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On a Periodic Solution for an Impulsive System of Differential Equations with Gerasimov–Caputo Fractional Operator and Maxima

Tursun K. Yuldashev, Khanlar R. Mamedov, Tohirjon A. Abduvahobov

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Abstract. Existence and uniqueness of $(\omega, c)-$periodic solution of boundary value problem for a system of ordinary differential equations with Gerasimov–Caputo operator, impulsive effects and maxima are investigated. This problem is reduced to the investigation of solvability of the system of nonlinear functional integral equations. The method of contracted mapping is used in the proof of one-valued solvability of nonlinear functional integral equations. Obtained some estimates for the $(\omega, c)-$periodic solution of the studying problem.

Key Words and Phrases: impulsive system of differential equations, Gerasimov–Caputo operator, $(\omega, c)-$periodic solution, contracted mapping, existence and uniqueness.

2010 Mathematics Subject Classifications: Primary 34B37, 34C25, 34B15

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1. Introduction

Differential equations, the solution of which is functions with first kind discontinuities at times, are called differential equations with impulse effects. One can see a lot of publications of studying on differential equations with impulsive effects, describing many natural and practical processes (for examples, [1, 2, 3, 4, 5, 6, 7]). The interest in the study of nonlocal problems for the impulsive differential equations is only increasing [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Fractional calculus plays an important role in the mathematical modeling of many problems in scientific and engineering disciplines [20, 21, 22, 23]. In the works [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], the problems of applying of the fractional order integro-differential operators to the theory of differential equations are considered.

In the works [18, 19], the periodic solutions of impulsive differential equations are studied. In the works [35, 36, 37], the problems of existence of $(\omega, c)-$periodic solutions for impulsive differential equations are considered.

According to the works [35, 36, 37], a continuous function $f : \mathbb{R} \to \mathbb{X}$ is $(\omega, c)-$periodic, if $f(t+\omega) = c \cdot f(t)$ for all $t \in \mathbb{R}$, where $0 < c = \text{const}, \ c \neq 1, \ 0 < \omega < \infty, \ \mathbb{X}$ is closed set.

In our present paper on the interval $\Omega \equiv [0,\omega] \setminus \left\{t_i\right\}$ for $i = 1,2,...,p$ we consider the questions of existence of the $(\omega, c)-$periodic solutions of the impulsive nonlinear system of $\alpha-$order differential equations with maxima

$$ {}_C D^{\alpha}_{0t} x(t) = f\left(t, x(t), \max\left\{x(\tau)\,|\tau \in [t-h,t]\right\}\right), \tag{1} $$

where $0 < h = \text{const}$ is delay.

The equation (1) we study with $(\omega, c)-$periodic condition

$$ x(\omega) = c \cdot x(0) \tag{2} $$

and nonlinear impulsive effects

$$ x\left(t_i^+\right) - x\left(t_i^-\right) = F_i\left(x\left(t_i\right)\right), \qquad i = 1,2,...,p, \tag{3} $$

where ${}_C D^{\alpha}_{0t}$ is the Gerasimov–Caputo $\alpha-$order fractional derivative for a function $x(t)$ and defined by

$$ {}_C D^{\alpha}_{0t} x(t) = I^{1-\alpha}_{0t} x'(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{x'(s)\,ds}{(t-s)^{\alpha}}, \ \ t \in (0,\,\omega), \tag{4} $$

$$ I^{\alpha}_{0t} x(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} x(s) ds, \quad 0 < \alpha \le 1, \tag{5} $$

$\Gamma(\alpha)$ is the Gamma-function, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = \omega$, $x \in \mathbb{X}$, $\mathbb{X}$ is the closed bounded domain in the space $\mathbb{R}^n$, $f \in \mathbb{R}^n$, $x\left(t_i^+\right) = \lim\limits_{\nu \to 0^+} x\left(t_i + \nu\right)$, $x\left(t_i^-\right) = \lim\limits_{\nu \to 0^-} x\left(t_i - \nu\right)$.

We note that this paper is further development of the work [18]. We recall that in the work [18] it is studied the differential equation (1) when $\alpha = c = 1$. However, the work [18] is not particular case of the present paper. Because in our work we assume that $c \neq 1$.

By $C\left([0,\omega], \mathbb{R}^n\right)$ is denoted the Banach space, which consists continuous vector functions $x(t)$, defined on the segment $[0,\omega]$, with the norm

$$ \parallel x(t) \parallel = \sqrt{\sum_{j=1}^n \max_{0 \le t \le \omega} |\, x_j(t)\,|}. $$

By $PC\left([0,\omega], \mathbb{R}^n\right)$ is denoted the following linear vector space

$$ PC\left([0,\omega], \mathbb{R}^n\right) = \Big\{x : [0,\omega] \to \mathbb{R}^n;\ x(t) \in C\left(\left(t_i, t_{i+1}\right], \mathbb{R}^n\right),\ i = 1,...,p\Big\}, $$

where $x\left(t_i^+\right)$ and $x\left(t_i^-\right)$ $(i = 0,1,...,p)$ exist and are bounded; $x\left(t_i^-\right) = x\left(t_i\right)$. Note, that the linear vector space $PC\left([0,\omega], \mathbb{R}^n\right)$ is Banach space with the norm

$$ \parallel x(t) \parallel_{PC} = \max\Big\{\parallel x(t) \parallel_{C(t_i, t_{i+1}]},\ i = 1,2,...,p\Big\}. $$

We use also the vector space $BD\left([0,\omega],\mathbb{R}^n\right)$, which is Banach space with the following norm

$$\|\, x(t)\,\|_{BD} = \|\, x(t)\,\|_{PC} + h\cdot \left\|\, x'(t)\,\right\|_{PC},$$

where $0 < h = \text{const}$ is delay given by equation (1).

Formulation of problem. To find the $(\omega, c)-$periodic function $x(t) \in BD\left([0,\omega],\mathbb{R}^n\right)$, which for all $t \in \Omega$ satisfies the system of differential equations (1) for $0 < \alpha \leq 1$, $(\omega, c)-$periodic condition (2) and for $t = t_i,\ \ i = 1, 2, ..., p,\ \ 0 < t_1 < t_2 < ... < t_p < \omega$, satisfies the nonlinear limit condition (3).

2. Reduction to functional integral equation

Let the function $x(t) \in BD\left([0,\omega],\mathbb{R}^n\right)$ is a solution of the $(\omega, c)-$periodic boundary value problem (1)–(3). Then, by virtue of (4) and (5), after integration on the intervals $(0,t_1]\,,\ (t_1,t_2]\,,\ \ldots,\ (t_p,t_{p+1}]$ we have:

$$I^{\alpha}_{0\,t_1}\,{}_C D^{\alpha}_{0\,t_1}\left(x(t)\right) = I^{\alpha}_{0\,t_1} I^{1-\alpha}_{0\,t_1} x'(t) = I^{1}_{0\,t_1} x'(t) = \int\limits_0^{t_1} x'(s)ds = x(t_1^-) - x(0^+), \tag{6}$$

$$I^{\alpha}_{0t_1} f\left(t, x(t), y(t)\right) = \frac{1}{\Gamma(\alpha)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-1} f\left(s, x(s), y(s)\right) ds, \tag{7}$$

$$\frac{1}{\Gamma(\alpha)} \int\limits_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f\left(s, x(s), y(s)\right) ds = x(t_2^-) - x(t_1^+), \tag{8}$$

$$\vdots$$

$$\frac{1}{\Gamma(\alpha)} \int\limits_{t_p}^{\omega} (T - s)^{\alpha-1} f\left(s, x(s), y(s)\right) ds = x(t_{p+1}^-) - x(t_p^+), \tag{9}$$

where $f\left(t, x(t), y(t)\right) = f\left(t, x(t), \max\left\{x(\tau)\,|\tau \in [t-h,t]\,\right\}\right).$

From the formulas (6)–(9) and $x(0^+) = x(0),\ \ x(t_{p+1}^-) = x(t)$, on the interval $(0,\omega]$ we have

$$\frac{1}{\Gamma(\alpha)} \int\limits_0^{t} (t - s)^{\alpha-1} f\left(s, x(s),\, y(s)\right) ds =$$

$$= -x(0) - \left[x\left(t_1^+\right) - x\left(t_1\right)\right] - \left[x\left(t_2^+\right) - x\left(t_2\right)\right] - \ \ldots\ - \left[x\left(t_p^+\right) - x\left(t_p\right)\right] + x(t).$$

Taking into account the impulsive condition (3) in the last equality, we obtain

$$x(t) = x(0) + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right) + \frac{1}{\Gamma(\alpha)} \int\limits_0^{t} (t - s)^{\alpha-1} f\left(s, x(s), y(s)\right) ds. \tag{10}$$

Hence, we have

$$x(\omega) = x(0) + \sum_{0<t_i<\omega} F_i\left(x\left(t_i\right)\right) + \frac{1}{\Gamma(\alpha)} \int\limits_0^{\omega} (\omega - s)^{\alpha-1} f\left(s, x(s), y(s)\right) ds. \tag{11}$$

Let the function $x(t) \in BD\left([0,\omega], \mathbb{R}^n\right)$ in (10), satisfies the boundary value condition (2). Then from (11) we have

$$x(0) = \frac{1}{c-1} \sum_{0<t_i<\omega} F_i\left(x\left(t_i\right)\right) + \frac{1}{(c-1)\Gamma(\alpha)} \int\limits_0^{\omega} (\omega - s)^{\alpha-1} f\left(s, x(s), y(s)\right) ds. \tag{12}$$

By virtue of (12), from (10) we obtain the functional differential equation

$$x(t) = J(t;x) \equiv \frac{1}{c-1} \sum_{0<t_i<\omega} F_i\left(x\left(t_i\right)\right) + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right) +$$

$$+ \frac{1}{(c-1)\Gamma(\alpha)} \int\limits_0^{\omega} (\omega - s)^{\alpha-1} f\left(s, x(s), \max\left\{x(\tau)\left|\tau \in [s-h, s]\right.\right\}\right) ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int\limits_0^{t} (t - s)^{\alpha-1} f\left(s, x(s), \max\left\{x(\tau)\left|\tau \in [s-h, s]\right.\right\}\right) ds. \tag{13}$$

Lemma 2.1. *For the equation* (13) *is true the following estimate*

$$\| J(t;x) \|_{PC} \leq \left| \frac{c}{c-1} \right| \left[\frac{\omega^{\alpha}}{\alpha\,\Gamma(\alpha)} M_1 + p \cdot M_2 \right], \tag{14}$$

where $M_1 = \| f \|, \ \ M_2 = \max\limits_{1 \leq i \leq p} \| F_i \|.$

Proof. From the equation (13) we obtain

$$\| J(t;x) \|_{PC} \leq \frac{1}{|\, c-1 \,|} \sum_{0<t_i<\omega} \|F_i\| + \sum_{0<t_i<t} \|F_i\| +$$

$$+ \frac{1}{|\, c-1 \,|\, \Gamma(\alpha)} \int\limits_0^{\omega} (\omega - s)^{\alpha-1} \, \| f \| \, ds + \frac{1}{\Gamma(\alpha)} \int\limits_0^{t} (t-s)^{\alpha-1} \, \| f \| \, ds \leq$$

$$\leq p \left| \frac{c}{c-1} \right| \max_{1 \leq i \leq p} \| F_i \| + \| f \| \left[\frac{1}{|\, c-1 \,|\, \Gamma(\alpha)} \int\limits_0^{\omega} (\omega - s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int\limits_0^{t} (t-s)^{\alpha-1} ds \right] \leq$$

$$\leq \left| \frac{c}{c-1} \right| \left[p \cdot \max_{1 \leq i \leq p} \| F_i \| + \frac{\omega^{\alpha}}{\alpha\,\Gamma(\alpha)} \| f \| \right]. \tag{15}$$

From the estimate (15) follows (14). Lemma 2.1 is proved. □

Lemma 2.2. *Let the following conditions to be met: there exist constants P_1, P_2, Q_1, Q_2 such that*

$$\| f\,(t, x(t), y(t))\,\| \leq P_1 \,\| x(t) \,\| + Q_1 < \infty, \quad \max_{1\leq i\leq p} \| \,F_i\,(x(t_i))\,\| \leq P_2 \max_{1\leq i\leq p} \| \,x(t_i)\,\| + Q_2 < \infty.$$

Then the following estimate is true

$$\| \,x(t)\,\|_{PC} \leq \frac{\mu}{1-\nu}, \tag{16}$$

where

$$\mu = p\,Q_2 \left| \frac{c}{c-1} \right| + Q_1 \frac{1 + |\,c-1\,|}{|\,c-1\,|} \frac{\omega^\alpha}{\alpha\,\Gamma(\alpha)}, \quad \nu = p \left| \frac{c}{c-1} \right| P_2 + \frac{1 + |\,c-1\,|}{|\,c-1\,|} \frac{\omega^\alpha}{\alpha\,\Gamma(\alpha)} P_1 < 1.$$

Proof. Similar to the proof of the Lemma 2.1 above, we obtain

$$\| \,J(t;x)\,\|_{PC} \leq \frac{1}{|\,c-1\,|} \sum_{0<t_i<\omega} \|F_i\| + \sum_{0<t_i<t} \|F_i\| +$$

$$+ \frac{1}{|\,c-1\,|\,\Gamma(\alpha)} \int\limits_0^\omega (\omega - s)^{\alpha-1} \,\|\, f \,\|\, ds + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \,\|\, f\,\|\, ds \leq$$

$$\leq p \left| \frac{c}{c-1} \right| \left[P_2 \max_{1\leq i\leq p} \| \,x(t_i)\,\| + Q_2 \right] +$$

$$+ \left[P_1 \,\| \,x(t)\,\| + Q_1 \right] \left[\frac{1}{|\,c-1\,|\,\Gamma(\alpha)} \int\limits_0^\omega (\omega - s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} ds \right] \leq$$

$$\leq p \left| \frac{c}{c-1} \right| P_2 \max_{1\leq i\leq p} \| \,x(t_i)\,\| + \frac{1 + |\,c-1\,|}{|\,c-1\,|} \frac{\omega^\alpha}{\alpha\,\Gamma(\alpha)} P_1 \,\| \,x(t)\,\| +$$

$$+ p\,Q_2 \left| \frac{c}{c-1} \right| + Q_1 \frac{1 + |\,c-1\,|}{|\,c-1\,|} \frac{\omega^\alpha}{\alpha\,\Gamma(\alpha)}.$$

Hence, we derive the estimate (16). The Lemma 2.2 is proved. □

Lemma 2.3 ([18])**.** *For the difference of two functions with maxima there holds the following estimate*

$$\| \max \,\{x(\tau)\,|\,\tau \in [t-h, t]\,\} - \max\,\{y(\tau)\,|\,\tau \in [t-h, t]\,\}\,\| \leq$$

$$\leq \| \,x(t) - y(t)\,\| + 2h \left\| \frac{\partial}{\partial\, t} \left[x(t) - y(t) \right] \right\|,$$

where $0 < h = \text{const}$.

Theorem 2.4. *Let* $f:\Omega\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *be a continuous function and* $f(t+\omega,cx,cy)=c\,f(t,x,y)$. *Assume that there exist positive quantities* M_1, M_2, L_1, L_{2i} *such that for all* $t\in\Omega$ *are fulfilled the following conditions:*
1. *For the positive integer* p, *there hold* $F_i=F_{i+p},\ \ t_{i+p}=t_i+\omega$;
2. $\|\,f\,(t,x(t),y(t))\,\|\le M_1<\infty,\ \ \max\limits_{1\le i\le p}\|\,F_i\,(x(t_i))\,\|\le M_2<\infty$;
3. $\|\,f\,(t,x_1,y_1)-f\,(t,x_2,y_2)\,\|\le L_1\big[\,\|\,x_1-x_2\,\|+\|\,y_1-y_2\,\|\,\big]$;
4. $\|\,F_i\,(t,x_1)-F_i\,(t,x_2)\,\|\le L_{2i}\,\|\,x_1-x_2\,\|$;
5. *The radius of the inscribed ball in* $\mathbb{X}$ *is greater than* $\left|\frac{c}{c-1}\right|\left[\frac{\omega^\alpha}{\alpha\,\Gamma(\alpha)}M_1+p\cdot M_2\right]$;
6. $\rho<1$, *where* $\rho=\max\,\{\beta_1+\beta_2;\ \gamma_1+\gamma_2\}$ *and* $\beta_1,\beta_2,\gamma_1,\gamma_2$ *are defined from* (20), (21), (23) *and* (24) *below.*

Then the problem (1)–(3) *has a unique* $(\omega,c)-$*periodic solution for all* $t\in\Omega$.

Proof. The theorem we proof by the fixed-point method. According to the theorem condition, we have

$$f(t+\omega,x(t+\omega),y(t+\omega))=f(t+\omega,\,c\,x(t),\,c\,y(t))=c\,f(t,x(t),y(t)).$$

We differentiate (13):

$$x'(t)=J(t;x')\equiv\frac{\alpha-1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-2}f\,(s,x(s),\max\,\{x(\tau)\,|\tau\in[s-h,s]\,\})\,ds. \qquad (17)$$

For the difference of two operators in (13), we have estimate

$$\|J(t;x)-J(t;y)\|\le$$

$$\le\frac{1}{|\,c-1\,|}\sum_{0<t_i<\omega}\|\,F_i\,(x\,(t_i))-F_i\,(y\,(t_i))\,\|+\sum_{0<t_i<t}\|\,F_i\,(x\,(t_i))-F_i\,(y\,(t_i))\,\|+$$

$$+\frac{1}{|\,c-1\,|\;\Gamma(\alpha)}\int\limits_0^\omega(\omega-s)^{\alpha-1}\,\|f\,(s,x(s),\max\,\{x(\tau)\,|\tau\in[s-h,s]\,\})-$$

$$-f\,(s,y(s),\max\,\{y(\tau)\,|\tau\in[s-h,s]\,\})\,\|\,ds+$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\,\|\,f\,(s,x(s),\max\,\{x(\tau)\,|\tau\in[s-h,s]\,\})-$$

$$-f\,(s,y(s),\max\,\{y(\tau)\,|\tau\in[s-h,s]\,\})\,\|\;ds.$$

Hence, using conditions of the theorem, we have

$$\|J(t;x)-J(t;y)\|\le$$

$$\le\frac{1}{|\,c-1\,|}\sum_{0<t_i<\omega}L_{\,2i}\,\|\,x\,(t_i)-y\,(t_i)\,\|+\sum_{0<t_i<t}L_{\,2i}\,\|\,x\,(t_i)-y\,(t_i)\,\|+$$

$$+\frac{L_1}{|\,c-1\,|\;\Gamma(\alpha)}\int\limits_0^{\omega}(\omega-s)^{\alpha-1}\big[\,\|\,x(s)-y(s)\,\|+\|\max\,\{x(\tau)\,|\tau\in[s-h,s]\,\}-$$

$$-\max\,\{y(\tau)\,|\tau\in[s-h,s]\,\}\,\|\,\big]ds+$$

$$+\frac{L_1}{\Gamma(\alpha)}\int\limits_0^{t}(t-s)^{\alpha-1}\big[\,\|\,x(s)-y(s)\,\|+\|\max\,\{x(\tau)\,|\tau\in[s-h,s]\,\}-$$

$$-\max\,\{y(\tau)\,|\tau\in[s-h,s]\,\}\,\|\,\big]ds. \tag{18}$$

By virtue of estimate given in the Lemma 2.3, from (18) we obtain that

$$\|J(t;x)-J(t;y)\|\le\left|\frac{c}{c-1}\right|\sum_{i=1}^{p}L_{2i}\,\|\,x\,(t_i)-y\,(t_i)\,\|+$$

$$+\frac{2L_1}{|\,c-1\,|\;\Gamma(\alpha)}\int\limits_0^{\omega}(\omega-s)^{\alpha-1}\big[\,\|\,x(s)-y(s)\,\|+h\cdot\,\left\|\,x'(s)-y'(s)\,\right\|\,\big]ds+$$

$$+\frac{2L_1}{\Gamma(\alpha)}\int\limits_0^{t}(t-s)^{\alpha-1}\big[\,\|\,x(s)-y(s)\,\|+h\cdot\,\left\|\,x'(s)-y'(s)\,\right\|\,\big]ds\le$$

$$\le\beta_1\,\|\,x(t)-y(t)\,\|+\gamma_1 h\,\left\|\,x'(t)-y'(t)\,\right\|, \tag{19}$$

where

$$\beta_1=\left|\frac{c}{c-1}\right|\sum_{i=1}^{p}L_{2i}+\frac{1+|\,c-1\,|}{|\,c-1\,|}\frac{2L_1}{\Gamma(\alpha)}\frac{\omega^{\alpha}}{\alpha}, \tag{20}$$

$$\gamma_1=2\frac{1+|\,c-1\,|}{|\,c-1\,|}\frac{L_1}{\Gamma(\alpha)}\frac{\omega^{\alpha}}{\alpha}. \tag{21}$$

Now for the difference of two operators in (17). Similarly, we have estimate

$$\left\|J(t;x')-J(t;y')\right\|\le$$

$$\le 2L_1\frac{\alpha-1}{\Gamma(\alpha)}\sup_{t\in\Omega}\int\limits_0^{t}(t-s)^{\alpha-2}\big[\,\|\,x(s)-y(s)\,\|+h\cdot\,\left\|\,x'(s)-y'(s)\,\right\|\,\big]ds\le$$

$$\le\beta_2\,\|\,x(t)-y(t)\,\|+\gamma_2 h\,\left\|\,x'(t)-y'(t)\,\right\|, \tag{22}$$

where

$$\beta_2=2hL_1\frac{\alpha-1}{\Gamma(\alpha)}\max_{t\in\Omega}\int\limits_0^{t}(t-s)^{\alpha-2}ds, \tag{23}$$

$$\gamma_2 = 2hL_1 \frac{\alpha-1}{\Gamma(\alpha)} \max_{t\in\Omega} \int\limits_0^t (t-s)^{\alpha-2} ds. \tag{24}$$

We multiply both sides of (22) to h term by term. Then, adding the estimates (19) and (22) term by term, we obtain that

$$\|J(t;x) - J(t;y)\|_{BD} \le \rho \cdot \|x(t) - y(t)\|_{BD}, \tag{25}$$

where $\rho = \max\left\{\beta_1 + \beta_2;\ \gamma_1 + \gamma_2\right\}$.

According to the last condition of the theorem $\rho < 1$, so right-hand side of (13) as an operator is contraction mapping. From the estimates (14), (16) and (25) implies that there exists a unique fixed point $x(t)$, satisfying equation (1) and $(\omega, c)-$periodic condition (2). Moreover, we obtain that

$$\|x(t)\| \le \frac{1}{|c-1|} \sum_{0<t_i<\omega} \left[L_{2i} \|x(t_i)\| + \|F_i(0)\|\right] + \sum_{0<t_i<t} \left[L_{2i} \|x(t_i)\| + \|F_i(0)\|\right] +$$

$$+ \frac{L_1}{|c-1|\,\Gamma(\alpha)} \int\limits_0^\omega (\omega - s)^{\alpha-1} \big[\, \|x(s)\| + \|\max\{x(\tau)\,|\tau \in [s-h, s]\}\| +$$

$$+ \|f(s,0,0)\| \,\big] ds +$$

$$+ \frac{L_1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \big[\, \|x(s)\| + \|\max\{x(\tau)\,|\tau \in [s-h, s]\}\| +$$

$$+ \|f(s,0,0)\| \,\big] ds \le \left|\frac{c}{c-1}\right| \sum_{i=1}^{p} L_{2i} \|x(t_i)\| +$$

$$+ \frac{1+|c-1|}{|c-1|} \frac{2L_1}{\Gamma(\alpha)} \int\limits_0^\omega (\omega - s)^{\alpha-1} \|x(s)\| ds +$$

$$+ p\left|\frac{c}{c-1}\right| M_2 + M_1 \frac{1+|c-1|}{|c-1|} \frac{2L_1}{\Gamma(\alpha)} \int\limits_0^\omega (\omega - s)^{\alpha-1} ds. \tag{26}$$

From (26) we derive the following estimate

$$\|x(t)\|_{BD} \le \frac{q}{1-\rho}, \tag{27}$$

where

$$q = p\left|\frac{c}{c-1}\right| M_2 + M_1 \frac{1+|c-1|}{|c-1|} \frac{2L_1}{\Gamma(\alpha)} \int\limits_0^\omega (\omega - s)^{\alpha-1} ds < \infty.$$

The theorem is proved. □

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Tursun K. Yuldashev
Tashkent State University of Economics, 100066, Karimov street, 49, TSUE, Tashkent, Uzbekistan
E-mail: tursun.k.yuldashev@gmail.com

Khanlar R. Mamedov
İgdir university, İgdir, Turkey
E-mail: hanlar.residoglu@igdir.edu.tr

Tohirjon A. Abduvahobov
Tashkent State University of Economics, 100066, Karimov street, 49, TSUE, Tashkent, Uzbekistan
E-mail: tohirjonabduvahopov2603@gmail.com

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