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Petal Numbers of Rational Maps

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Abstract. The aim in this paper, we investigate how changes the petal numbers of rational maps that arise relaxed Newton's map given by

$$N_{F,h}(z) = \frac{(n+5-h)z^{n+5} + c(n-h)z^n + (h-5)z^5 + ch}{(n+5)z^{n+4} + cnz^{n-1} - 5z^4}$$

where c complex parameter, n positive integer and $h \in]0,1]$. Although the dynamics on these maps are almost always very different, the number of petals depends on the degree of map $f(z) = z^n - 1$ which perturbs the complex polynomial $P(z) = z^5 + c$.

Key Words and Phrases: Relaxed Newton's Method, Perturbation, Rational Iterations, Julia Set, Attracting Petals

2010 Mathematics Subject Classifications: 7Fxx, 37F05, 37F10

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1. Introduction

This paper is about rational iteration for perturbed the complex polynomial $P(z) = z^5 + c$ by maps $f(z) = z^n - 1$ which examines the resulting dynamics when relaxed Newton's method is applied to $F(z) = (z^5 + c)(z^n - 1)$.

Newton's method to the form $N_f(x) = x - \frac{f(x)}{f'(x)}$, probably the oldest and most famous iterative process to be found in mathematics developed by Isaac Newton in the late 1600's. It can be used to approximate both real and complex solutions to the equation $f(x) = 0$, where f is a holomorphic or meromorphic function on $\mathbb{C}$. The key to understanding Newton's method lies in two observations: first, $N_f(x) = x - \frac{f(x)}{f'(x)}$ shows that the zeros of f correspond to fixed points of N_f and second, successive iterates under N_f have tendency to converge towards an attracting fixed point of N_f. We interest in this paper is the relaxed Newton's method $N_{f,h}(z) = z - h\frac{f(x)}{f'(x)}$ that adjustment of the parameter h can lead to improved convergence. If we know that f has a multiple root of order exactly h, we can apply Newton's method to any branch of $\sqrt[h]{f(z)}$ to obtain $N_{f,h}(z) = z - \frac{f(z)^{\frac{1}{h}}}{\frac{1}{h}f(z)^{\frac{1}{h}-1}f'(z)} = z - h\frac{f(z)}{f'(z)}$. This is called the *relaxed Newton's method.* In 1989, Flexor and Sentenac found that there exists an $h \in \mathbb{C}$ such that convergence could be guaranteed almost everywhere [9]. In 1990, Haesler and Kriete [10] studied relaxed Newton's method for polynomials and included deeper analysis of the parametrized cubic polynomials studied by Curry Garnett and Sullivan [5]. The behavior of orbits under $N_{f,h}$ has been studied in depth by Bergweiler et al in 1991 [3]. For detailed historical background, see article [1].

The goal of this paper is to investigate the behavior of relaxed Newton iteration function, $N_{F,h}$ applied to the polynomial $F(z) = (z^5+c)(z^n-1)$ where $n \geq 1$ and the parameter c will be chosen small enough. In this case, the map F is called a singular perturbation of $P(z) = z^5+c$ by $f(z) = z^n - 1$. The reason for the interest in singular perturbations arises from Newton's method. For example, consider the simple polynomial of $P_\lambda(z) = z^2 + \lambda$. When $\lambda = 0$ this polynomial has a multiple root at 0 and the Newton iteration function $N_0(z) = \frac{1}{2}z$. However, when $\lambda \neq 0$, the Newton iteration function becomes $N_\lambda(z) = \frac{z}{2} - \frac{\lambda}{2z}$ and it is clear that, as in the family P_λ, the degree jumps as we move away from $\lambda = 0$. In addition to this, instead of the attracting fixed point at the origin after the perturbation, the dynamical behavior of N_λ become much richer [8]. Most orbits of N_λ still do converge to one of the two roots of f_λ, that is, $\pm\sqrt{\lambda}$, but points on the straight line passing through the origin perpendicular to the line segment connecting $\pm\sqrt{\lambda}$ have orbits that do not converge the roots. Rather, all orbits on this line behave chaotically, so the dynamical behavior is quite different in this case [13].

In this paper, we examine the dynamics of the rational map which is obtained from perturbed complex polynomial $P(z) = z^5 + c$, where $c = -0.000001$ is small enough, by a simple complex polynomial $f(z) = z^n - 1$ and so we get a new map $F(z) = (z^5 - 0.000001)(z^n - 1)$ and then we apply to relaxed Newton's method

$$N_{F,h}(z) = \frac{(n+5-h)z^{n+5} + c(n-h)z^n + (h-5)z^5 + ch}{(n+5)z^{n+4} + cnz^{n-1} - 5z^4}$$

where the parameter $c = -0.000001$, $h \in]0,1]$, and n positive integer the degree of simple polynomial f is pivotal in this study because it determines the number of petals in rational iteration. The dynamics such a perturbation are very interesting because the relaxed Newton's map is a non-polynomial examples and also their dynamical behavior is changed when the degree of simple polynomial f is changed. In addition to this, the behavior of (relaxed) Newton's method is really amazing even in the case of polynomials. When generalized to the complex plane, leads to not only stunning pictures but also the dynamics of the map are changed dramatically [6],[7].

2. Preliminaries

We introduce some basic concepts of complex dynamics terms which are used in this study. The family of functions generated by relaxed Newton's method perturbed $P(z) = z^5 - 0.000001$ by complex polynomial $f(z) = z^n - 1$ where $n \geq 1$, is a family of rational maps of the form

$$N_{F,h}(z) = \frac{(n+5-h)z^{n+5} - (0.000001)(n-h)z^n + (h-5)z^5 - (0.000001)h}{(n+5)z^{n+4} - (0.000001)nz^{n-1} - 5z^4} \ ,$$

Where $h \in]0,1]$. A rational map is a function of the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ to itself of the form $R(z) = \frac{p(z)}{q(z)}$, where p and q are complex polynomials without a common divisor. We are interested in the holomorphic dynamical systems that arise by iterating rational maps. This means we will describe the behavior of iterates with a starting point $z_0 \in \mathbb{C}_\infty$, we obtain the sequence $\{z_n\}_{n\in\mathbb{N}}$ of iterates of z_0 by repeated application of rational map R. Then the points z_n are recursively defined by $z_{n+1} = R(z_n)$, where $n \in \mathbb{N} := \{0, 1, 2, \dots\}$. Using the abbreviations $R^0 := id$ and $R^n := \underbrace{R \circ R \circ \cdots \circ R}_{n-times}$ and so we obtain $z_n = R^n(z_0)$. A point $\xi \in \mathbb{C}_\infty$ is called a *fixed point* if $R(\xi) = \xi$. It is a *periodic point* if $R^n(\xi) = \xi$ holds for some $n \in \mathbb{N}^* := \{1, 2, \cdots\}$. The minimal such n is the period of ξ. The point ξ is called *pre-periodic* if $R^{(n+j)}(\xi) = R^j(\xi)$ for some $j \geq 1$ and $n \in \mathbb{N}^*$. A point ξ which is pre-periodic but not periodic will be called *strictly pre-periodic*. In order to classify the periodic points first we need to know the multiplier $M(\xi) := (R^n)'(\xi)$ of some periodic point $\xi \in \mathbb{C}$ of period $n \in \mathbb{N}^*$. If $\xi = \infty$ then we set $M(\xi) = \frac{1}{(R^n)'(\xi)}$. After that the classification of a periodic point ξ is as: ξ is attracting, i.e. $0 <\mid M(\xi) \mid< 1$, ξ is super-attracting, i.e. $M(\xi) = 0$, ξ is rationally indifferent(or neutral, or parabolic), i.e. $M(\xi) = e^{2\pi it}$ with some $t \in \mathbb{Q}$, ξ is irrationally indifferent, i.e. $M(\xi) = e^{2\pi it}$ with some $t \in \mathbb{R}\backslash\mathbb{Q}$, ξ is repelling, i.e. $\mid M(\xi) \mid> 1$.

The set of all repelling periodic points plays a crucial role in iteration theory. If ξ is a repelling periodic point, the nearby dynamical behavior is quite different. Let $R : \mathbb{C} \to \mathbb{C}$ be a rational function of degree at least 2. The closure of the set of all repelling periodic points of R is called the *Julia set* of R [11], denoted by $\mathfrak{J}(R)$. For a complex analytic map, most of interesting dynamics occur on the Julia set, $\mathfrak{J}(R)$. It consists of all points at which the family of iterates of R fails to be a normal family of functions. The complement, $\mathfrak{F}(R) := \mathbb{C}_\infty\backslash\mathfrak{J}(R)$, of the Julia set of some rational function R of degree larger than one is called the *Fatou set* of R [2].

If z_0 is a periodic point of period n, then the associated cycle $\mathfrak{C}$ is defined as $\mathfrak{C} := \{z_0, R(z_0), ..., R^{n-1}(z_0)\}$. It is called (super)attracting, (ir)rationally indifferent or repelling if z_0 is (super)attracting, (ir)rationally indifferent or repelling, respectively. In this study, we shall be mainly concerned with attracting cycles, but the main result of this paper focuses on the basins of attraction of fixed points which are created petals. Let z_0 be an attracting fixed point of some rational function of degree larger than one. Then its basin of attraction $\mathcal{A}(z_0)$ is defined by

$$\mathcal{A}(z_0) := \{z \in \mathbb{C} \mid R^n(z) \to z_0, \ n \to \infty\}.$$

The immediate basin of attraction $\mathcal{A}^*(z_0)$ is the component of $\mathcal{A}(z_0)$ which contains z_0. A parabolic basin is the basin of attraction for a rationally indifferent fixed point. If z_0 is an attracting periodic point, then the basin of attraction $\mathcal{A}(z_0)$ is a union of components of the Fatou set, and the boundary of $\mathcal{A}(z_0)$ coincides with the Julia set. The basin is contained in the Fatou set and the parabolic point lies on the boundary of the basin and is in the Julia set [4].

Basin of attraction for attracting cycles of period m are defined using R^m instead of R. If $\{z_0, z_1, \ldots, z_{m-1}\}$ is an attracting cycle of length m, then each z_j is an attracting fixed point for R^m, and we define the basin of attraction of the attracting cycle, or of z_0, to be the union of the basins of the attraction $\mathcal{A}(z_j, R^m)$ of the z_j's with respect to R^m. The immediate basin of attraction is again denoted by $\mathcal{A}^*(z_0)$, is the union of m component of $\mathcal{A}(z_0)$ which contain an element of $\{z_0, z_1, \ldots, z_{m-1}\}$. In addition to this, the sets $\mathcal{A}(z_0)$ and $\mathcal{A}^*(z_0)$ are always open and non-empty. $\mathcal{A}(z_0)$ and $\mathcal{A}^*(z_0)$ are (forward)invariant with respect to R, and also $\mathcal{A}(z_0)$ is backward invariant. If z_0 is an attracting periodic point, then the basin of attraction $\mathcal{A}(z_0)$ is a union of components of the Fatou set, and the boundary of $\mathcal{A}(z_0)$ coincides with the Julia set. If z_0 is an attracting fixed point for the function $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$, then there is an interval I that contains z_0 in this interior and in which the condition

$$\textit{if } z \in I \textit{ , then } F^n(z) \in I \textit{ for all } n \textit{ and } F^n(z) \to z_0,$$
$$\textit{where } F \textit{ is a differentiable function, } n \in \mathbb{N}^*$$

is satisfied(attracting fixed point theorem) [8]. The attracting fixed point theorem guarantees that attracting fixed points and cycles have immediate basin of attraction. The key of the result of this study is a relation between attracting and non-attracting fixed points.

$M(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots$ is defined and holomorphic in some neighborhood of the origin. Suppose that the multiplier λ at the fixed point is a root of unity, $\lambda^t = 1$. Such a fixed point is said to be parabolic. We have considered the special case $|\lambda| = 1$; $M(z) = z + az^{n+1} +$ *higher terms*, where $a \neq 0$. The integer $n+1$ is called the multiplicity of the fixed point and we are concerned with fixed points of multiplicity $n + 1 \geq 2$.

Theorem 1. *(Leau-Fatou Flower Theorem)* [12] If M is a holomorphic map of the form

$$M(z) = z + az^{n+1} + \textit{higher terms}, \quad a \neq 0, \ n \geq 1$$

defined in some neighborhood of the origin, then zero is a parabolic fixed point and there are exactly n attracting petals for the n attracting directions and n repelling petals for the n repelling directions at the parabolic fixed point zero.

The Leau- Fatou Flower Theorem provides an analytic description of the dynamics around a parabolic fixed point. We now consider the case that $z = 0$ is a parabolic fixed point of M, that is, multiplier is a root of unity. Choose a neighborhood N of the origin that is small enough so that M maps N conformally onto some neighborhood N_0 of the origin. A connected open set U, with compact closure $\bar{U} \subset N \cap N_0$, will be called an *attracting petal* for M at the origin if $M(\bar{U}) \subset U \cup \{0\}$, and $\bigcap_{k \geq 0} M^k(\bar{U}) = \{0\}$. Similarly, $V \subset N \cap N_0$ is a *repelling petal* for M if V is an attracting petal for M^{-1}.

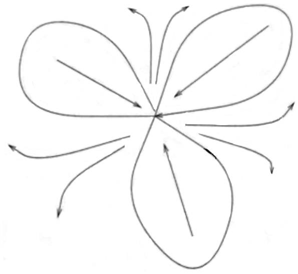


Figure 1: The Leau-Fatou Flower Theorem with three petals.

Figure 2: The Leau-Fatou Flower Theorem with four petals view from nature.

3. The dynamics of the perturbed map

In this section we examine the resulting dynamics of rational map which is arising from relaxed Newton's method applying to perturbed a complex polynomial $F(z) = (z^5 - 0.000001)(z^n - 1)$. In particular, we will draw attention to the change in the number of petals depending on the degree of the perturbing complex polynomial, while the dynamics of the Newton iteration transform changes when the degree of the perturbing complex polynomial changes.

Proposition 2. *∞ is a parabolic fixed point for relaxed Newton's method applied to perturbed complex polynomials $F(z) = P(z)f(z)$, with relaxation parameter $h \in [0,1]$.*

Proof. Let $P(z) = z^5 + c$ complex polynomial be perturbed by complex polynomials $f(z) = z^n - 1$. We obtain new maps $F(z) = (z^5 + c)(z^n - 1)$ that apply to relaxed Newton's method, $N_{F,h}(z) = z - h\frac{F(z)}{F'(z)}$ where c is any constant and $n \geq 1$, is a rational map, $N_{F,h}(z) = \frac{(n+5-h)z^{n+5}+c(n-h)z^n+(h-5)z^5+ch}{(n+5)z^{n+4}+cnz^{n-1}-5z^4}$. In $N_{F,h}(z)$, the degree of the numerator is $n+5$, and the degree of the denominator is $n+4$. Therefore, $\lim_{z\to\infty} N_{F,h}(z) = \infty$, so ∞ is a fixed point. To determine its nature, map ∞ to 0 via $g(z) = \frac{1}{z}$:

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{N_{F,h}} & \mathbb{C}_\infty \\ \downarrow g & & \downarrow g \\ \mathbb{C}_\infty & \xrightarrow[T]{} & \mathbb{C}_\infty \end{array}$$

The conjugate function T is given by $g \circ N_{F,h} = T \circ g$, thus we obtain:

$$T(v) = g(N_{F,h}(\frac{1}{v}) = \frac{1}{N_{F,h}(\frac{1}{v})} = \frac{(n+5)v + cnv^6 - 5v^{n+1}}{(n+5-h) + c(n-h)v^5 + (h-5)v^n + chv^{n+5}}.$$

Let $H(v) = \frac{h+chv^5-hv^n-chv^{n+5}}{(n+5-h)+c(n-h)v^5+(h-5)v^n+chv^{n+5}}$ be. Then $T(v) = v + vH(v)$. Calculating the value of the derivative, $T^{'}(v) = 1 + H(v) + vH^{'}(v)$ at 0 yields $T^{'}(0) = 1$. Thus ∞ is a parabolic fixed point for all $h \in]0, 1]$.

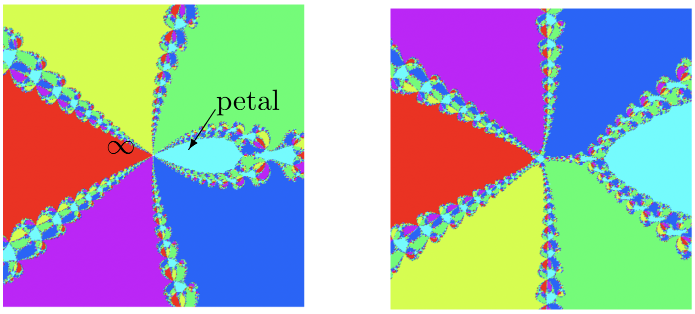
3.1. The arcane behind pedal numbers

In this section, we examine the dynamics of the rational iteration which obtain from perturbed polynomial family applying to relaxed Newton's method for choosing specific relaxing parameter $h \in]0, 1]$. We will deal with the dynamics of

$$N_{F,h}(z) = \frac{(n+5-h)z^{n+5} + c(n-h)z^n + (h-5)z^5 + ch}{(n+5)z^{n+4} + cnz^{n-1} - 5z^4}$$

for different value of the degree of perturbs complex polynomial $f(z) = z^n - 1$.

The dynamics of perturbed polynomial the case n=1: Relaxed Newton's method applied to the polynomial function $F(z) = P(z)f(z) = (z^5 - 0.000001)(z - 1)$ yields a rational map in this particular case $N_{F,h}(z) = \frac{5z^6-4z^5-0.000001}{6z^5-5z^4-0.000001}$ for $h = 1$ since $\frac{F^{'}}{F} = \frac{P^{'}}{P} + f^{'}$. The finite fixed points of $F(z)$ are 1, -0.615591, $-0.0207361 \pm 0.0590407i$, and $0.515147 \pm 0.386614i$. The finite fixed point 1 is a attracting fixed point of $N_{F,h}$ and other finite fixed point $-0.615591$is repelling fixed point of $N_{F,h}$. ∞ is also a fixed point since $\lim_{z\to\infty} N_{F,h}(z) = \infty$. Proposition 2 implies that ∞ is a parabolic fixed point of $F(z)$. So, by the conjugate function $T(v) = v + vH(v)$ where $H(v) = \frac{1+(0.000001)v^6-(0.000001)v^5-v}{5-4v-(0.000001)v^6}$. Thus we obtain $M(z) = z + z^2 + \textit{higher terms}$ by Leau-Fatou Flower Theorem implies that there is one attracting petal for the attracting direction and one repelling petal for the repelling direction at the parabolic fixed point.

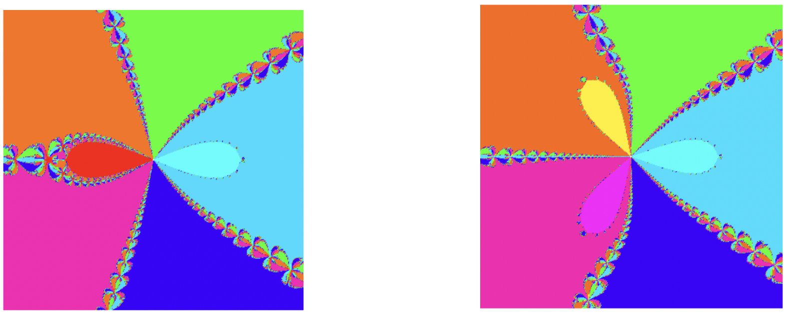


(a) **(b)**

Figure 3: The dynamical plane picture for $h = 1$

(a) The petal views from ∞, **(b)**The petal views from 0.

In Figure 3(a) the relaxed Newton map for the polynomial $F(z) = (z^5 - 1)(z - 1)$ has one petal at the parabolic fixed point. The roots of the perturbed function are divided into 6 regions, each representing a color. The petal is connected to ∞ within its parabolic basin. In Figure 3(b), same as the dynamical plane picture in Figure 3(a) which are fractals of relaxed iteration of the perturbed map $F(z) = (z^5 - 1)(z - 1)$.



(a) **(b)**

Figure 4: The dynamical plane picture for $h = 0.7$, the petals view from ∞,

(a)$F(z) = (z^5 - 0.000001)(z^2 - 1)$, **(b)**$F(z) = (z^5 - 0.000001)(z^3 - 1)$.

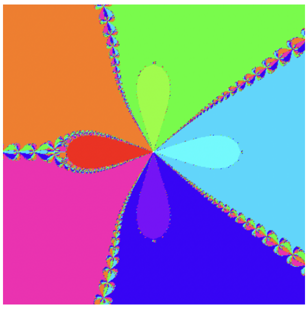


Figure 5:
The dynamical plane picture for $h = 0.5$ view from ∞.

Corollary 3. *If the degree of the perturbing polynomial has n, then there are exactly n attracting petals and n repelling petals at the parabolic fixed point ∞.*

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