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The $m-$Order Linear Recursive Quaternions

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Abstract. This study considers the $m-$order linear recursive sequences yielding some well-known sequences (such as the Fibonacci, Lucas, Pell, Jacobsthal, Padovan, and Perrin sequences). Also, the Binet-like formulas and generating functions of the $m-$order linear recursive sequences have been derived. Then, we define the $m-$order linear recursive quaternions, and give the Binet-like formulas and generating functions for them.

Key Words and Phrases: Linear Recursive, Fibonacci numbers, quaternions, generating functions.

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1. Introduction

Primarily, we will consider a linear recursion sequence that gives us some special sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, Perrin, and Tribonacci with certain initial conditions and coefficients. Then, we obtain the Binet-like formula and generating functions of the linear recursive sequence to find the Binet-like formulas and the generating functions of some special sequences by choosing certain initial conditions and coefficients. Thus, we will make it easier for us to prove the Binet-like formulas and generating functions of some special sequences as a result of this study. The $m-$order linear recursive sequence definition given below is given by Matyas and Szakacs in [1, 4]. Now, let's examine some identities by reminding this definition again.
For $a_0, a_1, \ldots, a_{m-1} \in \mathbb{Z}$ with $a_{m-1} \neq 0$ and $m \in \mathbb{Z}^+$, the $m-$order linear recursive sequence $\{S_n\}_{n\geq 0}$ are defined by reccurence relation

$$S_{n+m} = \sum_{k=0}^{m-1} a_k S_{n+k} \tag{1}$$

where the initial conditions $S_0, S_1, \ldots, S_{m-1}$ with $|S_0| + |S_1| + \cdots + |S_{m-1}| \neq 0$. The reccurence relation (1) involves the characteristic equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} - x^m = 0.$$

By the complex numbers $q_1, q_2, \ldots, q_m$, we donete the roots of the characteristic equation. Assume that the numbers a_i's are chosen such that the roots of the characteristic equation are distinct.

Linear recursive sequences have been studied by many authors [1, 2, 3, 4, 5, 42]. Matyas investigated some sequence transformations of $\{G_{n+d}/G_n\}_{n=0}^{\infty}$ of linear recursive sequences and linear recurrences and roots-finding methods in [1, 5] where $\{G_n\}$ is a linear sequence with m-order. Gatta and D'amito studied sequences H_n for which H_{n+1}/H_n approaches the golden ratio in [2] where $\{H_n\}$ is a third order linear sequence. Komatsu continued the work of Gatta and D'amito, and examined the sequence H_n for which H_{n+1}/H_n approaches an irrational number in [3]. Szakacs investigated sequence $\{G_{n+1}/G_n\}_{n=1}^{\infty}$ which are approaching the Golden Ratio, in case $\{G_n\}_{n=0}^{\infty}$ is defined the $k-$order linear recursive sequence of real numbers [4]. In the present work, we derive the Binet-like Formula and generating functions in the general case.

In [41, 43], the Binet-like formula of the m-order linear recursive sequences is

$$S_n = \sum_{r=1}^{m} p_r q_r^n, \quad (n \geq 0) \tag{2}$$

where

$$p_1 = \frac{\begin{vmatrix} S_0 & 1 & \ldots & 1 \\ S_1 & q_2 & \ldots & q_m \\ \ldots & \ldots & \ldots & \ldots \\ S_{m-1} & q_2^{m-1} & \ldots & q_m^{m-1} \end{vmatrix}}{\prod_{1 \leq j < i \leq m} (q_i - q_j)},$$

$$p_2 = \frac{\begin{vmatrix} 1 & S_0 & \ldots & 1 \\ q_1 & S_1 & \ldots & q_m \\ \ldots & \ldots & \ldots & \ldots \\ q_1^{m-1} & S_{m-1} & \ldots & q_m^{m-1} \end{vmatrix}}{\prod_{1 \leq j < i \leq m} (q_i - q_j)},$$

$$\ldots$$

$$p_m = \frac{\begin{vmatrix} 1 & 1 & \ldots & S_0 \\ q_1 & q_2 & \ldots & S_1 \\ \ldots & \ldots & \ldots & \ldots \\ q_1^{m-1} & q_2^{m-1} & \ldots & S_{m-1} \end{vmatrix}}{\prod_{1 \leq j < i \leq m} (q_i - q_j)}$$

By choosing suitable initial conditions and coefficients we obtain the Binet-like formulas for the well-known sequences as follows:

If the terms of the sequence (1) take $m = 2$, $S_0 = 0, S_1 = 1$ and $a_0 = 1, a_1 = 1$, the Binet-like formula for the Fibonacci numbers will be denoted by

$$S_n = \frac{q_2^n - q_1^n}{q_2 - q_1}.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence $S_{n+2} = S_{n+1} + S_n$.
If the terms of the sequence (1) take $m = 2,\ S_0 = 2, S_1 = 1$ and $a_0 = 1, a_1 = 1$, the Binet-like formula for the Lucas numbers will be denoted by

$$S_n = q_2^n + q_1^n.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Lucas sequence $S_{n+2} = S_{n+1} + S_n$.
If the terms of the sequence (1) take $m = 2,\ S_0 = 0, S_1 = 1$ and $a_0 = 1, a_1 = 2$, the Binet-like formula for the Pell numbers will be denoted by

$$S_n = \frac{q_2^n - q_1^n}{q_2 - q_1}.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - 2x - 1 = 0$ of the Pell sequence $S_{n+2} = 2S_{n+1} + S_n$.
If the terms of the sequence (1) take $m = 2,\ S_0 = 0, S_1 = 1$ and $a_0 = 2, a_1 = 1$, the Binet-like formula for the Jacobsthal numbers will be denoted by

$$S_n = \frac{q_2^n - q_1^n}{q_2 - q_1}.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - x - 2 = 0$ of the Jacobsthal sequence $S_{n+2} = S_{n+1} + 2S_n$.
If the terms of the sequence (1) take $m = 2,\ S_0 = a, S_1 = b$ and $a_0 = -q, a_1 = p$, the Binet-like formula for the Horadam numbers will be denoted by

$$S_n = \frac{(aq_1 - b)q_2^n - (aq_2 - b)q_1^n}{q_2 - q_1}.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - px + q = 0$ of the Horadam sequence $S_{n+2} = pS_{n+1} - qS_n$.
If the terms of the sequence (1) take $m = 2,\ S_0 = 1, S_1 = t$ and $a_0 = -1, a_1 = 2t$, the Binet-like formula for the Chebyshev polynomials will be denoted by

$$S_n = \frac{(t - q_1)q_2^n - (t - q_2)q_1^n}{q_2 - q_1}.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - 2tx + 1 = 0$ of the Chebyshev polynomial sequence $S_{n+2} = 2tS_{n+1} - S_n$.
If the terms of the sequence (1) take $m = 3,\ S_0 = 1, S_1 = 1, S_2 = 1$ and $a_0 = 1, a_1 = 1, a_2 = 0$, the Binet-like formula for the Padovan numbers will be denoted by

$$S_n = p_1 q_1^n + p_2 q_2^n + p_3 q_3^n,$$

where $p_1 = \dfrac{(q_2-1)(q_3-1)}{(q_1-q_2)(q_1-q_3)}$, $p_2 = \dfrac{(q_1-1)(q_3-1)}{(q_2-q_1)(q_2-q_3)}$, $p_3 = \dfrac{(q_1-1)(q_2-1)}{(q_3-q_1)(q_3-q_2)}$ and q_1, q_2, q_3 are the roots of the characteristic equation $x^3 - x - 1 = 0$ of the Padovan sequence $S_{n+3} = S_{n+1} + S_n$.
If the terms of the sequence (1) take $m = 3$, $S_0 = 3, S_1 = 0, S_2 = 2$ and $a_0 = 1, a_1 = 1, a_2 = 0$, the Binet-like formula for the Perrin numbers will be denoted by

$$S_n = q_1^n + q_2^n + q_3^n.$$

where q_1, q_2, q_3 are the roots of the characteristic equation $x^3 - x - 1 = 0$ of the Perrin sequence $S_{n+3} = S_{n+1} + S_n$.
If the terms of the sequence (1) take $m = 3$, $S_0 = 0, S_1 = 1, S_2 = 1$ and $a_0 = 1, a_1 = 1, a_2 = 1$, the Binet-like formula for the Tribonacci numbers will be denoted by

$$S_n = p_1 q_1^n + p_2 q_2^n + p_3 q_3^n,$$

where $p_1 = \dfrac{q_1^{n+2}}{(q_1-q_2)(q_1-q_3)}$, $p_2 = \dfrac{q_2^{n+2}}{(q_2-q_1)(q_2-q_3)}$, $p_3 = \dfrac{q_3^{n+2}}{(q_3-q_1)(q_3-q_2)}$ and q_1, q_2, q_3 are the roots of the characteristic equation $x^3 - x^2 - x - 1 = 0$ of the Tribonacci sequence $S_{n+3} = S_{n+2} + S_{n+1} + S_n$.
The Binet-like formulas and generating functions of some special sequences are available in the studies in [25, 27, 7, 20, 10, 15, 12, 23, 38, 17, 14, 31, 11, 9, 13, 33, 18, 8, 16, 30, 32, 6, 34, 36, 35, 39, 26, 19, 37, 21, 22, 28, 29, 24, 40]. Now we give generating function for the $m-$order linear recursive sequences.
In [43], the generating function of the $m-$order linear recursive sequences is

$$\sum_{n=0}^{\infty} S_n x^n = \frac{\sum_{i=0}^{m-1} S_i x^i \left(1 - \sum_{j=1}^{m-i-1} a_{m-j} x^j\right)}{1 - \sum_{k=0}^{m-1} a_k x^{m-k}}.$$

By choosing suitable initial conditions and coefficients we obtain the generating functions for the well-known sequences as follows:

m	$S_0, S_1, \ldots, S_{m-1}$	$a_0, a_1, \ldots, a_{m-1}$	Generating Functions	Names of sequence
2	$S_0 = 0, S_1 = 1$	$a_0 = 1, a_1 = 1$	$\frac{x}{1-x-x^2}$	Fibonacci
2	$S_0 = 2, S_1 = 1$	$a_0 = 1, a_1 = 1$	$\frac{2-x}{1-x-x^2}$	Lucas
2	$S_0 = 0, S_1 = 1$	$a_0 = 1, a_1 = 2$	$\frac{x}{1-2x-x^2}$	Pell
2	$S_0 = 0, S_1 = 1$	$a_0 = 2, a_1 = 1$	$\frac{x}{1-x-2x^2}$	Jacobsthal
2	$S_0 = a, S_1 = b$	$a_0 = q, a_1 = p$	$\frac{a+(b-ap)x}{1-px-qx^2}$	Horadam
2	$S_0 = 1, S_1 = t$	$a_0 = -1, a_1 = 2t$	$\frac{1-tx}{1-2tx+x^2}$	Chebyshev polynomials
3	$S_0 = 1, S_1 = 1, S_2 = 1$	$a_0 = 1, a_1 = 1, a_2 = 0$	$\frac{x+1}{1-x^2-x^3}$	Padovan
3	$S_0 = 3, S_1 = 0, S_2 = 2$	$a_0 = 1, a_1 = 1, a_2 = 0$	$\frac{3-x^2}{1-x^2-x^3}$	Perrin
3	$S_0 = 0, S_1 = 1, S_2 = 1$	$a_0 = 1, a_1 = 1, a_2 = 1$	$\frac{x}{1-x-x^2-x^3}$	Tribonacci

Now we give exponential generating function for the $m-$order linear recursive sequences.
The exponential generating function of the $m-$order linear recursive sequences is

$$\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum_{r=1}^{m} p_r e^{q_r x}.$$

By choosing suitable initial conditions and coefficients we obtain the generating functions for the well-known sequences as follows:

m	$S_0, S_1, \ldots, S_{m-1}$	$a_0, a_1, \ldots, a_{m-1}$	Exponential Generating Functions	Names of sequence
2	$S_0 = 0, S_1 = 1$	$a_0 = 1, a_1 = 1$	$\frac{e^{q_2 x} - e^{q_1 x}}{q_2 - q_1}$	Fibonacci
2	$S_0 = 2, S_1 = 1$	$a_0 = 1, a_1 = 1$	$e^{q_2 x} + e^{q_1 x}$	Lucas
2	$S_0 = 0, S_1 = 1$	$a_0 = 1, a_1 = 2$	$\frac{e^{q_2 x} - e^{q_1 x}}{q_2 - q_1}$	Pell
2	$S_0 = 0, S_1 = 1$	$a_0 = 2, a_1 = 1$	$\frac{e^{q_2 x} - e^{q_1 x}}{q_2 - q_1}$	Jacobsthal
2	$S_0 = a, S_1 = b$	$a_0 = q, a_1 = p$	$\frac{(aq_1 - b)e^{q_2 x} - (aq_2 - b)e^{q_1 x}}{q_2 - q_1}$	Horadam
2	$S_0 = 1, S_1 = t$	$a_0 = -1, a_1 = 2t$	$\frac{(t - q_1)e^{q_2 x} - (t - q_2)e^{q_1 x}}{q_2 - q_1}$	Chebyshev polynomials
3	$S_0 = 1, S_1 = 1, S_2 = 1$	$a_0 = 1, a_1 = 1, a_2 = 0$	$p_1 e^{q_1 x} + p_2 e^{q_2 x} + p_3 e^{q_3 x}$	Padovan
3	$S_0 = 3, S_1 = 0, S_2 = 2$	$a_0 = 1, a_1 = 1, a_2 = 0$	$e^{q_1 x} + e^{q_2 x} + e^{q_3 x}$	Perrin
3	$S_0 = 0, S_1 = 1, S_2 = 1$	$a_0 = 1, a_1 = 1, a_2 = 1$	$p_1 e^{q_1 x} + p_2 e^{q_2 x} + p_3 e^{q_3 x}$	Tribonacci

1.1. $m-$Order Linear Recursive Quaternions

A quaternion is defined by

$$q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

where a_0, a_1, a_2 and a_3 are real numbers and $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ are the standart basis in $\mathbb{R}^4$.
The quaternion multiplication is defined using the rules:

$$e_0^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1 \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2.$$

This algebra is associative and non-commutative.
Let $q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $p = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3$ be any two quaternions. Then the addition and subtraction of them is

$$q \mp p = (a_0 \mp b_0)e_0 + (a_1 \mp b_1)e_1 + (a_2 \mp b_2)e_2 + (a_3 \mp b_3)e_3$$

and for $k \in \mathbb{R}$, the multiplication by scalar is

$$kq = ka_0e_0 + ka_1e_1 + ka_2e_2 + ka_3e_3$$

and the conjugate and norm of a quaterion are

$$\overline{q} = a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3$$

and

$$N(q) = q\overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Addition, equality and multiplication by scalar of two quaternions can be found [1, 2, 5].

Definition 1.1. *The $m-$order linear recursive quaternion $\{\mathcal{Q}S_n\}_{n\geq 0}$ is defined by*

$$\mathcal{Q}S_n = S_ne_0 + S_{n+1}e_1 + S_{n+2}e_2 + S_{n+3}e_3 \tag{3}$$

where S_n is the $m-$order linear recursive numbers.

Theorem 1.2. *The Binet-like formula for the $m-$order linear recursive quaternion $\{\mathcal{Q}S_n\}_{n\geq 0}$ is*

$$\mathcal{Q}S_n = \sum_{r=1}^{m} p_r\hat{q_r}q_r^n \tag{4}$$

where $\hat{q_r} = e_0 + q_re_1 + q_r^2e_2 + q_r^3e_3$.

Proof. From the definition of the $m-$order linear recursive quaternion $\mathcal{Q}S_n$ in (3) and Binet-like formula for the $m-$order linear recursive number S_n, we write

$$\begin{aligned}
\mathcal{Q}S_n &= S_ne_0 + S_{n+1}e_1 + S_{n+2}e_2 + S_{n+3}e_3 \\
&= \sum_{r=1}^{m} p_rq_r^ne_0 + \sum_{r=1}^{m} p_rq_r^{n+1}e_1 + \sum_{r=1}^{m} p_rq_r^{n+2}e_2 + \sum_{r=1}^{m} p_rq_r^{n+3}e_3 \\
&= \sum_{r=1}^{m} p_r\left(e_0 + q_r^1e_1 + q_r^2e_2 + q_r^3e_3\right)q_r^n \\
&= \sum_{r=1}^{m} p_r\hat{q_r}q_r^n
\end{aligned}$$

□

As a special case of the equality (4), the Binet-like formula of Fibonacci quaternions can be given as follows:
For $m = 2$, the Binet-like formula for the Fibonacci quaternions will be denoted by

$$\mathcal{Q}S_n = \frac{\hat{q}_2 q_2^n - \hat{q}_1 q_1^n}{q_2 - q_1}.$$

where q_1 and q_2 are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence $S_{n+2} = S_{n+1} + S_n$, and $\hat{q}_1 = e_0 + q_1^1 e_1 + q_1^2 e_2 + q_1^3 e_3$, $\hat{q}_2 = e_0 + q_2^1 e_1 + q_2^2 e_2 + q_2^3 e_3$. Binet-like formulas of other special quaternion sequences can be obtained in a similar way using (4).

Theorem 1.3. *The generating function for $m-$order linear recursive quaternion $\{\mathcal{Q}S_n\}_{n\geq 0}$ is*

$$G_{\mathcal{Q}S}(x) = \frac{\left(e_0x^3 + e_1x^2 + e_2x + e_3\right) G_S(x) - \left(S_0(e_1x^2 + e_2x + e_3) + S_1(e_2x^2 + e_3x) + S_2(e_3x^2)\right)}{x^3}$$

where $G_S(x)$ is the generating function of the $m-$order linear recursive sequences

Proof. Let

$$G_{\mathcal{Q}S}(x) = \sum_{n=0}^{\infty} \mathcal{Q}S_n x^n \tag{5}$$

be generating function of the $m-$order linear recursive quaternion. We have

$$\begin{aligned}
G_{\mathcal{Q}S}(x) &= \sum_{n=0}^{\infty} \left(S_n e_0 + S_{n+1} e_1 + S_{n+2} e_2 + S_{n+3} e_3\right) x^n \\
&= e_0 \sum_{n=0}^{\infty} S_n x^n + e_1 \sum_{n=0}^{\infty} S_{n+1} x^n + e_2 \sum_{n=0}^{\infty} S_{n+2} x^n + e_3 \sum_{n=0}^{\infty} S_{n+3} x^n \\
&= e_0 G_S(x) + e_1 \left(G_S(x)\frac{1}{x} - \frac{S_0}{x}\right) + e_2 \left(G_S(x)\frac{1}{x^2} - \frac{S_0}{x^2} - \frac{S_1}{x}\right) \\
&+ e_3 \left(G_S(x)\frac{1}{x^3} - \frac{S_0}{x^3} - \frac{S_1}{x^2} - \frac{S_2}{x}\right) \\
&= \frac{\left(e_0x^3 + e_1x^2 + e_2x + e_3\right) G_S(x) - \left(S_0(e_1x^2 + e_2x + e_3) + S_1(e_2x^2 + e_3x) + S_2(e_3x^2)\right)}{x^3}
\end{aligned}$$

□

As a special case of the equality (5), the Binet-like formula of Fibonacci quaternions can be given as follows:
For $m = 2$, the generating function for the Fibonacci quaternions will be denoted by

$$G_{\mathcal{Q}S}(x) = \frac{x + e_1 + e_2(x+1) + e_3(x+2)}{1 - x - x^2}.$$

Generating functions of other special quaternion sequences can be obtained in a similar way using (5).

Theorem 1.4. *The exponential generating function of the $m-$order linear recursive quaternions is*

$$E_S(x) = \sum_{r=1}^{m} p_r \hat{q_r} e^{q_r x} \tag{6}$$

Proof. Let

$$E_S(x) = \sum_{n=0}^{\infty} \mathcal{Q}S_n \frac{x^n}{n!}$$

Using the identity (2), we get

$$E_S(x) = \sum_{n=0}^{\infty} \mathcal{Q}S_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=1}^{m} p_r \hat{q_r} q_r^n \frac{x^n}{n!} = \sum_{r=1}^{m} p_r \hat{q_r} \sum_{n=0}^{\infty} \frac{(q_r x)^n}{n!} = \sum_{r=1}^{m} p_r \hat{q_r} e^{q_r x}$$

□

For $m = 2$, the exponential generating function for the Fibonacci quaternions will be denoted by

$$E_S(x) = \frac{\hat{q_2} e^{q_2 x} - \hat{q_1} e^{q_1 x}}{q_2 - q_1}$$

Exponential generating functions of other special quaternion sequences can be obtained in a similar way using (6).

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