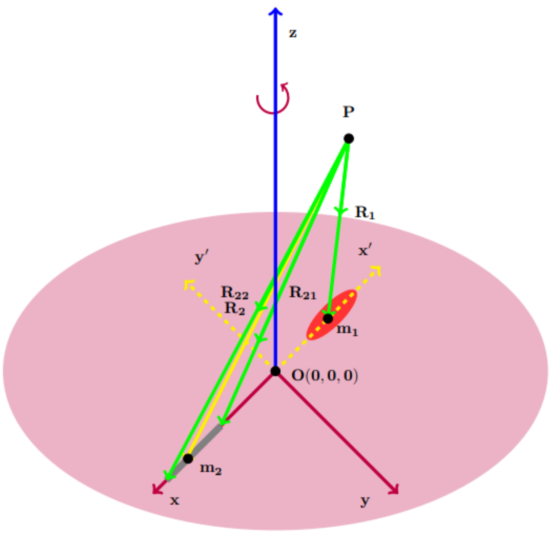


Figure 1: Interactions in the restricted three-body problem with radiating oblate primary and dipole secondary

This paper is arranged in many sections: The literature review is given in section 1. The equations of motion are determined in section 2, section 3 represents the numerical studies with various sub-sections. The paper is windup with conclusion in section 4.

2. Evaluation of Equations of motion

The perturbed circular restricted three-body problem (PCR3BP) is investigated by supposing primary as radiating oblate body of mass m_1 with radiation factor q as well as oblateness factor A, and secondary as dipole of mass m_2 (dipole is the combination of two masses m_{21} and m_{22} with separation distance $2\,\ell$). These bodies are moving in circular orbits around their common center of mass which is taken as origin while the third smallest body (infinitesimal body) of mass m is moving under the gravitational forces of the primaries and varies its mass according to Jeans law. The effects of interaction between these bodies with interaction parameter K and the effects of coriolis as well as centrifugal forces with parameters ϕ, ψ are considered.

Now fixing the units as the sum of the masses, the distance between both bigger bodies and G are separately considered as unity. Which yields $m_1 = 1 - 2\,\nu$, where $m_2 = m_{21} + m_{22}$, with $m_{21} = m_{22} = \nu$.

Utilizing the method used by J. Singh [30] and E.I. Abouelmagd [31], we can write the equations of motion of the variable mass body as

$$
\begin{aligned}
\ddot{\alpha} - 2\,n\,\phi\,\dot{\beta} \quad &+ \quad \frac{\dot{m}}{m}\,(\dot{\alpha} - \; n\,\phi\,\beta) &= \frac{\partial\,Q}{\partial\,\alpha},\\
\ddot{\beta} + 2\;n\,\phi\,\dot{\alpha} \quad &+ \quad \frac{\dot{m}}{m}\,(\dot{\beta} + \; n\,\phi\,\alpha) &= \frac{\partial\,Q}{\partial\,\beta},\\
\ddot{\gamma} \quad &+ \quad \frac{\dot{m}}{m}\,\dot{\gamma} &= \frac{\partial\,Q}{\partial\,\gamma},
\end{aligned}
\tag{1}
$$

where, dots represent the differentiations with respect to time t, and

$$
\begin{aligned}
Q &= \frac{n^2\,\psi}{2}(\alpha^2+\beta^2) + \frac{q(1-2\nu)}{r_1} + \frac{q(1-2\nu)\,A}{2r_1^3} + \frac{\nu}{r_{21}} + \frac{\nu}{r_{22}} + \frac{K}{r_1\,r_{21}\,r_{22}},\\
n^2 &= 1 + \ell^2 + \frac{3\,A}{2},\\
r_1^2 &= (\alpha + 2\,\nu)^2 + \beta^2 + \gamma^2,\\
r_{21}^2 &= (\alpha + 2\,\nu + \,\ell - 1)^2 + \beta^2 + \gamma^2,\\
r_{22}^2 &= (\alpha + 2\,\nu - \,\ell\, - 1)^2 + \beta^2 + \gamma^2.
\end{aligned}
\tag{2}
$$

Jean's law (J.H. Jeans [32]) and Meshcherskii space-time transformations (I.V. Meshcherskii [33]) will be used due to variable mass of the infinitesimal body as:

$$
\begin{aligned}
m &= m_0\, e^{-\epsilon_1 t}, \\
(\alpha, \quad \dot{\alpha}, \quad \ddot{\alpha}) &= \epsilon_2^{-1/2}\,(\xi, \quad \dot{\xi} + \frac{1}{2}\,\epsilon_1\,\xi, \quad \ddot{\xi} + \epsilon_1\,\dot{\xi} + \frac{1}{4}\,\epsilon_1^2\,\xi), \\
(\beta, \quad \dot{\beta}, \quad \ddot{\beta}) &= \epsilon_2^{-1/2}\,(\eta, \quad \dot{\eta} + \frac{1}{2}\,\epsilon_1\,\eta, \quad \ddot{\eta} + \epsilon_1\,\dot{\eta} + \frac{1}{4}\,\epsilon_1^2\,\eta), \\
(\gamma, \quad \dot{\gamma}, \quad \ddot{\gamma}) &= \epsilon_2^{-1/2}\,(\zeta, \quad \dot{\zeta} + \frac{1}{2}\,\epsilon_1\,\zeta, \quad \ddot{\zeta} + \epsilon_1\,\dot{\zeta} + \frac{1}{4}\,\epsilon_1^2\,\zeta),
\end{aligned} \tag{3}
$$

where ϵ_1 is variation constant, $\epsilon_2 = \dfrac{m}{m_0}$, the initial mass m_0.
Utilizing Eqs. (1) and (3), we obtain

$$
\begin{aligned}
\ddot{\xi} - 2\,n\,\phi\,\dot{\eta} &= \frac{\partial\,P}{\partial\xi}, \\
\ddot{\eta} + 2\,n\,\phi\,\dot{\xi} &= \frac{\partial\,P}{\partial\eta}, \\
\ddot{\zeta} &= \frac{\partial\,P}{\partial\zeta},
\end{aligned} \tag{4}
$$

where,

$$
\begin{aligned}
P &= \frac{n^2\,\psi}{2}(\xi^2 + \eta^2) + \frac{\epsilon_1^2}{8}(\xi^2 + \eta^2 + \zeta^2) + \epsilon_2^{3/2}\left\{\frac{q\,(1-2\nu)}{\ell_1} + \frac{q\,(1-2\nu)\,A\,\epsilon_2}{2\,\ell_1^3}\right. \\
&\quad \left. + \frac{\nu}{\ell_{21}} + \frac{\nu}{\ell_{22}} + \frac{K\,\epsilon_2}{\ell_1\,\ell_{21}\,\ell_{22}}\right\}, \\
\ell_1^2 &= (\xi + 2\,\nu\,\sqrt{\epsilon_2})^2 + \eta^2 + \zeta^2, \qquad \ell_{21}^2 = \{\xi + (2\nu + \ell - 1)\sqrt{\epsilon_2}\}^2 + \eta^2 + \zeta^2, \\
\ell_{22}^2 &= \{\xi + (2\nu - \ell - 1)\sqrt{\epsilon_2}\}^2 + \eta^2 + \zeta^2.
\end{aligned} \tag{5}
$$

The quasi-Jacobian integral for this model can be written as:

$$
\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 2\,P + C + 2\int_{t_0}^{t}\left(\frac{\partial P}{\partial t}\right)dt, \tag{6}
$$

where C is the quasi-Jacobian constant.

3. Numerical studies

Here we will investigate the motion dynamical properties of the infinitesimal body such as stationary points, regions of motion, Poincaré surfaces of section and periodic orbits in

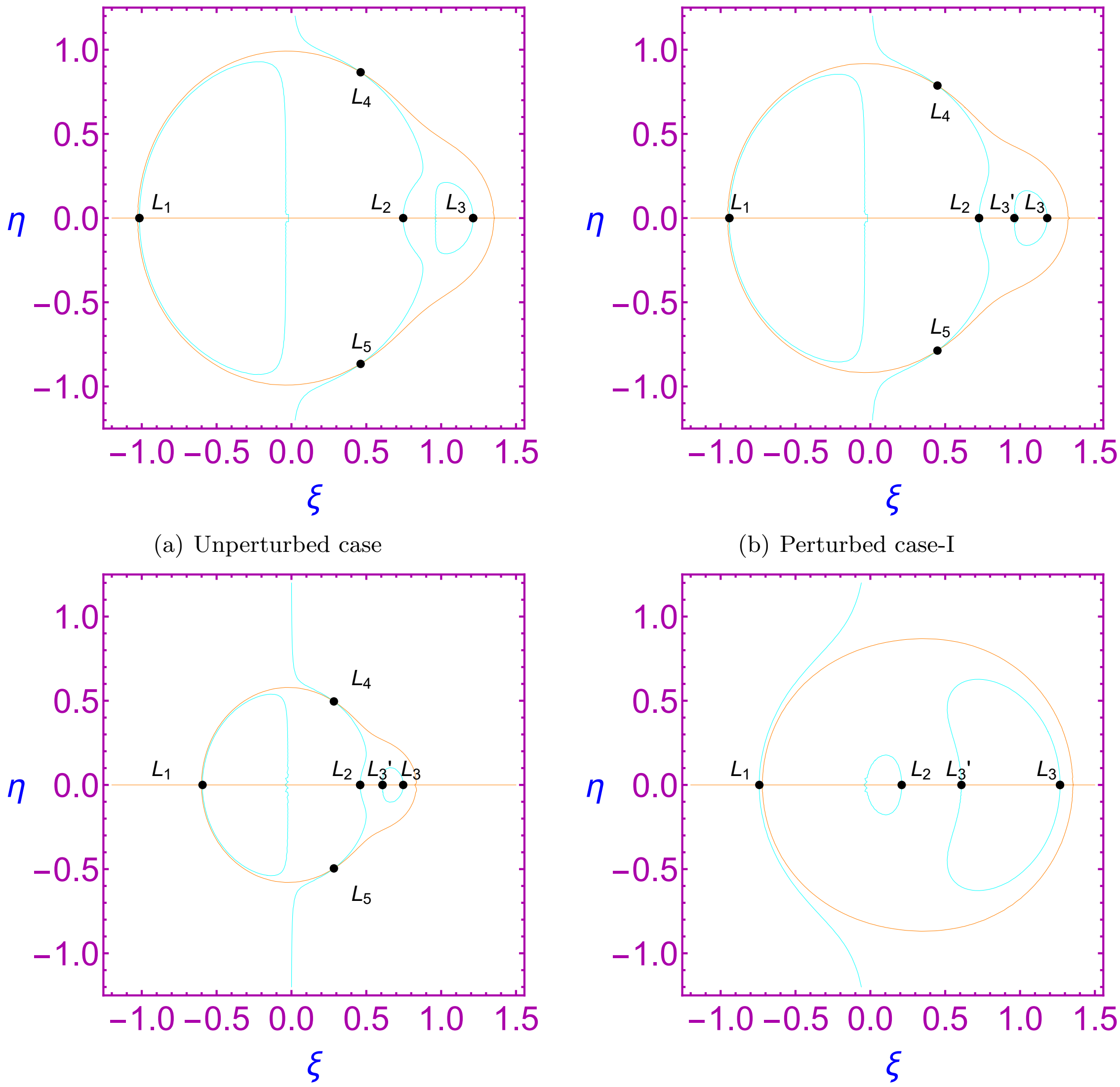
four cases.
1. Unperturbed case ($\epsilon_1 = 0, \epsilon_2 = 1, K = 0, \nu = 0.019, A = 0, q = 1, d = 0, \phi = \psi = 1$).
2. Perturbed case-I ($\epsilon_1 = 0, \epsilon_2 = 1, K = 0, \nu = 0.019, A = 0.002, q = 0.95, d = 0.02, \phi = \psi = 1.2$).
3. Perturbed case-II ($\epsilon_1 = 0.2, \epsilon_2 = 0.4, K = 0, \nu = 0.019, A = 0.002, q = 0.95, d = 0.02, \phi = \psi = 1.2$).
4. Perturbed case-III ($\epsilon_1 = 0.2, \epsilon_2 = 0.4, K = 2, \nu = 0.019, A = 0.002, q = 0.95, d = 0.02, \phi = \psi = 1.2$).

3.1. Stationary points

For stationary points, we will put zero to all the derivatives with respect to time t in Eq. (4) and then we get

$$\begin{aligned} \frac{\partial P}{\partial \xi} &= 0, \\ \frac{\partial P}{\partial \eta} &= 0, \\ \frac{\partial P}{\partial \zeta} &= 0. \end{aligned} \tag{7}$$

When we will solve first two equations of Eq. (7) by taking $\zeta = 0$, we will get in-plane stationary points and when we will solve first and last equations of Eq. (7) by taking $\eta = 0$, we will get out-of-plane stationary points. We have studied these in four cases as defined above.
At the study of in-plane stationary points, in unperturbed case, there are five stationary points out of which three are collinear and two are non-collinear in sub-figure (2(a)). In perturbed case-I, there are six stationary points out of which four are collinear and two are non-collinear in sub-figure (2(b)). In perturbed case-II, there are six stationary points out of which four are collinear and two are non-collinear, in this case all the stationary points move towards the origin in sub-figure (2(c)). In perturbed case-III, there are only four collinear stationary points, in this case non-collinear stationary points are no more exist in sub-figure (2(d)).



(a) Unperturbed case

(b) Perturbed case-I

(c) Perturbed case-II

(d) Perturbed case-III

Figure 2: In-plane locations of stationary points

At the study of out-of-plane stationary points, in unperturbed case and perturbed case-I, there are no out-of-plane stationary points exists in sub-figures (3(a)) and (3(b)) respectively. In perturbed case-II and III, there are two out-of-plane stationary points exists in sub-figures (3(c)) and (3(d)) respectively.

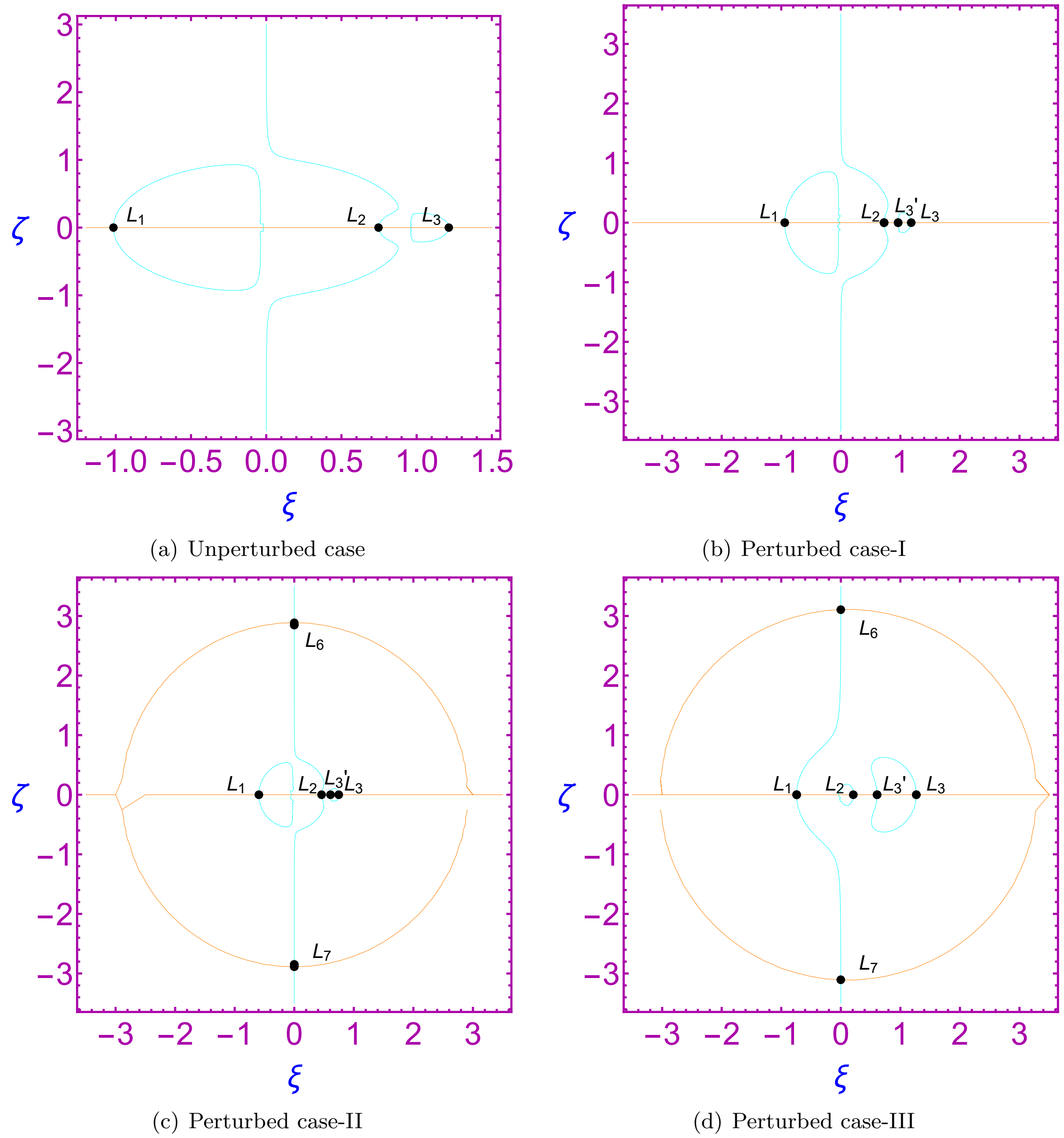
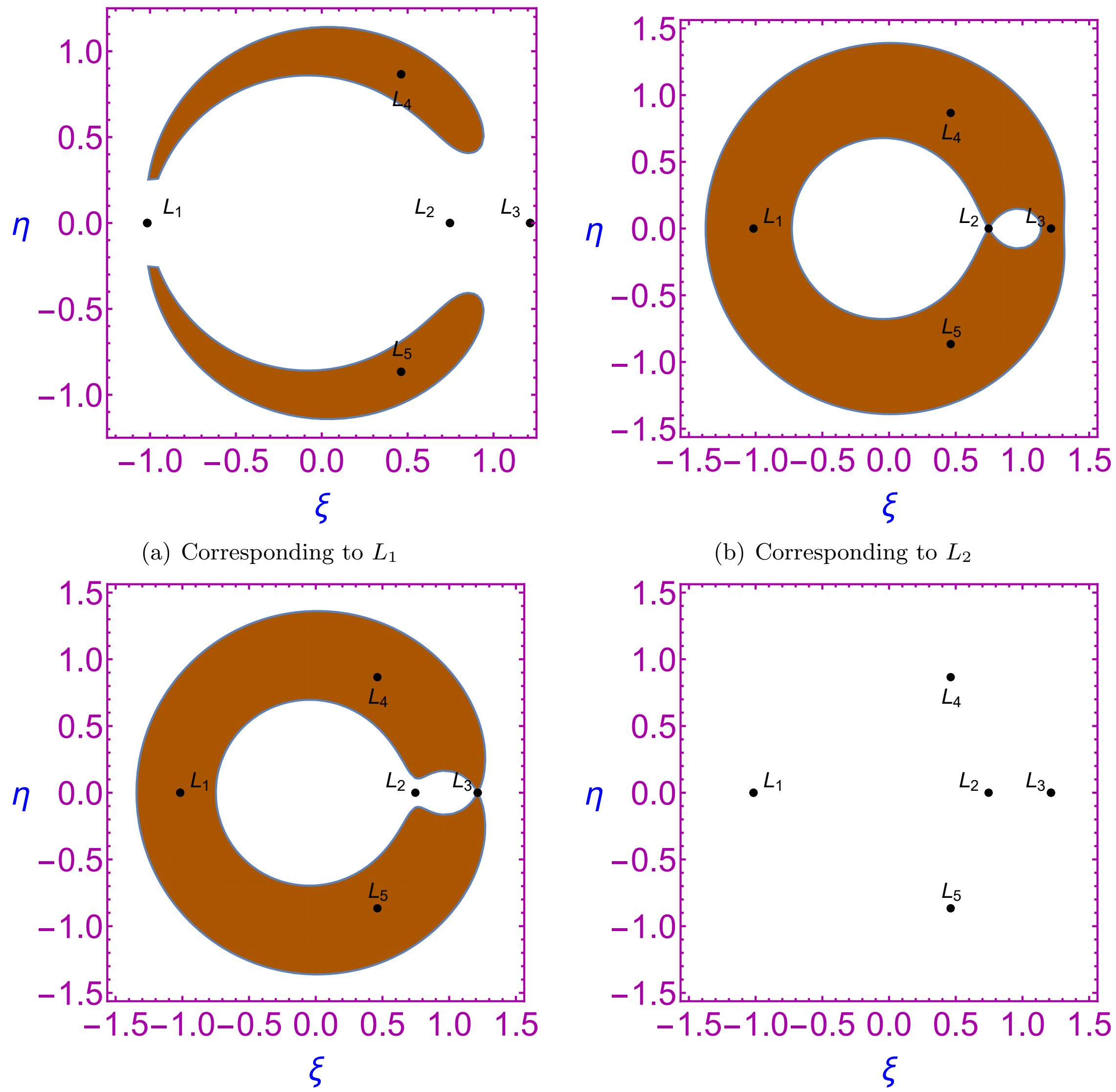


Figure 3: Out-of-plane locations of stationary points

3.2. Regions of motion

Using the procedure given by L.G. Lukyanov [34], we have performed the regions of motion for our model. For this firstly, we have calculated the value of jacobian constant C corresponding to each stationary points with the help of Eq. (6) and then we have illustrated the regions of motion in unperturbed case (figure (4)) and in perturbed case-III (figure (5)). In both the figures (4) and (5), colored regions are prohibited regions while white regions are allowed regions.



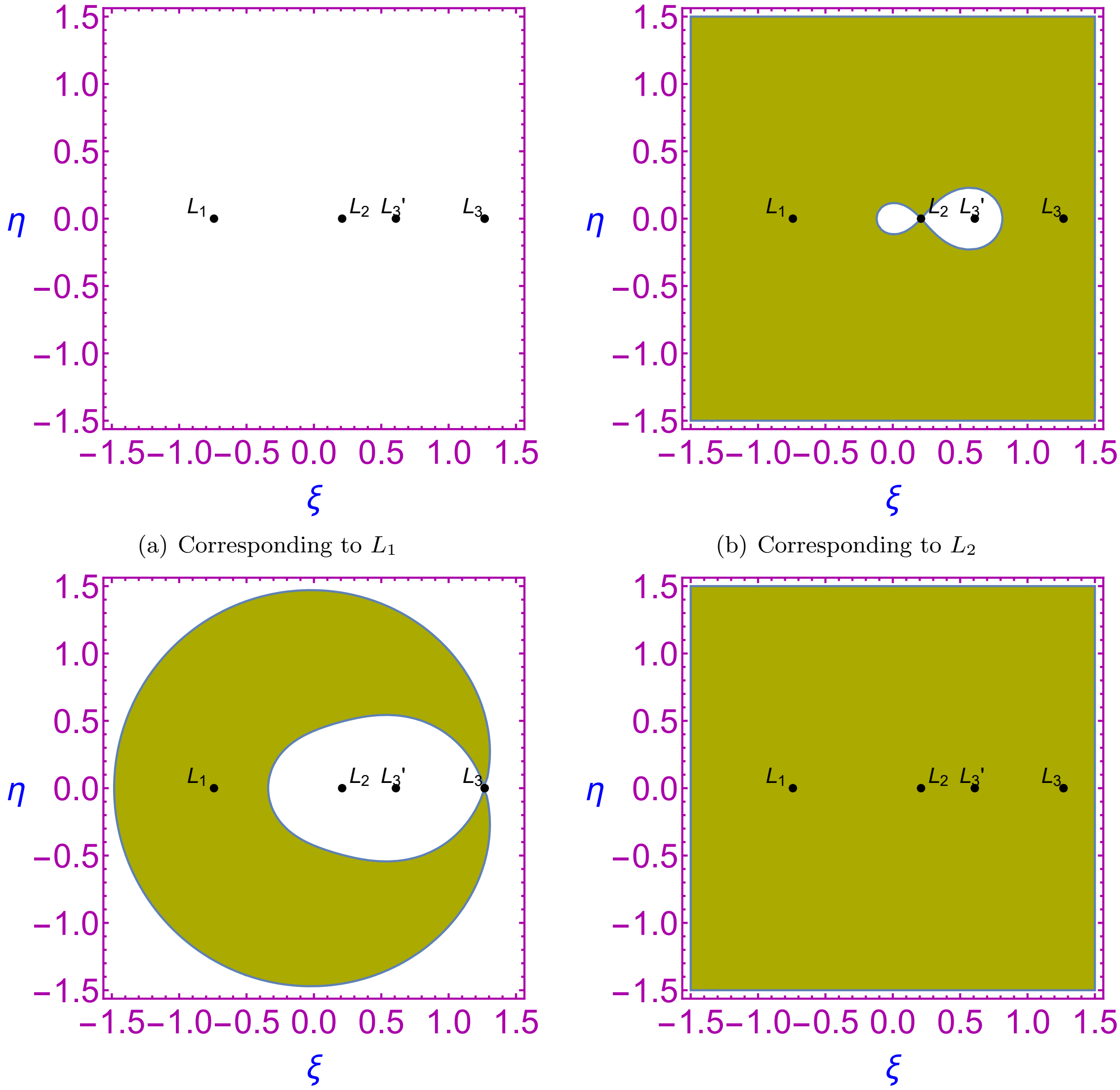
(a) Corresponding to L_1

(b) Corresponding to L_2

(c) Corresponding to L_3

(d) Corresponding to $L_{4,\,5}$

Figure 4: Regions of motion corresponding to stationary points in unperturbed case



(a) Corresponding to L_1

(b) Corresponding to L_2

(c) Corresponding to L_3

(d) Corresponding to L_3'

Figure 5: Regions of motion corresponding to stationary points in perturbed case-III

3.3. Poincaré surfaces of section

Here we have performed the Poincaré surfaces of section in unperturbed case and perturbed case-III. There are chaos in unperturbed case (figures (6(a)) and (6(b))) while there are no chaos in perturbed case-III (figures (6(c)) and (6(d))).

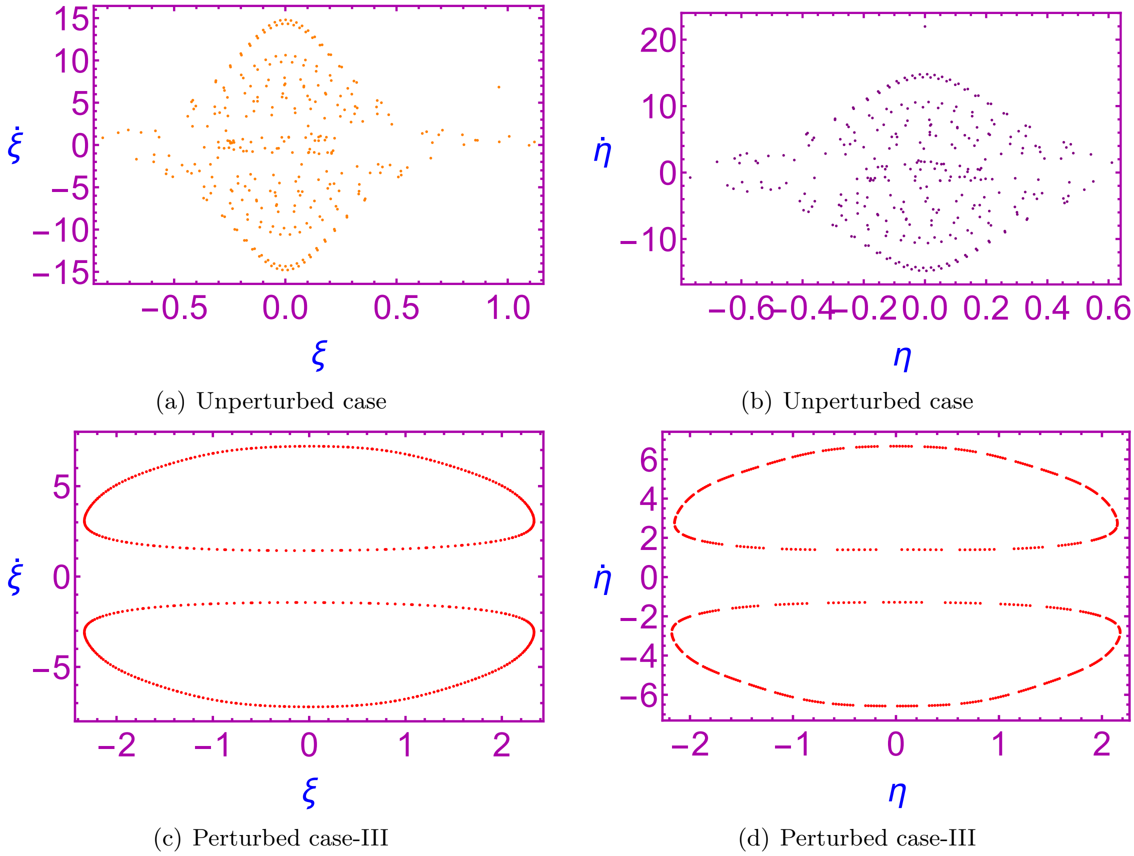
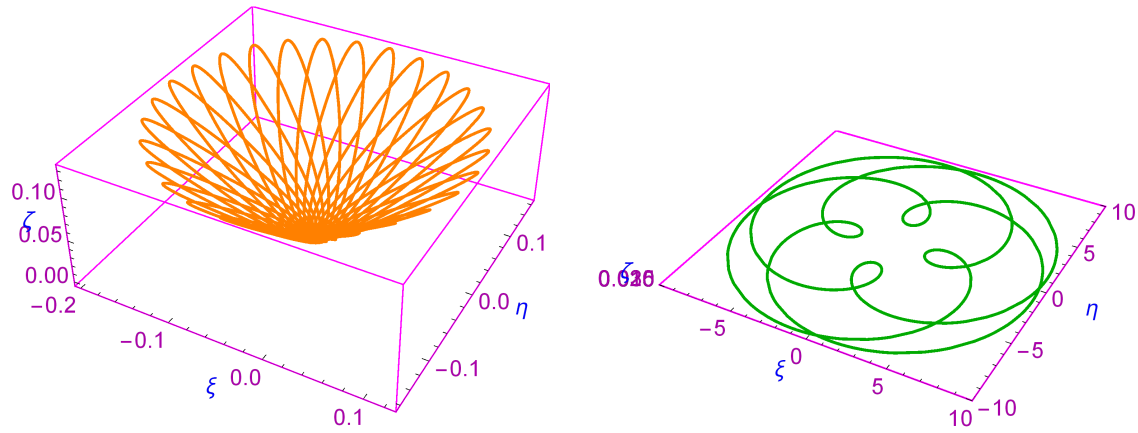


Figure 6: Poincaré surfaces of section

3.4. Periodic orbits

Utilizing the equations of motion (i.e. Eq. (4)), we have illustrated the periodic orbits in unperturbed case (figure (7(a))) and in perturbed case-III (figure (7(b))). In the unperturbed case, we used the initial conditions $\xi(0)$ = -0.2, $\eta(0) = 0$, $\zeta(0) = 0.134$, u(0) = 0, v(0) = - 0.2456, w(0) = 0 with time period 6.3 units. While in the perturbed case-III, we used the initial conditions $\xi(0)$ = -2.2, $\eta(0) = 0$, $\zeta(0) = 0.01$, u(0) = 0, v(0) = - 2.2, w(0) = 0 with time period 26.1 units.



(a) Unperturbed case

(b) Perturbed case-III

Figure 7: Periodic Orbits

3.5. Stability of stationary points

The stability of stationary points can be examined for this model for which we assume the motion near stationary point $(\xi_0,\, \eta_0\, \zeta_0)$ as $(\xi_0+\xi_{01},\, \eta_0+\eta_{01},\, \zeta_0+\zeta_{01})$, where $(\xi_{01},\, \eta_{01},\, \zeta_{01})$ are minor shift from the stationary point.

Eq. (4) can be written in phase space as:

$$\begin{aligned}
\dot{\xi}_{01} &= \xi_{02}, \\
\dot{\eta}_{01} &= \eta_{02}, \\
\dot{\zeta}_{01} &= \zeta_{02}, \\
\dot{\xi}_{02} &= 2\,n\,\phi\,\eta_{02} + (P_{\xi\,\xi})^0\,\xi_{01} + (P_{\xi\,\eta})^0\,\eta_{01} + (P_{\xi\,\zeta})^0\,\zeta_{01}, \\
\dot{\eta}_{02} &= -2\,n\,\phi\,\xi_{02} + (P_{\eta\,\xi})^0\,\xi_{01} + (P_{\eta\,\eta})^0\,\eta_{01} + (P_{\eta\,\zeta})^0\,\zeta_{01}, \\
\dot{\zeta}_{02} &= (P_{\zeta\,\xi})^0\,\xi_{01} + (P_{\zeta\,\eta})^0\,\eta_{01} + (P_{\zeta\,\zeta})^0\,\zeta_{01},
\end{aligned} \tag{8}$$

where the superscript 0 represents the value of the 2^{nd} derivatives of P at the corresponding stationary point $(\xi_0,\, \eta_0,\, \zeta_0)$ from Eq. (5).
Following the procedure given in F. Bouaziz [35], the characteristic equation for the system (8) can be written as:

$$\lambda^6 + D_5\,\lambda^5 + D_4\,\lambda^4 + D_3\,\lambda^3 + D_2\,\lambda^2 + D_1\,\lambda + D_0 = 0, \tag{9}$$

where

$$
\begin{aligned}
D_5 &= -3\,\phi, \\
D_4 &= 4\,\epsilon_1^2\,n^2 - (P_{\xi\,\xi})^0 - (P_{\eta\,\eta})^0 - (P_{\zeta\,\zeta})^0 + \frac{15}{4}\,\phi^2, \\
D_3 &= \frac{2}{3}\,D_4\,D_5, \\
D_2 &= \frac{15}{16}\,\phi^4 + \frac{3}{2}\,\phi^2\,(4\,\epsilon_1^2\,n^2 - (P_{\xi\,\xi})^0 - (P_{\eta\,\eta})^0 - (P_{\zeta\,\zeta})^0\,) \\
&\quad -[\,\{4\,\epsilon_1^2\,n^2 - (P_{\xi\,\xi})^0 - (P_{\eta\,\eta})^0\}\,(P_{\zeta\,\zeta})^0 \\
&\quad +((P_{\xi\,\eta})^0)^2 + ((P_{\xi\,\zeta})^0)^2 + ((P_{\eta\,\zeta})^0)^2 - (P_{\xi\,\xi})^0\,(P_{\eta\,\eta})^0\,], \\
D_1 &= -\frac{3}{16}\,\phi^5 + \frac{\phi^3}{2}\{\,(P_{\xi\,\xi})^0 + (P_{\eta\,\eta})^0 + (P_{\zeta\,\zeta})^0 - 4\,\epsilon_1^2\,n^2\} \qquad (10) \\
&\quad +\phi\,[((P_{\xi\,\eta})^0)^2 + ((P_{\xi\,\zeta})^0)^2 + ((P_{\eta\,\zeta})^0)^2 - (P_{\xi\,\xi})^0\;(P_{\eta\,\eta})^0 \\
&\quad \{\,4\,\epsilon_1^2\,n^2 - (P_{\xi\,\xi})^0 - (P_{\eta\,\eta})^0\,\}\,(P_{\zeta\,\zeta})^0\,], \\
D_0 &= \frac{1}{64}\,\phi^6 + \frac{1}{16}\,\phi^4\{4\,\epsilon_1^2\,n^2 - (P_{\xi\,\xi})^0 - (P_{\eta\,\eta})^0 - (P_{\zeta\,\zeta})^0\} \\
&\quad -\frac{1}{4}\,\phi^2\,[\{4\,\epsilon_1^2\,n^2 - (P_{\xi\,\xi})^0 - (P_{\eta\,\eta})^0\,\}\,(P_{\zeta\,\zeta})^0 + ((P_{\xi\,\eta})^0)^2 + ((P_{\xi\,\zeta})^0)^2 \\
&\quad +((P_{\eta\,\zeta})^0)^2 - (P_{\xi\,\xi})^0\;(P_{\eta\,\eta})^0\,] + ((P_{\xi\,\zeta})^0)^2\,(P_{\eta\,\eta})^0 + (P_{\xi\,\xi})^0\,((P_{\eta\,\zeta})^0)^2 \\
&\quad +((P_{\xi\,\eta})^0)^2\,(P_{\zeta\,\zeta})^0 - (P_{\xi\,\xi})^0\,(P_{\eta\,\eta})^0\,(P_{\zeta\,\zeta})^0 - (P_{\xi\,\eta})^0\,(P_{\xi\,\zeta})^0\,(P_{\eta\,\zeta})^0\,.
\end{aligned}
$$

The characteristic roots are evaluated numerically for the characteristic equation (9) and given in the tables (1, 2) from where we got that all the stationary points are unstable because all the roots have either at-least one positive real root or a positive real part of the complex roots.

4. Conclusion

The effects of perturbations (radiating oblate, dipole, variable mass, interactions between the bodies, coriolis and centrifugal forces) have investigated on the motion of infinitesimal body. The equations of motion and quasi-jacobian integral have determined under above said perturbations. The important dynamical properties (locations of stationary points, regions of motion, Poincaré surfaces of section and periodic orbits) have studied numerically in four cases (unperturbed case and perturbed cases-I, II, III). In the unperturbed case, we got five stationary points while in the perturbed cases-I, II, III, we got four collinear stationary points in three cases but two non-collinear stationary points in first two cases (I, II) in in-plane. In the out-of-plane, there are no stationary points exists in unperturbed case and perturbed case-I while two stationary points exists in perturbed cases I, II. In the regions of motion, colored regions are prohibited regions while white regions are allowed regions in the unperturbed case and perturbed case-III. In the Poincaré surfaces of section, there are chaos exists in unperturbed case while no chaos in

Table 1: Nature of stationary points for Perturbed case-III in $\xi-\eta$-plane.

Stationary Point		*Characteristic Roots*	*Nature*
$\xi - Co.$	$\eta - Co.$		
$-$ 0.7415901976	0.0000000000	$0.0999999999 \pm 1.0448057624\,i$ $0.0999999999 \pm 0.7181961471\,i$ $0.1000000001 \pm 0.9602580857\,i$	*Unstable*
0.2095130200	0.0000000000	$0.0999999999 \pm 13.6877489804\,i$ $0.1000000001 \pm 13.7433644537\,i$ 14.5218191189 -14.3218191189	*Unstable*
0.6085486206	0.0000000000	$0.0999999999 \pm 5001.5143072229\,i$ $0.1000000000 \pm 5001.6349575365\,i$ 5001.8441885653 -5001.6441885653	*Unstable*
1.2671447710	0.0000000000	$0.0999999999 \pm 1.3475574029\,i$ $0.1000000000 \pm 1.4196498707\,i$ 2.0637711582 -1.8637711582	*Unstable*

Table 2: Nature of stationary points for Perturbed case-III in $\xi-\zeta$-plane.

Stationary Point		*Characteristic Roots*	*Nature*
$\xi - Co.$	$\zeta - Co.$		
$-$ 0.0006425196	± 3.1052477154	$0.0999999999 \pm 1.6926544868\,i$ $0.1000000003 \pm 0.7114209254\,i$ 0.2835009113 -0.0835009113	*Unstable*

perturbed case-III. Further, in the unperturbed case and perturbed case-III, we got the periodic orbits with time period 6.3 units and 26.1 units respectively. Finally, we have examined the stability of the stationary points in the perturbed case-III and found that all the stationary points are unstable. In this way, the perturbations, we considered have excellent influence on the motion properties.

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