Journal of Contemporary Applied Mathematics
V. 13, No 2, 2023, December
ISSN 2222-5498
DOI 10.5281/zenodo.8378402

The Relation Between the Number of Related-Matrices and the Number of Topologies on a Finite Set

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Abstract. A new approach on counting the topologies on a finite set is given by using a characterisation of topological spaces and an upper bound is given by reducing the problem into an another problem. Moreover, the results of the well-known paper of Krishnamurthy published in 1966 are reproved.

Key Words and Phrases: Number of Topologies, Quasi-Metrics, 0-1-valued Quasi-Metrics, Number of Matrices

2010 Mathematics Subject Classifications: Primary 54E99, Secondary 54A05

1. Introduction

Counting is one of the heights of ambition of human-beings. For the mathematicians, counting the structures on sets. The subject of the present paper is counting the topologies on a finite set by using a new characterization of topological spaces. For the convenience, the reader may follow the problem of counting the topologies on a finite set from the papers [1, 2, 3, 4, 5, 6]. Generally, the counting process is carried out using other structures related to topology such as graphs and quasi-orders that use directly combinatorics. But, in the paper of V.Krishnamurthy [1], a relation is set between $n \times n$ matrices with 0-1-entries and topologies on a set with n elements. Counting the topologies on a finite set is an old problem and has been studied extensively. Thus, we are going to focus on the results of the paper of V. Krishnamurthy and our goal is going to be offering a new approach on the counting the topologies on a finite set that reproves and improves the results of [1].

A function $d : X \times X \to [0, \infty)$ is called a quasi-metric (The terminology belongs to [7] but it is called hemi-metric, quasi-pseudo-metric in [8, 9] but all of them refers to T_0 hemi-metric.) if $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x,\ y,\ z \in X$. A 0-1-valued generalized quasi-metric on a set X is a function d from $X \times X$ into $\{0, 1\}^I$ for some non-empty set I if for each $i \in I$ the function $d_i : X \times X \to \{0, 1\}$ defined by $d_i(x, y) = d(x, y)(i)$ is a quasi-metric. A set X equipped with a 0-1-valued generalised quasi-metric d is called a 0-1-valued generalized quasi-metric space. This structure (space) is denoted by (X, d, I), and a subset $U \subseteq X$ is called open if for each $x \in U$ there exists a finite set $J \subseteq I$ such that

$$\bigcap_{i \in J} \{y \in X : d(x,y)(i) = 0\} \subseteq U$$

[7] asserts as a main result that the set of open sets with respect to (X, d, I), denoted by $To(X, d, I)$, defines a topology and every topological spaces comes from a 0-1-valued generalised quasi-metric space. That is, if (X, τ) is a topological space, then there exists a 0-1-valued generalized quasi-metric on X such that $\tau = To(X, d, I)$ with $I = \tau$ as an index set and $d = (d_U)_{U \in \tau}$ where for each $U \in \tau$, the map $d_U : X \times X \to \mathbb{R}$ defined by

$$d_U(x,y) := \begin{cases} 0, & t(x \in U \implies y \in U) = 1 \\ 1, & t(x \in U \implies y \in U) = 0 \end{cases}$$

defines a 0-1-valued quasi-metric where t is the truth value of a propositon, or equivalently $d_U(x,y) = \chi_{\{U\}}(x)\chi_{\{U^c\}}(y)$ where χ is the characteristic function and $U^c := X \setminus U$. These quasi-metrics characterize the topology since

$$\mathcal{B} = \{\{y : d_i(x,y) = 0\} : x \in X, i \in I\}$$

is the subbase for τ. Now, let X be a finite set with n elements and τ be any topology on X. We can enumarate the elements of X as $x_1, x_2, ..., x_n$ and for any $U \in \tau$, consider the map d_U defined as above with the table

d_U	x_1	x_2	$\dots$	x_n
x_1	$d_U(x_1,x_1)$	$d_U(x_1,x_2)$	$\dots$	$d_U(x_1,x_n)$
x_2	$d_U(x_2,x_1)$	$d_U(x_2,x_2)$	$\dots$	$d_U(x_2,x_n)$
.	.	.		.
.	.	.		.
.	.	.		.
x_n	$d_U(x_n,x_1)$	$d_U(x_n,x_2)$	$\dots$	$d_U(x_n,x_n)$

It is seen that d_U defines an $n \times n$ matrix with 0-1-entries. Let we call this matrix the related matrix and denote it by $M^{d_U}_{n\times n}$. Moreover, if a 0-1-valued quasi-metric d on a finite set X is given, then we can define a matrix with the entries $a_{ij} := d(x_i, x_j)$, denoted by $M^d_{n\times n}$.

Theorem 1.1. *Distinct open sets define distinct related-matrices except X and $\emptyset$ and all entries of $M^{d_\emptyset}_{n\times n}$ and $M^{d_X}_{n\times n}$ are zero.*

Proof. For any $U, V \in \tau$ with $U \neq V$. Let assume

$$a_{ij} = b_{ij}, \text{ for all } i, j$$

where a_{ij}, b_{ij} are the entries of the matrices derived by using U and V respectively. If $a_{ij} = b_{ij}$, for all i, j, then

$$d_U(x_i, x_j) = d_V(x_i, x_j)$$

If $U \neq V$, without loss of generality there exist k such that $x_k \in U$ but $x_k \notin V$. For any element $x_i \in V$,

$$1 = d_V(x_i, x_k) = d_U(x_i, x_k) = 0$$

It is a contradiction. Moreover, for any entry a_{ij} of $M^{d_\emptyset}_{n\times n}$

$$a_{ij} = d_\emptyset(x_i, x_j) = \chi_{\{\emptyset\}}(x_i)\chi_{\{X\}}(x_j) = 0$$

similarly, all entries of $M^{d_X}_{n\times n}$ are zero. □

Here is the main idea that if we determine the number of 0-1-valued quasi-metrics on X, then we can estimate the number of topologies. To the determine the number of 0-1-valued quasi-metrics, we use the the $n \times n$ matrices with 0-1 entries due to the above observation and Theorem 1.1.

2. Number of 0-1-valued Quasi-Metrics on a Finite Set

In this section, we will deal with counting the all 0-1-valued quasi-metrics on a set X with the labelled elements $x_1, x_2, ..., x_n$ for a fixed $n \in \mathbb{N}$, without considering the notion of topology. Then, let us start with the investigation of the properties of $n \times n$ matrices that are produced by 0-1-valued quasi-metrics.

Theorem 2.1. *Let d be a 0-1-valued quasi-metric on X and $M^d_{n\times n}$ be the related matrix then $a_{ii} = 0$ for all $i \in \{1, 2, ..., n\}$.*

Proof. d is a 0-1-valued quasi-metric. Therefore, $a_{ii} = d(x_i, x_i) = 0$ □

Theorem 2.2. *Let d be a 0-1-valued quasi-metric on X and $M^d_{n\times n}$ be the related-matrix. If $a_{ij} = 1$ and $a_{ik} = 0$, then $a_{kj} = 1$, for all $i, j, k \in \{1, 2, ..., n\}$.*

Proof. For any $i, j, k \in \{1, 2, ..., n\}$, if $a_{ij} = 1$ and $a_{ik} = 0$, it means that $d(x_i, x_j) = 1$ and $d(x_i, x_k) = 0$, then the inequality

$$d(x_i, x_j) \leq d(x_i, x_k) + d(x_k, x_j)$$

must hold since d is a 0-1-valued quasi-metric. Hence, $d(x_k, x_j) = a_{kj} = 1$, otherwise we get a contradiction. □

Theorem 2.3. *Let X be a finite set, $\{x_1, x_2, ..., x_n\}$. If an $n \times n$ matrix $M_{n\times n}$ with 0-1 entries satisfies the properties*

i. $a_{ii} = 0$, *for all* $i \in \{1, 2, ..., n\}$

ii. $a_{ij} = 1$ *and* $a_{ik} = 0$ *implies* $a_{kj} = 1$, *for all* $i, j, k \in \{1, 2, ..., n\}$

then there exists unique 0-1-valued quasi-metric d on X with $M_{n\times n} = M^d_{n\times n}$

Proof. Let $M_{n\times n}$ be a matrix with 0-1-entries and satisfies the property (i) and (ii), then consider the map $d : X\times X \to \{0,1\}, (x_i, x_j) \mapsto a_{ij}$. This map is a 0-1-valued quasi-metric and it is easily seen that it is unique by the argument of Theorem 1.1. □

Hence, we reduce the problem of counting 0-1-valued quasi-metrics on a finite set X into the counting the matrices satisying the properties (i) and (ii) in Theorem 2.3. For each entry there two possibility and the diagonal of the each matrices is zero. Hence n^2-n entries have two possibilities. Therefore, one can easily prove the following theorem.

Theorem 2.4. *The number of the* $n\times n$ *matrices with* 0-1-*entries which satisfy the property* (i) *is* 2^{n^2-n}.

Therefore, the number of 0-1-valued quasi-metrics on a finite set X with n elements is less than 2^{n^2-n} by Theorem 2.3 and Theorem 2.4. Then, if we discart the matrices that do not holds the property (ii) among 2^{n^2-n} matrices,we get the $n \times n$ matrices with 0-1-entries that holds properties $(i) - (ii)$. And the surplus is the number of all possible 0-1-valued quasi-metrics on X. If a given matrix does not hold the property (ii), the proposition "$\neg(\forall i, j, k \;\; a_{ij} = 1, a_{ik} = 0 \implies a_{kj} = 1)$" is true. Thus, there exists at least one i_0, j_0 and k_0 such that the entries of the matrix $a_{i_0j_0}, a_{i_0k_0}$ and $a_{k_0j_0}$ are equal to $1, 0, 0$ respectively. The each triple (i, j, k) that makes the above proposition true is named as unwelcome-situation.

Let first consider the 3×3 matrices before make our generalization to $n \times n$ matrices. Let X denotes the set of all 3×3 matrices with 0-1-entries. It is trivial that $|X| = 2^9$ and among this matrices the number of the matrices that holds property (i) is $2^{9-3} = 2^6$ according to Theorem 2.4. Now, we extract the matrices with

$$a_{12} = 1, a_{13} = 0, a_{32} = 0$$

entries because the matrices with these entries do not hold the property (ii) since

$$\neg(a_{ij} = 1, a_{ik} = 0 \implies a_{kj} = 1) \equiv a_{12} = 1, a_{13} = 0 \text{ and } a_{32} = 0$$

$$\begin{matrix} 0 & 1 & 0 \\ ? & 0 & ? \\ ? & 0 & 0 \end{matrix}$$

The number of matrices with the abovementioned entries is 2^3. Therefore, the number of the 3×3 matrices with 0-1-entries which hold the property (i)-(ii) is less than $2^6 - 2^3$. This result is valid only for the values $i = 1, j = 2, k = 3$. Now, we can give the following theorem.

Theorem 2.5. *Let consider the set of* $n \times n$ *matrix with* 0-1-*entries and property* (i). *Then, the number of matrices with* 0-1-*entries which hold the property* (i)-(ii) *is less than* $2^{n^2-n} - 2^{n^2-n-3}$ *whenever* $n > 2$.

Proof. Let (i_0, j_0, k_0) be a fixed unwelcome-situation for M. Then,

$$a_{i_0 j_0} = 1, a_{i_0 k_0} = 0, a_{k_0 j_0} = 1 \text{ and } a_{ii} = 0, \ \ 1 \leq i \leq n$$

Hence the number of matrices with these entries are 2^{n^2-n-3} that can not hold property (ii) because of the fixed unwelcome-situation (i_0, j_0, k_0). □

3. The Relation Between 0-1-valued Quasi-Metrics and Topologies

Let X be a set with labelled elements $x_1, x_2, ..., x_n$ for a fixed $n \in \mathbb{N}$ and $\mathbb{Q}$ be the set of all distinct 0-1-valued quasi-metrics on X. An equivalence relation can be defined on $\mathcal{P}(\mathbb{Q})$ as follows. For any $\mathbb{D}, \mathbb{P} \in \mathcal{P}(\mathbb{Q})$,

$$\mathbb{D} \sim \mathbb{P} \Longleftrightarrow < \mathbb{D} > = < \mathbb{P} >$$

where $< \mathbb{D} >$ and $< \mathbb{P} >$ denotes the topologies generated by $\mathbb{D}$ and $\mathbb{P}$. Let $\mathbb{T}$ be the set of all distinct topologies on X. The map

$$\mathbb{T} \to \mathcal{P}(\mathbb{Q}) / \sim$$

is a bijection map as a result of [7].

Theorem 3.1. *Let d and p be distinct 0-1-valued quasi-metrics on X with n elements. The map*

$$d \diamond p : X \times X \to \{0, 1\}$$

given by

$$(d \diamond p)(x, y) := \begin{cases} 0, & b_{ij} = a_{ij} = 0 \\ 1, & b_{ij} = 1 \ \textit{or} \ a_{ij} = 1 \end{cases}$$

is also a 0-1-valued quasi-metric and $< d, p > = < d \diamond p >$.

Proof. Let c_{ij} denotes the entries of the related-matrix of $d \diamond p$. For any i, $c_{ii} = 0$ since $b_{ii} = a_{ii} = 0$. If $c_{ij} = 1$ and $c_{ik} = 0$, then ($b_{ij} = 1$ or $a_{ij} = 1$) and ($b_{ik} = a_{ik} = 0$), in either case $c_{kj} = 1$. Therefore, $d \diamond p$ is a 0-1-valued quasi-metric. Moreover,

$$\{y : d(x, y) = 0\} \cap \{y : p(x, y) = 0\} = \{y : d \diamond p)(x, y) = 0\}$$

equality holds, therefore; $< d, p > = < d \diamond p >$. □

As a result of Theorem 3.1 it is easily seen that every finite set of 0-1-valued quasi-metrics, there exists only one 0-1-valued quasi-metric which generates the same topology.

4. Conclusion

Each 0-1-valued quasi-metric on a finite set X defines a different topology; moreover as a corallary of Theorem 3.1,

$$|\mathcal{P}(\mathbb{Q})/\sim| = |\mathbb{Q}|$$

It is proved that the number of topologies on X with n elements is less than

$$2^{n^2-n} - 2^{n^2-n-3}$$

whenever $n > 2$ by Theorem 2.5. And, It is the reprove of the result of V.Krishnamurthy in the paper [1] since $2^{n^2-n} - 2^{n^2-n-3} < 2^{n^2-n} - 1$ where the number $2^{n^2-n} - 1$ is the result of [1]. To ponder if further studies in this relation is possible, the following question can be put: Can the unwelcome-situations are determined? Since Theorem 2.5 is given just by taking into consideration one unwelcome-situation and determining the unwelcome-situations lessen the upper bound $2^{n^2-n} - 2^{n^2-n-3}$.

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Received 17 June 2023
Accepted 25 September 2023