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A New Approximate Method For Hypersingular Integral Operators With Hilbert Kernel

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Abstract. In the present paper, the hypersingular integral operator with Hilbert kernel is approximated by a sequence of operators of the special form and is obtained the estimate of the convergence rate in Hölder spaces.

Key Words and Phrases: hypersingular integral, Hölder space, approximating operators, convergence rate.

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1. Introduction

An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. (see [6,7,12,14,16,20,22,23,26,27]). Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works [3-11,13,15,16-19,21-25,27-31] are devoted to their development. In the present paper the hypersingular integral operator

$$\left(S^{(\lambda)}\varphi\right)(t) = \frac{1}{4\pi}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)\,d\tau, 0 \le \lambda < 1,$$

is approximated by a sequences of operators of the form

$$\left(S_n^{(\lambda)}\varphi\right)(t) = \sum_{k=0}^{2n-1}\alpha_k^{(n)}(t)\,\varphi\left(t+\frac{\pi k}{n}\right), \qquad t \in T_0 = [0, 2\pi],$$

where $k = \overline{0,\, 2n-1},\ n \in N,\ \alpha_k^{(n)}(t)$ are continuous functions on T_0. It should be noted that, the determination of the inverse operator $\left[S_n^{(\lambda)}\right]^{-1}$ is equivalent to the study of the equation

$$\sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right) = f(t)\ , \quad t \in T_0$$

at the points $t, t+\frac{\pi}{n}, ..., t+\frac{\pi(2k-1)}{n}$, because by solving the resulting system of linear algebraic equations with respect to $\left(\varphi(t)\ ,\ \varphi\left(t+\frac{\pi}{n}\right), ..., \varphi\left(t+\frac{\pi(2k-1)}{n}\right)\right)$ we can obtain the function $\varphi(t)$.

Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and their applications to the singular integral equations are given in the papers [1] and [2], analogous approximations for hypersingular integral operators with Cauchy kernel are given in [3,4].

2. Hypersingular integral operators with Hilbert kernel

Consider the following integrals

$$\int_0^{2\pi} \left|\csc\frac{\tau-t}{2}\right|^m \varphi(\tau)\,d\tau, m \in N, t \in T_0 = [0, 2\pi] \tag{1}$$

$$\int_0^{2\pi} \left|\csc\frac{\tau-t}{2}\right|^{m+\lambda} \varphi(\tau)\,d\tau\ , \qquad m \in N,\ \ \lambda \in [0,1)\,, t \in T_0 = [0, 2\pi]\,, \tag{2}$$

where the function $\varphi(t)$ is Lebesque integrable in the interval T_0. If we define these integrals similar to the Cauchy integral, even if $\varphi \equiv 1$ we get the divergent integrals. Therefore, using the idea of Hadamard finite part integral [16], we will define the integrals (1) and (2) as follows.

Definition 1.1 Let $m \in N$ 2π-periodic function $\varphi(t)$ is Lebesgue integrable on T_0 and $(m-1)$ times differentiable at the point $t \in T_0$. If a finite limit

$$\lim_{\varepsilon\to 0+}\left[\int_{t-\pi}^{t-\varepsilon} \left|\csc\frac{\tau-t}{2}\right|^m \varphi(\tau)\,d\tau + \int_{t+\varepsilon}^{t+\pi} \left|\csc\frac{\tau-t}{2}\right|^m \varphi(\tau)\,d\tau - \right.$$

$$-2\sum_{k=0}^{p-1} \frac{\left[\varphi(s)\left((s-t)\csc\frac{s-t}{2}\right)^m\right]^{(2k)}(t)}{(2k)!\,(p-k)\,\varepsilon^{2p-2k}} - \frac{2}{(m-1)!}\times$$

$$\left.\times\left[\varphi(s)\cdot\left((s-t)\csc\frac{s-t}{2}\right)^m\right]^{(m-1)}(t)\ln\frac{1}{\varepsilon}\right]$$

when $m = 2p+1,\ p \in N$,

$$\lim_{\varepsilon\to 0+}\left[\int_{t-\pi}^{t-\varepsilon} \left|\csc\frac{\tau-t}{2}\right|^m \varphi(\tau)\,d\tau + \right.$$

$$+\int_{t+\varepsilon}^{t+\pi}\left|\csc\frac{\tau-t}{2}\right|^{m}\varphi(\tau)\,d\tau-2\sum_{k=0}^{p-1}\frac{\left[\varphi(s)\left|(s-t)\csc\frac{s-t}{2}\right|^{m}\right]^{(2k)}(t)}{(2k)!\,(2p-2k-1)\,\varepsilon^{2p-2k-1}}\right]$$

when $m=2p,p\in N$,

$$\lim_{\varepsilon\to0+}\left[\int_{t-\pi}^{t-\varepsilon}\left|\csc\frac{\tau-t}{2}\right|^{m}\varphi(\tau)\,d\tau+\int_{t+\varepsilon}^{t+\pi}\left|\csc\frac{\tau-t}{2}\right|^{m}\varphi(\tau)\,d\tau-4\varphi(t)\ln\frac{1}{\varepsilon}\right]$$

when $m=1$,

exists, then the value of this limit is referred to as the hypersingular integral of the function $\left|\csc\frac{\tau-t}{2}\right|^{m}\varphi(\tau)$ on T_0 and is denoted by $\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{m}\varphi(\tau)\,d\tau$.

Definition 1.2. Let $m\in N$ $0<\lambda<1$, 2π-periodic function $\varphi(t)$ is Lebesgue integrable on T_0 and $(m-1)$ times differentiable at the point $t\in T_0$. If a finite limit

$$\lim_{\varepsilon\to0+}\left[\int_{t-\pi}^{t-\varepsilon}\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)\,d\tau+\int_{t+\varepsilon}^{t+\pi}\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)\,d\tau-\right.$$

$$\left.-2\sum_{k=0}^{p-1}\frac{\left[\varphi(s)\left((s-t)\csc\frac{s-t}{2}\right)^{m+\lambda}\right]^{(2k)}(t)}{(2k)!\,(2p-2k-1+\lambda)\,\varepsilon^{2p-2k-1+\lambda}}\right]$$

when $m=2p$, $p\in N$,

$$\lim_{\varepsilon\to0+}\left[\int_{t-\pi}^{t-\varepsilon}\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)\,d\tau+\int_{t+\varepsilon}^{t+\pi}\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)\,d\tau-\right.$$

$$\left.-2\sum_{k=0}^{p-1}\frac{\left[\varphi(s)\left|(s-t)\csc\frac{s-t}{2}\right|^{m+\lambda}\right]^{(2k)}(t)}{(2k)!\,(2p-2k-2+\lambda)\,\varepsilon^{2p-2k-2+\lambda}}\right]$$

when $m=2p-1$, $p\in N$,

exists, then the value of this limit is referred to as the hypersingular integral of the function $\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)$, on T_0 and is denoted by $\int\limits_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)\,d\tau$.

Now consider the following integral

$$\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{\lambda+1}\varphi(\tau)\,d\tau,\quad\lambda\in[0,1),\ t\in T_0=[0,2\pi]\tag{3}$$

where $\varphi(t)$ is Lebesgue integrable on T_0.

Using definition 1.2, we define the integral (3) as follows:

Definition 1.3. Let $0<\lambda<1$, 2π-periodic function $\varphi(t)$ is Lebesgue integrable on T_0. If a finite limit

$$\lim_{\varepsilon\to0+}\left[\int_{t-\pi}^{t-\varepsilon}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)\,d\tau+\int_{t+\varepsilon}^{t+\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)\,d\tau-\frac{2^{2+\lambda}}{\lambda\varepsilon^{\lambda}}\cdot\varphi(t)\right]$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\left|\csc \frac{\tau - t}{2}\right|^{1+\lambda} \varphi(\tau)$, $t \in T_0$ on T_0 and is denoted by $\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right|^{1+\lambda} \varphi(\tau)\, d\tau$.

Let $H_\alpha(T_0)$, $0 < \alpha \le 1$ be the space of 2π-periodic and Hölder continuous functions with exponent α on the number axis, i.e. the space of the functions which satisfies the following condition

$$\exists M > 0 \quad \forall t_1,\ t_2 \in R: \quad |\varphi(t_1) - \varphi(t_2)| \le M \cdot |t_1 - t_2|^\alpha,$$

with the norm

$$\|\varphi\|_\alpha = \|\varphi\|_\infty + h(\varphi; \alpha),$$

where

$$\|\varphi\|_\infty = \max_{t \in T_0} |\varphi(t)|, h(\varphi; \alpha) = \sup \left\{ \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\alpha} :\ t_1, t_2 \in R,\ t_1 \ne t_2 \right\}.$$

Show that, if $\varphi \in H_\alpha(T_0)$, $0 \le \lambda < \alpha \le 1$, then hypersingular integral $\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right|^{\lambda+1} \varphi(\tau)\, d\tau$ exists for all $t \in T_0$.

Indeed, if $\varphi \in H_\alpha(T_0)$, then according to the definition of the Hölder space for any $\tau,\ t \in R$, $\tau \ne t$ the following relation is true.

$$\left|\csc \frac{\tau - t}{2}\right|^{\lambda+1} \cdot |\varphi(\tau) - \varphi(t)| \le \left(\frac{\pi}{|\tau - t|}\right)^{\lambda+1} \cdot M \cdot |\tau - t|^\alpha \le \frac{M \cdot \pi^{\lambda+1}}{|\tau - t|^{1+\lambda-\alpha}} \tag{4}$$

From inequality (4) it follows that, the integral

$$\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right|^{1+\lambda} (\varphi(\tau) - \varphi(t))\, d\tau$$

converges in the sense of Lebesque. Hence and from the existence of the hypersingular integral $\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right|^{1+\lambda} d\tau$ follows the existence of the hypersingular integral $\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right|^{1+\lambda} \varphi(\tau)\, d\tau$ at all $t \in T_0$.

If we calculate the following integrals

$$\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right| d\tau \text{ and } \int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right|^{1+\lambda} d\tau$$

using definitions 1.1and 1.2, then we get

$$\int_0^{2\pi} \left|\csc \frac{\tau - t}{2}\right| d\tau = \lim_{\varepsilon \to 0+} \left[\int_{t-\pi}^{t-\varepsilon} \left|\csc \frac{\tau - t}{2}\right| d\tau + \int_{t+\varepsilon}^{t+\pi} \left|\csc \frac{\tau - t}{2}\right| d\tau - 4 \ln \frac{1}{\varepsilon} \right] =$$

$$= \lim_{\varepsilon \to 0+} \left[\int_{t-\pi}^{t-\varepsilon} \csc \frac{\tau - t}{2} d\tau + \int_{t+\varepsilon}^{t+\pi} \csc \frac{\tau - t}{2} d\tau - 4 \ln \frac{1}{\varepsilon} \right] =$$

$$= \lim_{\varepsilon \to 0+} \left[-2 \ln \left(tg \frac{t - \tau}{4} \right) \Bigg|_{t-\pi}^{t-\varepsilon} + 2 \ln \left(tg \frac{\tau - t}{4} \right) \Bigg|_{t+\varepsilon}^{t+\pi} - 4 \ln \frac{1}{\varepsilon} \right] =$$

$$= \lim_{\varepsilon\to 0+}\left[-4\ln\left(tg\frac{\varepsilon}{4}\right)-4\ln\frac{1}{\varepsilon}\right]=4\ln 4=\tilde{r}_0, \tag{5}$$

$$\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}d\tau=\lim_{\varepsilon\to 0+}\left[\int_{t-\pi}^{t-\varepsilon}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}d\tau+\right.$$

$$\left.+\int_{t+\varepsilon}^{t+\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}d\tau-\frac{2^{2+\lambda}}{\lambda\varepsilon^{\lambda}}\right]=$$

$$=2\lim_{\varepsilon\to 0+}\left[\int_{t+\varepsilon}^{t+\pi}\frac{d\tau}{\left(\sin\frac{\tau-t}{2}\right)^{1+\lambda}}-\frac{2^{1+\lambda}}{\lambda\varepsilon^{\lambda}}\right]=4\lim_{\varepsilon\to 0+}\left[\int_{\varepsilon/2}^{\pi/2}\frac{ds}{(\sin s)^{1+\lambda}}-\frac{2^{\lambda}}{\lambda\varepsilon^{\lambda}}\right]=\tilde{r}_\lambda \tag{6}$$

where $0<\lambda<1$.

From (5), (6) it follows that

$$\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)\,d\tau=\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}[\varphi(\tau)-\varphi(t)]\,d\tau+\tilde{r}_\lambda\varphi(t)\,. \tag{7}$$

From relation (7) it follows that, if $\varphi\in H_\alpha(T_0)$, then hypersingular integral $\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)\,d\tau$ exists for all $t\in T_0$.

Consider the hypersingular integral operator:

$$\left(S^{(\lambda)}\varphi\right)(t)=\frac{1}{4\pi}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)\,d\tau, 0\le\lambda<1,$$

where $\varphi\in H_\alpha(T_0)$, $\lambda<\alpha\le 1$.

Theorem 1. Hypersingular integral operator $S^{(\lambda)}$ is bounded from the space $H_\alpha(T_0)$ into the space $H_{\alpha-\lambda-\varepsilon}(T_0)$ for all $0\le\lambda<\alpha\le 1$ and $0<\varepsilon<\alpha-\lambda$.

Proof. From equation (7) it follows that, it is sufficient to prove the stated theorem for the following operator:

$$\left(\tilde{T}\varphi\right)(t)=\frac{1}{4\pi}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}[\varphi(\tau)-\varphi(t)]\,d\tau.$$

Let $\varphi\in H_\alpha(T_0)$. Then

$$\left\|\tilde{T}\varphi\right\|_\infty=\max_{t\in T_0}\left|\frac{1}{4\pi}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}[\varphi(\tau)-\varphi(t)]\,d\tau\right|\le$$

$$\le\max_{t\in T_0}\frac{1}{4\pi}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\cdot|\varphi(\tau)-\varphi(t)|\,d\tau\le$$

$$\le\frac{h(\varphi;\alpha)}{4\pi}\max_{t\in T_0}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\cdot|\tau-t|^{\alpha}\,d\tau\le$$

$$\le \frac{h(\varphi;\alpha)}{4\pi}\max_{t\in T_0}\int_0^{2\pi}\left|\frac{\pi}{\tau-t}\right|^{1+\lambda}\cdot|\tau-t|^{\alpha}\,d\tau\le C_1\cdot h(\varphi;\alpha)\le C_1\cdot\|\varphi\|_{\alpha}\,. \tag{8}$$

where C_1- constant which only depends on α.

Estimate the difference $\left(\tilde{T}\varphi\right)(t_1)-\left(\tilde{T}\varphi\right)(t_2)$ for any two points $t_1,\ t_2\in T_0,\ t_1\ne t_2$. If $|t_1-t_2|\ge\frac{1}{2}$, then from inequality (8) it follows that

$$\left|\left(\tilde{T}\varphi\right)(t_1)-\left(\tilde{T}\varphi\right)(t_2)\right|\le 2C_1\cdot\|\varphi\|_{\alpha}\le 4C_1\cdot\|\varphi\|_{\alpha}\cdot|t_1-t_2|^{\alpha-\lambda-\varepsilon}\,. \tag{9}$$

Consider the case $|t_1-t_2|\ge\frac{1}{2}$. Without loss of generality, we can assume that $t_1<t_2$ and denote $\delta=2|t_1-t_2|$. Represent the difference $\left(\tilde{T}\varphi\right)(t_1)-\left(\tilde{T}\varphi\right)(t_2)$ as follows:

$$\left(\tilde{T}\varphi\right)(t_1)-\left(\tilde{T}\varphi\right)(t_2)=\frac{1}{4\pi}\int_{t_1-2\delta}^{t_1+2\delta}\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}\cdot[\varphi(\tau)-\varphi(t_1)]\,d\tau-$$

$$-\frac{1}{4\pi}\int_{t_1-2\delta}^{t_1+2\delta}\left|\csc\frac{\tau-t_2}{2}\right|^{1+\lambda}\cdot[\varphi(\tau)-\varphi(t_2)]\,d\tau+$$

$$+\frac{1}{4\pi}\int_{[t_1-\pi;t_1+\pi]/\,[t_1-2\delta;t_1+2\delta]}\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}\cdot[\varphi(t_2)-\varphi(t_1)]\,d\tau+$$

$$+\frac{1}{4\pi}\int_{[t_1-\pi;t_1+\pi]/\setminus[t_1-2\delta;t_1+2\delta]}[\varphi(\tau)-\varphi(t_2)]\cdot\left\{\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}-\left|\csc\frac{\tau-t_2}{2}\right|^{1+\lambda}\right\}d\tau=$$

$$=\tilde{J}_1+\tilde{J}_2+\tilde{J}_3+\tilde{J}_4. \tag{10}$$

From $\varphi\in H_{\alpha}(T_0)$, $0\le\lambda<\alpha\le 1$ we get the following estimate

$$\left|\tilde{J}_1\right|\le\frac{1}{4\pi}\int\limits_{t_1-2\delta}^{t_1+2\delta}\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}\cdot|\varphi(\tau)-\varphi(t_1)|\,d\tau\le$$

$$\le\frac{1}{4\pi}\int_{t_1-2\delta}^{t_1+2\delta}\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}\cdot h(\varphi;\alpha)\cdot|\tau-t_1|^{\alpha}\,d\tau\le$$

$$\le\frac{h(\varphi;\alpha)}{4\pi}\cdot\left(\frac{4}{\pi}\right)^{1+\lambda}\cdot\int_{t_1-2\delta}^{t_1+2\delta}\frac{d\tau}{|\tau-t_1|^{1+\lambda-\alpha}}\le C_2\cdot\|\varphi\|_{\alpha}\cdot|t_1-t_2|^{\alpha-\lambda-\varepsilon}$$

where C_2- constant which only depends on α.

The integral $\tilde{J}_2$ is estimated absolutely analogously:

$$\left|\tilde{J}_2\right|\le C_3\cdot\|\varphi\|_{\alpha}\cdot|t_1-t_2|^{\alpha-\lambda-\varepsilon}\,.$$

Estimate the integral $\tilde{J}_3$ as follows:

$$\left|\tilde{J}_3\right|\le\frac{|\varphi(t_2)-\varphi(t_1)|}{4\pi}\cdot\int_{[t_1-\pi;t_1+\pi]\setminus[t_1-2\delta;t_1+2\delta]}\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}\le$$

$$\le \frac{h(\varphi;\alpha)\cdot|t_1-t_2|^\alpha}{4\pi}\cdot\left(\frac{4}{\pi}\right)^{1+\lambda}\cdot\int_{[t_1-\pi;t_1+\pi]\setminus[t_1-2\delta;t_1+2\delta]}\frac{d\tau}{|\tau-t_1|^{1+\lambda}}\le$$

$$\le \frac{h(\varphi;\alpha)\cdot|t_1-t_2|^\alpha}{4\pi}\cdot\left(\frac{4}{\pi}\right)^{1+\lambda}\cdot\frac{C_4}{|t_1-t_2|^{\lambda+\varepsilon}}\le C_5\cdot\|\varphi\|_\alpha\cdot|t_1-t_2|^{\alpha-\lambda-\varepsilon}$$

where C_4,C_5- constants which only depend on α.

We now turn to the estimate of the integral $\tilde{J}_4$.

$$\left|\tilde{J}_4\right|\le\frac{1}{4\pi}\cdot\int_{[t_1-\pi;t_1+\pi]\setminus[t_1-2\delta;t_1+2\delta]}|\varphi(\tau)-\varphi(t_2)|\times$$

$$\times\left|\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}-\left|\csc\frac{\tau-t_2}{2}\right|^{1+\lambda}\right|d\tau\le$$

$$\le\frac{h(\varphi;\alpha)}{4\pi}\cdot\int_{[t_1-\pi;t_1+\pi]\setminus[t_1-2\delta;t_1+2\delta]}|\tau-t_2|^\alpha\cdot\left|\csc\frac{\tau-t_1}{2}\right|^{1+\lambda}\times$$

$$\times\left|\csc\frac{\tau-t_2}{2}\right|^{1+\lambda}\cdot\left|\left|\sin\frac{\tau-t_2}{2}\right|^{1+\lambda}-\left|\sin\frac{\tau-t_1}{2}\right|^{1+\lambda}\right|d\tau$$

Since for any $\tau\in[t_1-\pi;t_1+\pi]\setminus[t_1-2\delta;t_1+2\delta]$ the following inequality holds $|\tau-t_1|\le 3|\tau-t_2|$,$\left|\left|\sin\frac{\tau-t_2}{2}\right|^{1+\lambda}-\left|\sin\frac{\tau-t_1}{2}\right|^{1+\lambda}\right|\le C_6\cdot|t_1-t_2|\cdot|\tau-t_1|^\lambda$, then we get,

$$\left|\tilde{J}_4\right|\le\frac{h(\varphi;\alpha)}{4\pi}\cdot C_6\cdot|t_1-t_2|\cdot\left(\frac{4}{\pi}\right)^{2+2\lambda}\cdot\int\limits_{[t_1-\pi;t_1+\pi]\setminus[t_1-2\delta;t_1+2\delta]}\frac{d\tau}{|\tau-t_1|^{2+\lambda-\alpha}}\le$$

$$\le C_7\cdot\|\varphi\|_\alpha\cdot|t_1-t_2|^{\alpha-\lambda-\varepsilon}$$

where C_6,C_7 -constants which only depend on α.

Comparing obtained estimates for $\tilde{J}_1,\tilde{J}_2,\tilde{J}_3$ and $\tilde{J}_4$, from relation (10) and inequality (9) follows the validity of the theorem. This completes the proof of the theorem.

3. Approximation of hypersingular integral operator with Hilbert kernel.

Consider the sequences of operators

$$\left(S_n^{(\lambda)}\varphi\right)(t)=\frac{1}{2n}\sum_{k=0}^{n-1}\left|\csc\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\left[\varphi\left(t+\frac{\pi(2k+1)}{n}\right)-\varphi(t)\right]+\frac{\tilde{r}_\lambda}{4\pi}\varphi(t)\ ,$$

$t\in T_0$, $n\in N$, where $\tilde{r}_0=4\ln 4$, $\tilde{r}_\lambda=4\lim\limits_{\varepsilon\to0+}\left[\int\limits_{\varepsilon/2}^{\pi/2}\frac{ds}{(\sin s)^{1+\lambda}}-\frac{2^\lambda}{\lambda\varepsilon^\lambda}\right]$, $0<\lambda<1$.

It is clear that, operators $S_n^{(\lambda)}$, $n = 1, 2, ...$ is bounded from the space $H_\alpha(T_0)$, into the space $H_\alpha(T_0)$ for all $0 < \alpha \le 1$.

Theorem 2.1. For any $\varphi \in H_\alpha(T_0)$, the following estimates hold

$$\left\| S^{(\lambda)}\varphi - S_n^{(\lambda)}\varphi \right\|_\infty \le \frac{C_8}{n^{\alpha-\lambda}} \cdot h(\varphi; \alpha), n = 1, 2, ... \tag{11}$$

when $0 < \lambda < 1$ and

$$\left\| S^{(0)}\varphi - S_n^{(0)}\varphi \right\|_\infty \le \frac{C_9 \ln(n+1)}{n^{\alpha}} \cdot h(\varphi; \alpha), n = 1, 2, ... \tag{12}$$

when $\lambda = 0$, where C_8 and C_9 are constants which depend on α.

Proof. From (7) it follows that, for any $t \in T_0$

$$\left| \left(S^{(\lambda)}\varphi\right)(t) - \left(S_n^{(\lambda)}\varphi\right)(t) \right| = \frac{1}{4\pi} \left| \int_0^{2\pi} \left| \csc\frac{\tau - t}{2} \right|^{1+\lambda} \cdot [\varphi(\tau) - \varphi(t)]\, d\tau - \right.$$

$$\left. -\frac{2\pi}{n} \sum_{k=0}^{n-1} \left| \csc\frac{\pi(2k+1)}{2n} \right|^{1+\lambda} \left[\varphi\left(t + \frac{\pi(2k+1)}{n}\right) - \varphi(t) \right] \right| \le$$

$$\le \frac{1}{4\pi} \sum_{k=0}^{n-1} \left| \int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}} \left| \csc\frac{\tau - t}{2} \right|^{1+\lambda} \cdot [\varphi(\tau) - \varphi(t)]\, d\tau - \right.$$

$$\left. -\frac{2\pi}{n} \left| \csc\frac{\pi(2k+1)}{2n} \right|^{1+\lambda} \left[\varphi\left(t + \frac{\pi(2k+1)}{n}\right) - \varphi(t) \right] \right| = \frac{1}{4\pi} \sum_{k=0}^{n-1} \tilde{I}_k. \tag{13}$$

Estimate the difference $\tilde{I}_k$, $k = \overline{0, n-1}$. For difference $\tilde{I}_0$ we get

$$\tilde{I}_0 \le \int_t^{t+\frac{2\pi}{n}} \left| \csc\frac{\tau - t}{2} \right|^{1+\lambda} \cdot |\varphi(\tau) - \varphi(t)|\, d\tau + \frac{2\pi}{n} \left| \csc\frac{\pi}{2n} \right|^{1+\lambda} \left| \varphi\left(t + \frac{\pi}{n}\right) - \varphi(t) \right| \le$$

$$\le h(\varphi; \alpha) \cdot \left[\int_t^{t+\frac{2\pi}{n}} \left(\frac{4}{\pi}\right)^{1+\lambda} \frac{d\tau}{|\tau - t|^{1+\lambda-\alpha}} + \frac{2\pi}{n} \cdot \left(\frac{4}{\pi}\right)^{1+\lambda} \cdot \left(\frac{2n}{\pi}\right)^{1+\lambda} \cdot \left(\frac{\pi}{n}\right)^{\alpha} \right] \le$$

$$\le \frac{C_{10}}{n^{\alpha-\lambda}} \cdot h(\varphi; \alpha)$$

where C_{10}- constant which depends on α. Analogously it follows that,

$$\tilde{I}_{n-1} \le \frac{C_{10}}{n^{\alpha-\lambda}} \cdot h(\varphi; \alpha).$$

For $\tilde{I}_k$, $k = \overline{1, [n/2]}$, where $[n/2]$ is the integer part of the number $n/2$, we have

$$\tilde{I}_: \le \int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}} \left| \csc\frac{\tau - t}{2} \right|^{1+\lambda} \cdot \left[\varphi\left(t + \frac{\pi(2k+1)}{n}\right) - \varphi(t) \right] d\tau -$$

$$-\frac{2\pi}{n}\left|\csc\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\left[\varphi\left(t+\frac{\pi(2k+1)}{n}\right)-\varphi(t)\right]+$$

$$+\left|\int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\cdot\left[\varphi(\tau)-\varphi\left(t+\frac{\pi(2k+1)}{n}\right)\right]d\tau\right|=\tilde{I}_k^{(1)}+\tilde{I}_k^{(2)}. \qquad (14)$$

Now estimate the difference $\tilde{I}_k^{(1)}$ as follows:

$$\tilde{I}_k^{(1)}\le\left|\varphi\left(t+\frac{\pi(2k+1)}{n}\right)-\varphi(t)\right|\times$$

$$\times\left|\int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}d\tau-\frac{2\pi}{n}\left|\csc\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\right|\le$$

$$\le h(\varphi;\alpha)\cdot\left(\frac{\pi(2k+1)}{n}\right)^{\alpha}\cdot\left|\int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}-\left|\csc\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\right|d\tau. \qquad (15)$$

Since for any $\tau\in\left[t+\frac{2\pi k}{n};t+\frac{2\pi(k+1)}{n}\right]$ the following inequality holds

$$\left|\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}-\left|\csc\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\right|=$$

$$=\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\cdot\left|\csc\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\cdot\left|\left|\sin\frac{\tau-t}{2}\right|^{1+\lambda}-\left|\sin\frac{\pi(2k+1)}{2n}\right|^{1+\lambda}\right|\le$$

$$\le\left(\frac{4}{\pi}\right)^{2+2\lambda}\cdot\frac{1}{|\tau-t|^{1+\lambda}}\cdot\left(\frac{n}{\pi(2k+1)}\right)^{\lambda+1}\cdot C_{11}\cdot\left|\frac{\tau-t}{2}-\frac{\pi(2k+1)}{2n}\right|\cdot\left|\frac{\tau-t}{2}\right|^{\lambda}\le$$

$$\le C_{12}\cdot\frac{n^{\lambda+1}}{k^{\lambda+2}},$$

where C_{11},C_{12}- constants which only depend on α.

Then from the inequality (15) we get the following inequality

$$\tilde{I}_k^{(1)}\le C_{13}h(\varphi;\alpha)\cdot\frac{k^{\alpha}}{n^{\alpha}}\cdot\frac{n^{\lambda+1}}{k^{\lambda+2}}\cdot\frac{1}{n}=C_{13}h(\varphi;\alpha)\cdot\frac{1}{n^{\alpha-\lambda}}\cdot\frac{1}{k^{\lambda+2-\alpha}}$$

where C_{13}- constant which only depends on α.

Now turn to the estimate of the integral $\tilde{I}_k^{(2)}$.

$$\tilde{I}_k^{(2)}\le\int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\cdot\left|\varphi(\tau)-\varphi\left(t+\frac{\pi(2k+1)}{n}\right)\right|d\tau\le$$

$$\le\left(\frac{4}{\pi}\right)^{1+\lambda}\cdot h(\varphi;\alpha)\cdot\int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}}\frac{\left|\tau-t-\frac{\pi(2k+1)}{n}\right|^{\alpha}}{|\tau-t|^{1+\lambda}}d\tau\le$$

$$\leq \left(\frac{4}{\pi}\right)^{1+\lambda} \cdot h(\varphi;\alpha) \cdot \left(\frac{\pi}{n}\right)^{\alpha} \cdot \int_{t+\frac{2\pi k}{n}}^{t+\frac{2\pi(k+1)}{n}} \frac{d\tau}{|\tau - t|^{1+\lambda}} \leq C_{14} h(\varphi;\alpha) \cdot \frac{1}{n^{\alpha-\lambda}} \cdot \frac{1}{k^{1+\lambda}}$$

where C_{14}- constant which only depends on α.

Comparing obtained estimates for $\tilde{I}_k^{(1)}$ and $\tilde{I}_k^{(2)}$from inequality (14) it follows the following inequality

$$\tilde{I}_k \leq C_{13} \cdot h(\varphi;\alpha) \cdot \frac{1}{n^{\alpha-\lambda}} \cdot \frac{1}{k^{2+\lambda-\alpha}} + C_{14} \cdot h(\varphi;\alpha) \cdot \frac{1}{n^{\alpha-\lambda}} \cdot \frac{1}{k^{1+\lambda}} \leq C_{15} \cdot h(\varphi;\alpha) \cdot \frac{1}{n^{\alpha-\lambda}} \cdot \frac{1}{k^{1+\lambda}},$$

where $k = \overline{1, [n/2]}$ and C_{15}- constant which only depends on α. But, for $k = \overline{[n/2]+1, n-2}$ analogously we get

$$\tilde{I}_k \leq C_{15} \cdot h(\varphi;\alpha) \cdot \frac{1}{n^{\alpha-\lambda}} \cdot \frac{1}{(n-1-k)^{1+\lambda}}.$$

Comparing obtained estimates for $\tilde{I}_k$, $k = \overline{0, n-1}$, from inequality (13) follows that

$$\left| \left(S^{(\lambda)}\right) \varphi(t) - \left(S_n^{(\lambda)}\right) \varphi(t) \right| \leq \frac{2C_{10}}{n^{\alpha-\lambda}} h(\varphi;\alpha) + \frac{2C_{15}}{n^{\alpha-\lambda}} h(\varphi;\alpha) \sum_{k=1}^{[n/2]} \frac{1}{k^{1+\lambda}}. \qquad (16)$$

Since we get $\sum_{k=1}^{[n/2]} \frac{1}{k^{1+\lambda}} \leq \ln(1+n)$, when $\lambda = 0$ and $\sum_{k=1}^{[n/2]} \frac{1}{k^{1+\lambda}} \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+\lambda}} = C_{27}$, when $0 < \lambda < 1$

Then from (16) follows the estimates (11) and (12). This completes the proof of the stated theorem.

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