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Estimates for the abstract Boussinesq equations

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Abstract. This paper obtains the existence and uniqueness of the solution of the integral boundary value problem for the abstract Boussinesq equations. The equations include a linear operator A defined in a Banach space E, in which by choosing E and A we can obtain numerous classes of nonlocal initial value problems for Boussinesq equations which occur in a wide variety of physical systems.

Key Words and Phrases: Boussinesq equations, semigroups of operators, UMD space, Minkowski's inequality, cosine and sine operator functions, operator-valued L^p-Fourier multipliers.

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1. Introduction

The subject of this paper is to study the estimates for solutions of value problems for the Boussinesq – operator equation

$$u_{tt} - \Delta u_{tt} + Au = \Delta g\left(x, t\right), \; x \in R^n, \; t \in (0, T), \tag{1}$$

$$u\left(0, x\right) = \varphi\left(x\right) + \int_0^T \alpha\left(\sigma\right) u\left(\sigma, x\right) d\sigma, \tag{2}$$

$$u_t\left(0, x\right) = \psi\left(x\right) + \int_0^T \beta\left(\sigma\right) u_t\left(\sigma, x\right) d\sigma,$$

where A is a linear operator in a Banach space E, $\alpha\left(s\right)$ and $\beta\left(s\right)$ are measurable functions on $(0, T)$, $u\left(x, t\right)$ denotes the E – valued unknown function, $\varphi\left(x\right)$ and $\psi\left(x\right)$ are the given initial value functions, n is the dimension of space variable x and Δ denotes the Laplace operator in R^n.
By choosing E and A , integral conditions, we can obtain numerous classis of nonlocal boundary value problems for generalized Boussinesq type equations which occur in a wide variety of physical systems, particularly in the propagation of longitudinal deformation

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waves in an elastic rod, hydro – a dynamical process in plasma, in materials science which describe spinodal decomposition, in the absence of mechanical stresses. For example, if we choose $E = L^q(\Omega)$, $\Omega \in R^m$, $D(A) = W_0^{2,q}(\Omega)$, $Au = -\Delta_y u$, $\alpha_k = 0$ *and* $\beta_k = 0$, we obtain the Cauchy problem for Boussinesq type equation

$$u_{tt} - \Delta u_{tt} - \Delta_y u = \Delta g(x,t), \ x \in R^n, \ t \in (0,T), \tag{3}$$

$$u(0,x,y) = \varphi(x,y), \ u_t(0,x,y) = \psi(x,y) \tag{4}$$

where

$$\Delta_y u = \sum_{k=1}^{m} \frac{\partial^2 u}{\partial y_k^2}, \ u = u(t,x,y), \ y = (y_1, y_2, \ \ldots, \ y_m) \in \Omega.$$

The equation (3) arises in different situations. For example, equation (3) for $n = 1$ describes a limit of a one – one-dimensional nonlinear lattice, shallow-water waves and the propagation of longitudinal deformation waves in an elastic rod.
Here, differential operator equations were studied e.g. in [2-6,8,9,11,12,15,17,18]. Cauchy problems for abstract hyperbolic equations were treated e.g. in [4]. In this paper, the key step is the derivation of the uniform estimate for the solutions of the nonlocal boundary value problems for the linearized Boussinesq equation. Harmonic analysis, the method of operator theory, interpolation of Banach spaces, and embedding theorems in abstract Sobolev spaces are the main tools implemented to carry out the analysis.
In order to state our results precisely, we introduce some notations and some function spaces.

2. Definitions and Background

Let E be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable E – valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega;E)} = \left(\int_{\Omega} \|f(x)\|_E^p dx\right)^{\frac{1}{p}}, \ 1 \leq p < \infty,$$

$$\|f\|_{L^\infty} = ess\sup_{x\in\Omega} \|f(x)\|_E \ .$$

The Banach space E is called an UMD space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L^p(R,E)$, $p \in (1,\infty)$ (see e.g. [18]). UMD spaces include e.g. L^p, l_p spaces and Lorentz spaces L_{pq}, $p,q \in (1,\infty)$.
Here,

$$S_\psi = \{\lambda \in \mathbb{C}, \ |arg\lambda| \leq \omega, \ 0 \leq \omega < \pi\},$$

$$S_{\omega,\varkappa} = \{\lambda \in S_\omega, \ |\lambda| > \ \varkappa > 0\}.$$

Let $E\left(A^{\theta}\right)$ denote the space $D\left(A^{\theta}\right)$ with the graphical norm

$$\|u\|_{E\left(A^{\theta}\right)} = \left(\|u\|^{p} + \left\|A^{\theta}u\right\|^{p}\right)^{\frac{1}{p}},\ 1 \leq p < \infty,\ 0 < \theta < \infty.$$

A closed linear operator A in a Banach space E belongs to $\sigma\left(C_0, \omega, E\right)$ (see [4], §11.2) if $D(A)$ is dense on E, the resolvent $\left(A - \lambda^2 I\right)^{-1}$ exists for $Re\lambda > \omega$ and

$$\left\|\left(A - \lambda^2 I\right)^{-1}\right\|_{B(E)} \leq C_0 |Re\lambda - \omega|^{-1}.$$

Here,

$$\alpha = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right),\ D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},\ |\alpha| = \sum_{k=1}^{n} \alpha_k.$$

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E. Let Ω be a domain in R^n and m is a positive integer. $W^{m,p}\left(\Omega; E_0, E\right)$ denotes the space of all functions $u \in L^p\left(\Omega; E_0\right)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p\left(\Omega; E\right),\ 1 \leq p \leq \infty$ with the norm

$$\|u\|_{W^{m,p}\left(\Omega; E_0, E\right)} = \|u\|_{L^p\left(\Omega; E_0\right)} + \sum_{k=1}^{n} \left\|\frac{\partial^m u}{\partial x_k^m}\right\|_{L^p(\Omega; E)} < \infty.$$

For $E_0 = E$ the space $W^{m,p}\left(\Omega; E_0, E\right)$ denotes by $W^{m,p}\left(\Omega; E\right)$. Here, $H^{s,p}\left(R^n; E\right),\ -\infty < s < \infty$ denotes the E – valued Sobolev space of order s which is defined as

$$H^{s,p} = H^{s,p}\left(R^n; E\right) = (I - \Delta)^{-\frac{s}{2}} L^p\left(R^n; E\right),$$

with the norm

$$\|u\|_{H^{s,p}} = \left\|(I - \Delta)^{\frac{s}{2}} u\right\|_{L^p(R^n; E)}.$$

It is clear that $H^{0,p}\left(R^n; E\right) = L^p\left(R^n; E\right)$. It is known that if E is a UMD space, then $H^{m,p}\left(R^n; E\right) = W^{m,p}\left(R^n; E\right)$ for positive integer m. $H^{s,p}\left(R^n; E_0, E\right)$ denotes the Sobolev – Lions type space, i.e.,

$$H^{s,p}\left(R^n; E_0, E\right) == \left\{u \in H^{s,p}\left(R^n; E\right) \cap L^p\left(R^n; E\right),\ \|u\|_{H^{s,p}\left(R^n; E_0, E\right)} = \right.$$

$$\left. = \|u\|_{L^p\left(R^n; E_0\right)} + \|u\|_{H^{s,p}\left(R^n; E\right)} < \infty\right\}.$$

$S\left(R^n; E\right)$ denotes the Schwartz class, i.e., the space of E – valued rapidly decreasing smooth functions on R^n, equipped with its usual topology generated by seminorms. Here,$S^{‘}\left(R^n; E\right)$ denotes the space of all continuous linear operators $L : S\left(R^n; E\right) \rightarrow E$, equipped with the bounded convergence topology. Recall $S\left(R^n; E\right)$ is norm dense in $L^p\left(R^n; E\right)$ when $1 \leq p < \infty$.

Here, $s=(s_1,s_2,\ldots,s_3)\,,\ s_k>0$ and F denotes the Fourier transform.
Let the operator A be a generator of a strongly continuous cosine operator function in a Banach space E defined by the formula

$$C\left(t\right)=\frac{1}{2}\left(e^{itA^{\frac{1}{2}}}+e^{-itA^{\frac{1}{2}}}\right).$$

Then, from the definition of sine operator – function $S\left(t\right)$

$$S\left(t\right)u=\int_0^t C\left(\sigma\right)ud\sigma$$

and it follows that

$$S\left(t\right)u=\frac{1}{2i}A^{-\frac{1}{2}}\left(e^{itA^{\frac{1}{2}}}-e^{-itA^{\frac{1}{2}}}\right).$$

By virtue of [4], [11] we have
Lemma A_1. The following estimates hold:

$$\left\|C\left(t\right)\right\|_{B(E)}\leq 1,\ \left\|A^{\frac{1}{2}}S\left(t\right)\right\|_{B(E)}\leq 1.$$

In a similar way, as in [1] we obtain
Lemma 1.1. Let

$$\left|1+\int_0^T\alpha\left(\sigma\right)\beta\left(\sigma\right)d\sigma\right|>\int_0^T\left(|\alpha(\sigma)|+|\beta(\sigma)|\right)d\sigma.$$

Then the operator O defined by

$$O=\left[1+\int_0^T\int_0^T\alpha\left(\sigma\right)\beta\left(\tau\right)d\sigma d\tau\right]I-\int_0^T\left(\alpha\left(s\right)+\beta\left(s\right)\right)C\left(s\right)ds$$

has an inverse O^{-1} and the following estimate is satisfied

$$\|\ O^{-1}\|_{B(E)}\leq\left[\left|1+\int_0^T\alpha\left(s\right)\beta\left(s\right)ds\right|-\int_0^T\left(\left|\alpha\left(s\right)\right|+\left|\beta\left(s\right)\right|\right)ds\right]^{-1}.$$

The embedding theorems in vector-valued spaces play a key role in the theory of DOEs. For estimating lower-order derivatives we use the following embedding theorem that is obtained from [12, Theorem 1]:
Theorem A_1. Suppose the following conditions are satisfied:

1. E is a UMD space and A is an R – positive operator in E;

2. $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is a n – tuples of nonnegative integer number and s is a positive number such that

$$\varkappa=\frac{1}{s}\left[|\alpha|+n\left(\frac{1}{p}-\frac{1}{q}\right)\right]\leq 1,\ 0\leq\mu\leq 1-\varkappa,\ 1<p\leq q<\infty;0<h\leq h_0,$$

where h_0 is a fixed positive number;
Then the embedding $D^{\alpha} H^{s,p}\left(R^n; E(A), E\right) \subset L^q\left(R^n; E(A^{1-\varkappa-\mu})\right)$ is continuous and for $u \in H^{s,p}\left(R^n; E(A), E\right)$ the following uniform estimate holds

$$\|D^{\alpha} u\|_{L^q(R^n; E(A^{1-\varkappa-\mu}))} \leq h^{\mu} \|u\|_{H^{s,p}(R^n; E(A), E)} + h^{-(1-\mu)} \|u\|_{L^p(R^n; E)}.$$

In a similar way , we obtain:
Proposition A_1. Let $1 < p \leq q < \infty$ and E be a UMD space. Suppose $\Psi_h \in C^n\left(R^n \backslash \{0\}; B(E)\right)$ and there is a positive constant K such that

$$\sup_{h \in Q} \left(\left\{ |\xi|^{|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right)} D^{\beta} \Psi_h(\xi) : \xi \in R^n \backslash \{0\},\ \beta_k \in \{0, 1\} \right\} \right) \leq K.$$

Then Ψ_h is a uniformly bounded collection of Fourier multiplier from $L^p\left(R^n; E\right)$ to $L^q\left(R^n; E\right)$.
Proof. First, in a similar way we show that Ψ_h is a uniformly bounded collection of Fourier multiplier from $L^p\left(R^n; E\right)$ to $L^q\left(R^n; E\right)$. Moreover, by Theorem A_1 we get that, for $s \geq n\left(\frac{1}{p}\frac{1}{q}\right)$ the embedding $H^{s,p}\left(R^n; E\right) \subset L^q\left(R^n; E\right)$ is continuous. From this two fact, we obtain the conclusion.
The paper is organized as follows: The first section contains an introduction. In section 2, some definitions and background are given. In section 3, we obtain the existence of a unique solution and a priory estimate for the solution of the linearized problem $(5) - (6)$. Sometimes we use one and the same symbol C without distinction to denote positive constants that may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_{α}.

3. Estimates for the linearized equation.

In this section, we make the necessary estimates for solutions of initial value problems for the linearized abstract Boussinesq equation

$$u_{tt} - \Delta u_{tt} + Au = \Delta g(x,t),\ x \in R^n,\ t \in (0,T), \tag{5}$$

$$u(0,x) = \varphi(x) + \int_0^T \alpha(\sigma) u(\sigma, x) d\sigma, \tag{6}$$

$$u_t(0,x) = \psi(x) + \int_0^T \beta(\sigma) u_t(\sigma, x) d\sigma.$$

Let

$$X_p = L^p\left(R^n; E\right),\ Y^{s,p} = H^{s,p}\left(R^n; E\right),\ Y_1^{s,p} = H^{s,p}\left(R^n; E\right) \cap L^1\left(R^n; E\right),$$

$$Y_{\infty}^{s,p} = H^{s,p}\left(R^n; E\right) \cap L^{\infty}\left(R^n; E\right).$$

Condition 3.1. Assume:

1. $\left|1+\int_0^T \alpha(\sigma)\beta(\sigma)\,d\sigma\right| > \int_0^T (|\alpha(\sigma)|+|\beta(\sigma)|)\,d\sigma;$

2. E is a UMD space and linear operator A belongs to $\sigma(C_0,\omega,E)$;

3. $\varphi,\psi \in Y_\infty^{s,p}$ and $g(.,t) \in Y_\infty^{s,p}$ for $t\in(0,T)$ and $s>\frac{n}{p}$ for $1<p<\infty$.

First we need the following lemmas.

Lemma 3.1. Suppose the Condition 3.1 hold. Then the problem $(5)-(6)$ has a unique generalized solution.

Proof. By using of the Fourier transform we get from $(5)-(6)$

$$\widehat{u_{tt}}(t,\xi)+A_\xi\hat{u}(t,\xi)=|\xi|^2\hat{g}(t,\xi),$$

$$\hat{u}(0,\xi)=\widehat{\varphi}(\xi)+\int_0^T\alpha(\sigma)\,\hat{u}(\sigma,\xi)\,d\sigma, \tag{7}$$

$$\widehat{u_t}(0,\xi)=\widehat{\psi}(\xi)+\int_0^T\beta(\sigma)\,\hat{u}(\sigma,\xi)\,d\sigma,\ \xi\in R^n,\ t\in(0,T),$$

where $\hat{u}(t,\xi)$ is a Fourier transform of $u(x,t)$ with respect to x, where

$$A_\xi=\left(1+|\xi|^2\right)^{-1}A,\ \xi\in R^n.$$

Consider the problem

$$\widehat{u_{tt}}(t,\xi)+A_\xi\hat{u}(t,\xi)=|\xi|^2\left(1+|\xi|^2\right)^{-1}\hat{g}(t,\xi), \tag{8}$$

$$\hat{u}(0,\xi)=u_0(\xi),\ \widehat{u_t}(0,\xi)=u_1(\xi),\ \xi\in R^n,\ t\in(0,T),$$

where $u_0(\xi)\in D(A)$ and $u_1(\xi)\in D(A^{\frac{1}{2}})$ for $\xi\in R^n$. By virtue of [4, §11.2, 11.4] we obtain that A_ξ is a generator of a strongly continuous cosine operator function and problem (8) has a unique solution for all $\xi\in R^n$, moreover, the solution can be written as

$$\hat{u}(t,\xi)=C(t,\xi,A)\,u_0(\xi)+S(t,\xi,A)\,u_1(\xi)+$$

$$+\int_0^t S(t-\tau,\xi,A)|\xi|^2\left(1+|\xi|^2\right)^{-1}\hat{g}(\tau,\xi)\,d\tau,\quad t\in(0,T), \tag{9}$$

where $C(t,\xi,A)$ is a cosine and $S(t,\xi,A)$ is a sine operator – function (see e.g. [4]) generated by parameter dependent operator A_ξ. Using formula (9) and nonlocal boundary condition

$$u_0(\xi)=\widehat{\varphi}(\xi)+\int_0^T\alpha(\sigma)\,\hat{u}(\sigma,\xi)\,d\sigma$$

we get

$$u_0(\xi)=\widehat{\varphi}(\xi)+\int_0^T\alpha(\sigma)\left[C(\sigma,\xi,A)\,u_0(\xi)+S(\sigma,\xi,A)\,u_1(\xi)+\right.$$

$$+\int_0^{\sigma} S(\sigma-\tau,\xi,A)|\xi|^2\Big(1+|\xi|^2\Big)^{-1}\hat{g}\left(\tau,\xi\right)d\tau\Bigg]\,d\sigma,\ \tau\in(0,T)\,.$$

Then,

$$\left[I-\int_0^T \alpha\left(\sigma\right)C\left(\sigma,\xi,A\right)u_0\left(\xi\right)d\sigma\right]u_0\left(\xi\right)-\left[\int_0^T \alpha\left(\sigma\right)S\left(\sigma,\xi,A\right)u_0\left(\xi\right)d\sigma\right]u_1\left(\xi\right)=$$

$$=\int_0^T\int_0^{\sigma} \alpha\left(\sigma\right)S\left(\sigma-\tau,\xi,A\right)|\xi|^2\Big(1+|\xi|^2\Big)^{-1}\hat{g}\left(\tau,\xi\right)d\tau d\sigma++\widehat{\varphi}\left(\xi\right). \tag{10}$$

Differentiating both sides of formula (9) with respect to t we obtain

$$\widehat{u_t}\left(t,\xi\right)=-AS\left(t,\xi,A\right)u_0\left(\xi\right)+C\left(t,\xi,A\right)u_1\left(\xi\right)q+$$

$$+\int_0^t C\left(t-\tau,\xi,A\right)|\xi|^2\Big(1+|\xi|^2\Big)^{-1}\hat{g}\left(\tau,\xi\right)d\tau,\ t\in(0,T)\,.$$

Using this formula and integral condition

$$u_1\left(\xi\right)=\widehat{\psi}\left(\xi\right)+\int_0^T \beta\left(\sigma\right)\widehat{u_t}\left(\sigma,\xi\right)d\sigma$$

we obtain

$$C\left(\sigma,\xi,A\right)=\widehat{\psi}\left(\xi\right)+\int_0^T \beta\left(\sigma\right)\left[-AS\left(\sigma,\xi,A\right)u_0\left(\xi\right)+C\left(\sigma,\xi,A\right)u_1\left(\xi\right)+\right.$$

$$\left.+\int_0^{\sigma} C\left(\sigma-\tau,\xi,A\right)|\xi|^2\Big(1+|\xi|^2\Big)^{-1}\hat{g}\left(\tau,\xi\right)d\tau\right]d\sigma.$$

Thus,

$$\int_0^T \beta\left(\sigma\right)AS\left(\sigma,\xi,A\right)d\sigma u_0\left(\xi\right)+\left[I-\int_0^T \beta\left(\sigma\right)C\left(\sigma,\xi,A\right)u_0\left(\xi\right)d\sigma\right]u_1\left(\xi\right)=$$

$$=\int_0^T\int_0^{\sigma} \beta\left(\sigma\right)C\left(\sigma-\tau,\xi,A\right)|\xi|^2\Big(1+|\xi|^2\Big)^{-1}\hat{g}\left(\tau,\xi\right)d\tau d\sigma++\widehat{\psi}\left(\xi\right). \tag{11}$$

Now, we have a system of equations (10) and (11) for $u_0\left(\xi\right)$ and $u_1\left(\xi\right)$. The determinant of this system $O(\xi)$ is

$$D\left(\xi\right)=\begin{vmatrix}\alpha_{11}(\xi) & \alpha_{12}(\xi)\\ \alpha_{21}(\xi) & \alpha_{22}(\xi)\end{vmatrix},$$

where

$$\alpha_{11}\left(\xi\right)=I-\int_0^T \alpha\left(\sigma\right)C\left(\sigma,\xi,A\right)d\sigma,\quad \alpha_{12}\left(\xi\right)=-\int_0^T \alpha\left(\sigma\right)S\left(\sigma,\xi,A\right)d\sigma,$$

$$\alpha_{21}(\xi) = \int_0^T \beta(\sigma) AS(\sigma,\xi,A)\, d\sigma, \quad \alpha_{22}(\xi) = I - \int_0^T \beta(\sigma) S(\sigma,\xi,A)\, d\sigma.$$

Then by using the properties

$$[C(\sigma,A)\, C(\tau,A) + AS(\sigma,A)\, S(\tau,A)] = I$$

of sine and cosine operator functions [4, §11.2, 11.4] we obtain

$$D(\xi) = I - \int_0^T [\alpha(\sigma) + \beta(\sigma)]\, C(\sigma) d\sigma +$$

$$+ \int_0^T \int_0^T \alpha(\sigma)\, \beta(\tau)\, [C(\sigma,\xi,A)\, C(\tau,\xi,A) + AS(\sigma,\xi,A)\, S(\tau,\xi,A)] d\sigma d\tau =$$

$$= I - \int_0^T [\alpha(\sigma) + \beta(\sigma)]\, C(\sigma) d\sigma + \int_0^T \int_0^T \alpha(\sigma)\, \beta(\tau) d\sigma d\tau = O(\xi).$$

Solving the system (10) $-$ (11), we get

$$u_0(\xi) = O^{-1}(\xi) \left\{ \left[I - \int_0^T \beta(\sigma)\, C(\sigma,\xi,A)\, d\sigma \right] f_1 + \int_0^T \alpha(\sigma)\, S(\sigma,\xi,A)\, d\sigma f_2 \right\} \tag{12}$$

and

$$u_1(\xi) = O^{-1}(\xi) \left\{ \left[I - \int_0^T \alpha(\sigma)\, C(\sigma,\xi,A)\, d\sigma \right] f_2 - \int_0^T \beta(\sigma)\, A_\xi S(\sigma,\xi,A)\, d\sigma f_1 \right\} \tag{13}$$

where

$$f_1 = \int_0^T \int_0^\sigma \alpha(\sigma)\, S(\sigma-\tau,\xi,A) |\xi|^2 \left(1 + |\xi|^2\right)^{-1} \hat{g}(\tau,\xi)\, d\tau d\sigma + \widehat{\varphi}(\xi),$$

$$f_2 = \int_0^T \int_0^\sigma \beta(\sigma)\, C(\sigma-\tau,\xi,A) |\xi|^2 \left(1 + |\xi|^2\right)^{-1} \hat{g}(\tau,\xi)\, d\tau d\sigma + \widehat{\psi}(\xi). \tag{14}$$

From (9), (12) and (13) we get that, the solution of the problem (8) can be expressed as

$$\hat{u}(t,\xi) = O^{-1}(\xi) \left\{ C(t,\xi,A) \left[\left(I - \int_0^T \beta(\sigma)\, C(\sigma,\xi,A)\, d\sigma \right) f_1 + \int_0^T \alpha(\sigma)\, S(\sigma,\xi,A)\, d\sigma f_2 \right] + \right.$$

$$\left. + S(t,\xi,A) \left[\left(I - \int_0^T \alpha(\sigma)\, C(\sigma,\xi,A)\, d\sigma \right) f_2 - \int_0^T \beta(\sigma)\, A_\xi S(\sigma,\xi,A)\, d\sigma f_1 \right] \right\} +$$

$$+ \int_0^t S(t-\tau,\xi,A) |\xi|^2 \left(1 + |\xi|^2\right)^{-1} \hat{g}(\tau,\xi)\, d\tau, \; t \in (0,T). \tag{15}$$

Taking into account (14) we obtain from (15) that there is a generalized solution of the problem (5) $-$ (6) which given by

$$u(x,t) = S_1(t,A)\varphi(x) + S_2(t,A)\psi(x) + \varPhi(t,x), \tag{16}$$

where $S_1(t,A)$ and $S_2(t,A)$ are linear operator functions in E defined by

$$S_1(t,A)\varphi = (2\pi)^{-\frac{1}{n}} \int_{R^n} \left\{ e^{ix\xi} O^{-1}(\xi) \left[C(t,\xi,A) \left(I - \int_0^T \beta(\sigma) C(\sigma,\xi,A)\, d\sigma \right) - \right.\right.$$

$$\left.\left. - A_\xi S(\sigma,\xi,A)\, d\sigma \right] \widehat{\varphi}(\xi)\, d\xi \right\},$$

$$S_2(t,A)\psi = (2\pi)^{-\frac{1}{n}} \int_{R^n} \left\{ e^{ix\xi} O^{-1}(\xi) \left[C(t,\xi,A) \int_0^T \alpha(\sigma) S(\sigma,\xi,A) + \right.\right.$$

$$\left. + S(\sigma,\xi,A) \left(I - \int_0^T \alpha(\sigma) C(\sigma,\xi,A)\, d\sigma \right) \right] \widehat{\varphi}(\xi)\, d\sigma$$

$$\left. \left(I - \int_0^T \beta(\sigma) C(\sigma,\xi,A) - A_\xi S(\sigma,\xi,A)\, d\sigma \right) \widehat{\varphi}(\xi) \right\} d\xi, \tag{17}$$

$$\varPhi(t,x) = (2\pi)^{-\frac{1}{n}} \int_{R^n} e^{ix\xi} O^{-1}(\xi) \left\{ \int_0^t S(t-\tau,\xi,A) |\xi|^2 \left(1+|\xi|^2\right)^{-1} \hat{g}(\tau,\xi)\, d\tau + \right.$$

$$+ \left[C(t,\xi,A) \left(I - \int_0^T \beta(\sigma) C(\sigma,\xi,A) d\sigma \right) + S(t,\xi,A) \int_0^T \beta(\sigma) A_\xi S(\sigma,\xi,A)\, d\sigma \right] g_1(\xi) +$$

$$+ \left[C(t,\xi,A) \int_0^T \alpha(\sigma) S(\sigma,\xi,A)\, d\sigma + \right.$$

$$\left.\left. + S(t,\xi,A) \left(I - \int_0^T \alpha(\sigma) C(\sigma,\xi,A) d\sigma \right) \right] g_2(\xi) \right\} d\xi$$

here

$$g_1(\xi) = \int_0^T \int_0^\sigma \alpha(\sigma) S(\sigma-\tau,\xi,A) |\xi|^2 \left(1+|\xi|^2\right)^{-1} \hat{g}(\tau,\xi)\, d\tau d\sigma, \tag{18}$$

$$g_2(\xi) = \int_0^T \int_0^\sigma \beta(\sigma) C(\sigma-\tau,\xi,A) |\xi|^2 \left(1+|\xi|^2\right)^{-1} \hat{g}(\tau,\xi)\, d\tau d\sigma.$$

Lemma 3.2. Suppose the Condition 3.1 hold. Then the solution of the problem (5) $-$ (6) satisfies the following estimate

$$\left(\|u\|_{X_\infty} + \|u_t\|_{X_\infty}\right) \leq C\left(\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1} + \right.$$

$$+\int_0^t \left(\|\Delta g(.,\tau)\|_{Y^{s,p}}+\|\Delta g(.,\tau)\|_{X_1}\right)d\tau\Bigg) \tag{19}$$

uniformly in $0\in[0,T]$.

Proof. Let $N\in\mathbb{N}$ and

$$\prod\nolimits_N=\{\xi:\xi\in R^n,|\xi|\le N\},\ \prod\nolimits'_N=\{\xi:\xi\in R^n,|\xi|\ge N\}.$$

It is clear to see that

$$\|u(.,t)\|_{L^\infty(R^n;E)}=\left\|F^{-1}u(\xi,t)\right\|_{L^\infty(R^n;E)}\le\left\|F^{-1}C(t,\xi,A)\widehat{\varphi}(\xi)\right\|_{X_\infty}+$$

$$+\left\|F^{-1}S(t,\xi,A)\widehat{\psi}(\xi)\right\|_{X_\infty}\le\left\|F^{-1}C(t,\xi,A)\widehat{\varphi}(\xi)\right\|_{L^\infty(\prod_N;E)}+$$

$$+\left\|F^{-1}S(t,\xi,A)\widehat{\psi}(\xi)\right\|_{L^\infty(\prod_N;E)}+\left\|F^{-1}C(t,\xi,A)\widehat{\varphi}(\xi)\right\|_{L^\infty(\prod'_N;E)}+$$

$$+\left\|F^{-1}S(t,\xi,A)\widehat{\psi}(\xi)\right\|_{L^\infty(\prod'_N;E)}, \tag{20}$$

$$\left\|F^{-1}C(t,\xi,A)\widehat{\varphi}(\xi)\right\|_{L^\infty(\prod'_N;E)}+\left\|F^{-1}S(t,\xi,A)\widehat{\psi}(\xi)\right\|_{L^\infty(\prod'_N;E)}=$$

$$=\left\|F^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}}C(t,\xi,A)\left(1+|\xi|^2\right)^{\frac{s}{2}}\widehat{\varphi}(\xi)\right\|_{L^\infty(\prod'_N;E)}+$$

$$+\left\|F^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(t,\xi,A)\left(1+|\xi|^2\right)^{\frac{s}{2}}\widehat{\psi}(\xi)\right\|_{L^\infty(\prod'_N;E)}.$$

Using Hölder inequality we have

$$\left\|F^{-1}C(t,\xi,A)\widehat{\varphi}(\xi)\right\|_{L^\infty(\prod_N;E)}+\left\|F^{-1}S(t,\xi,A)\widehat{\psi}(\xi)\right\|_{L^\infty(\prod_N;E)}\le C\left[\|\varphi\|_{X_1}+\|\psi\|_{X_1}\right]. \tag{21}$$

By using the resolvent properties of operator A, representation of $C(t,\xi,A)$, $S(t,\xi,A)$ we get

$$|\xi|^{|\alpha|+\frac{n}{p}}\left\|D^\alpha\left[\left(1+|\xi|^2\right)^{-\frac{s}{2}}C(t,\xi,A)\right]\right\|_{B(E)}\le C_1,$$

$$|\xi|^{|\alpha|+\frac{n}{p}}\left\|D^\alpha\left[\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(t,\xi,A)\right]\right\|_{B(E)}\le C_2, \tag{22}$$

for $s>\frac{n}{p}$ and all $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, $\alpha_k\in\{0,1\}$, $\xi\in R^n$, $\xi\ne0$, $t\in[0,T]$. By Proposition A_1 from (22) we get that, the operator – valued functions

$\left(1+|\xi|^2\right)^{-\frac{s}{2}}C(t,\xi,A)$, $\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(t,\xi,A)$ are $L^p(R^n;E)\rightarrow L^\infty(R^n;E)$ Fourier multipliers uniformly in $t\in[0,T]$. Then by Minkowski's inequality for integrals, the semigroups estimates and (20) we obtain

$$\left\|F^{-1}C(t,\xi,A)\,\widehat{\varphi}(\xi)\right\|_{L^\infty(\Pi'_N;E)}+\left\|F^{-1}S(t,\xi,A)\,\widehat{\psi}(\xi)\right\|_{L^\infty(\Pi'_N;E)}\leq C\left[\|\varphi\|_{Y^{s,p}}+\|\psi\|_{Y^{s,p}}\right]. \tag{23}$$

By reasoning as the above we get

$$\left\|F^{-1}\int_0^t S(t-\tau,\xi,A)\,\hat{g}(\xi,\tau)\,d\tau\right\|_{X_\infty}\leq C\int_0^t\left(\|\Delta g(.,\tau)\|_{Y^{s,p}}+\|\Delta g(.,\tau)\|_{X_1}\right)d\tau. \tag{24}$$

By differentiating, in view of (10) we obtain from (9) the estimate of type (21), (23), (24) for u_t. Then by using the estimates (20), (21), (23) we get the estimate (19).

Lemma 3.3. Assume the Condition 3.1 hold. Then the solution of the problem (5) $-$ $-$ (6) satisfies the following uniform estimate

$$\left(\|u\|_{Y^{s,p}}+\|u_t\|_{Y^{s,p}}\right)\leq C\left(\|\varphi\|_{Y^{s,p}}+\|\psi\|_{Y^{s,p}}+\int_0^t\|\Delta g(.,\tau)\|_{Y^{s,p}}d\tau\right). \tag{25}$$

Proof. From (8) we have the following estimate

$$\left(\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}\right\|_{X_p}+\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\widehat{u_t}\right\|_{X_p}\right)\leq$$

$$\leq C\left\{\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}C(t,\xi,A)\,\widehat{\varphi}\right\|_{X_p}+\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}S(t,\xi,A)\,\widehat{\psi}\right\|_{X_p}+\right.$$

$$\left.+\int_0^t\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}S(t-\tau,\xi,A)\,\hat{g}(.,\tau)\right\|_{X_p}d\tau\right\}. \tag{26}$$

By properties of operator – valued functions $C(t,\xi,A)$, $S(t,\xi,A)$ and in view of Proposition A_1 we get $C(t,\xi,A)$, $S(t,\xi,A)$ are $L^p(R^n;E)$ uniform Fourier multipliers. So, the estimate (26) by using the Minkowski's inequality for integrals implies (25).

From Lemmas 3.1 - 3.3 we obtain

Theorem 3.1. Let the Condition 3.1 hold. Then the problem (5) $-$ (6) has a unique solution $u\in C^{(2)}\left([0,T];Y_1^{s,p}\right)$ and the following estimates holds

$$\left(\|u\|_{X_\infty}+\|u_t\|_{X_\infty}\right)\leq$$

$$\leq C\left(\|\varphi\|_{Y^{s,p}}+\|\varphi\|_{X_1}+\|\psi\|_{Y^{s,p}}+\|\psi\|_{X_1}+\int_0^t\left(\|\Delta g(.,\tau)\|_{Y^{s,p}}+\|\Delta g(.,\tau)\|_{X_1}\right)d\tau\right), \tag{27}$$

$$(\|u\|_{Y^{s,p}} + \|u_t\|_{Y^{s,p}}) \leq C \left(\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|\Delta g\,(.,\tau)\|_{Y^{s,p}} d\tau \right) \tag{28}$$

uniformly in $t \in [0, T]$.

Proof. From Lemma 3.1 we obtain that, problem (5) $-$ (6) has a unique generalized solution. From the representation of solution (9) and Lemma 3.2, 3.3 we have $u \in C^{(2)}\left([0,T]\,; Y_1^{s,p}\right)$ and estimates (27), (28) hold.

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