Journal of Contemporary Applied Mathematics
V. 13, No 2, 2023, December
ISSN 2222-5498
DOI 10.5281/zenodo.8416972

Optimal Control Synthesis for the Process described by the Third Order Partial Differential Equation

R.S.Mamedov, Sh.I.Kuliyeva, N. Bagirli

Abstract. The problem of optimal control synthesis for the process described by the third order partial differential equation is considered. The quality criterion is a quadratic functional characterizing the deviation of the system state from the desired one and the energy spent on process management. Combining the Fourier method with the dynamic programming method, we obtain an analytic expression for the optimal control in the form of a functional defined on the set of states of control object, and the matrix of the feedback operator is found as a solution of the Riccati-type matrix differential equation.

Key Words and Phrases: optimal control synthesis, Fourier method, dynamic programming, Bellman equation.

2010 Mathematics Subject Classifications: 49K20, 49J20, 93D15

1. Problem statement

Let the controlled process be described by the function $u(t,x)$, which satisfies the equation

$$\beta\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+\xi\frac{\partial^3 u}{\partial t\partial x^2}+p(t,x) \tag{1}$$

inside the domain $\bar{Q}=[0,\ T]\times[0,\ l]$ and the conditions

$$u(0,x)=u^0(x),\qquad \frac{\partial u(0,x)}{\partial t}=u^1(x), \tag{2}$$

$$\frac{\partial u(t,0)}{\partial x}=0,\quad \frac{\partial u(t,l)}{\partial x}+\alpha u(t,l)=0 \tag{3}$$

on the boundary, where $a,\ \alpha>0$ are real constants, β is a relaxation coefficient, and $\xi=const>0$ is a heat-conduction viscosity coefficient.

$p(t,x)$ is a control parameter, and we assume that every function $p(t,x)\in L_2(0,T)$ is an admissible control.

Note that the equation (1) describes the heat exchange processes in a viscous heat-conducting medium [1].

According to [2], [3] under the above assumptions, the problem (1)-(3) has a unique solution $u(t,x) \in W_2^1(Q)$. Using the source function, we can represent this solution as

$$\int_0^T \int_0^l \left\{ -\beta \frac{\partial u(t,x)}{\partial t} \frac{\partial \phi(t,x)}{\partial t} + a^2 \frac{\partial u(t,x)}{\partial x} \frac{\partial \phi(t,x)}{\partial x} + \xi \frac{\partial^2 u(t,x)}{\partial t \partial x} \frac{\partial \phi(t,x)}{\partial x} \right\} dxdt+ \\ +\beta \int_0^l \phi(0,x) u^1(x) dx + \alpha \int_0^T (a^2 u(t,l) + \xi \frac{\partial u(t,l)}{\partial t}) \phi(t,l) dt = 0$$

for every function $\Phi(t,x) \in W_2^1$, which satisfies the condition $\Phi(T,x) = 0$, and the satisfaction of conditions (2) is understood in the weak sense.

Optimal control problem is to determine a control parameter such that the functional

$$J[p] = \int_0^l (\gamma_1 [u(T,x) - \varphi^0(x)]^2 + \gamma_2 [u_t(T,x) - \varphi^1(x)]^2) dx + \gamma \int_0^T \int_0^l p^2(t,x) dxdt \quad (4)$$

takes the smallest possible value, where $\varphi^0(x)$ and $\varphi^1(x)$ are the given functions from $L_2(0,l)$.

2. Problem statement in infinite-dimensional phase space

Using Fourier method, we can restate our problem in an infinite-dimensional phase space. For this, we will seek for the solution of the problem (1)-(3) in the following form:

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) X_k(x), u_k(t) = \int_0^l u(t,x) X_k(x) dx \quad (5)$$

where $X_k(x)$ is a non-trivial solution of the problem

$$X''(x) + \lambda^2 X(x) = 0, 0 < x < l, X'(0) = 0, X'(l) + \alpha X(l) = 0, \quad (6)$$

which is also an orthonormal system of functions in $L_2(0,l)$ [4].

Substituting the function (5) into the equation (1), we obtain an infinite system of ordinary differential equations with respect to the Fourier coefficients $u_n(t),\ n = 1, 2, 3, ...$:

$$\beta \frac{d^2 u_n(t)}{dt^2} + (1 + \xi \lambda_n^2) \frac{du_n(t)}{dt} + a^2 \lambda_n^2 + u_n(t) = p_n(t), n = 1, 2, 3, ... \quad (7)$$

As the function (5) must satisfy the initial conditions (2), we have

$$u_n(0) = u_n^0, \frac{du_n(0)}{dt} = u_n^1, n = 1, 2, 3, ..., \quad (8)$$

where $p_n(t)$, u_n^0 and u_n^1 are the coefficients of the functions $p\,(t,x)$, $u^0(x)$ and $u^1(x)$, respectively.

Since the system of functions $\{X_n(x)\}$ is orthonormal, the functional (4) can be represented as follows:

$$I[p] = \sum_{n=1}^{\infty} I_n[p_n], \quad (9)$$

$$I_n[p_n] = \gamma_1[u_n(T) - \varphi_n^0]^2 + \gamma_2[\frac{du_n(T)}{dt} - \varphi_n']^2 + \gamma \int_0^T p_n^2(t)dt. \quad (10)$$

Thus, the above problem of optimal control synthesis is reduced to finding a control such that the functional (9) takes the smallest possible value.

It is seen from the system (7) that the coefficients $u_n(t),\ n = 1, 2, 3, ...$ are defined independently of each other. Therefore, it suffices to consider the minimizations of the functional $I_n[p_n]$ for every n under restrictions (7)-(8).

Denote

$$u_{1n}(t) = u_n(t), u_{2n}(t) = \frac{du_n(t)}{dt}. \quad (11)$$

Then the problem (7)-(8) can be written in canonical form

$$\begin{cases} \frac{du_{1n}(t)}{dt} = u_{2n}(t) \\ \frac{du_{2n}(t)}{dt} = -\frac{a^2}{\beta}\lambda_n^2 u_1(t) - \frac{1+\xi\lambda n^2}{\beta} u_{2n}(t) + \frac{1}{\beta} p_n(t), n = 1, 2, 3, ... \end{cases} \quad (12)$$

$$u_{1n}(0) = u_n^0, u_{2n}(0) = u_n^1, n = 1, 2, 3, ..., \quad (13)$$

and the functional (10) becomes

$$I_n[p_n] = \gamma_1[u_{1n}(T) - \varphi_n^0]^2 + \gamma_2[u_{2n}(T) - \varphi_n^1]^2 + \gamma \int_0^T p_n^2(t)dt \quad (14)$$

Introduce the following vectors and matrices:

$$\bar{u}_n(t) = \begin{pmatrix} u_{1n}(t) \\ u_{2n}(t) \end{pmatrix}, \bar{u}_n^0 = \begin{pmatrix} u_n^0 \\ u_n^1 \end{pmatrix}, \bar{u}_n^0 = \begin{pmatrix} u_n^0 \\ u_n^1 \end{pmatrix}, \bar{\varphi}_n = \begin{pmatrix} \varphi_n^0 \\ \varphi_n^1 \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 \\ \frac{1}{\beta} \end{pmatrix}, A_n = \begin{pmatrix} 0 & 1 \\ -\frac{a^2}{\beta}\lambda_n^2 & -\frac{1+\xi\lambda_n^2}{\beta} \end{pmatrix}, G = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}. \quad (15)$$

Then the problem (12)-(13) with a quality criterion (14) can be rewritten in matrix form:

$$\begin{cases} \frac{d\bar{u}_n}{dt} = A_n\bar{u}_n + B_n p_n(t) \\ \bar{u}_n(0) = \bar{u}_n^0 \end{cases} \quad (16)$$

$$I[\bar{p}_n] = [\bar{u}_n(T) - \bar{\varphi}_n]^* G[\bar{u}_n(T) - \bar{\varphi}_n] + \gamma \int_0^T p_n^2(t)dt \quad (17)$$

3. Finding optimal control

To determine the optimal control in the form of feedback, we will use the dynamic programming method. Therefore, following [5], we introduce the Bellman functional

$$S[t, \bar{u}_n] = \min_{p_n(T)\in L_2(0,T)} \left\{ [\bar{u}_n(T) - \bar{\varphi}_n]^* G[\bar{u}_n(T) - \varphi_n] + \gamma \int_t^T p_n^2(T)dt \right\} \quad (18)$$

Applying the well-known Bellman method [6], we find that this functional satisfies the equation

$$-\frac{\partial S}{\partial t} = -\frac{1}{4\gamma}(\frac{\partial S}{\partial \bar{u}_n})^* B_n B_n^* \frac{\partial S}{\partial \bar{u}_n} + \frac{\partial S}{\bar{u}_n} A_n \bar{u}_n \tag{19}$$

with the additional condition

$$S[T, \bar{u}_n] = [\bar{u}_n(T) - \bar{\varphi}_n]^* G[u_n(T) - \varphi_n], \tag{20}$$

and the control parameter is defined as follows:

$$p_n(t) = -\frac{1}{2\gamma} B_n^* \frac{\partial S}{\partial \bar{u}_n}. \tag{21}$$

The solution of the functional equation is sought in the following form:

$$S[t, \bar{u}_n] = [\bar{u}_n(t) - \bar{\varphi}_n{}^* K_n(t)[\bar{u}_n(t) - \bar{\varphi}_n], \tag{22}$$

where the symmetric positive definite matrix

$$K_n(t) = \begin{pmatrix} k_{0n}(t) & k_{1n}(t) \\ k_{1n}(t) & k_{2n}(t) \end{pmatrix}$$

is a solution of the Riccati-type matrix differential equation [7], [8]

$$\frac{dK_n}{dt} = -A_n^* K_n - A_n K_n + \frac{1}{\gamma} K B_n B_n^* K \tag{23}$$

with an additional condition

$$K(t) = G. \tag{24}$$

Then from (22) we obtain $\frac{\partial S}{\partial \bar{u}_n} = 2K_n(t)[\bar{u}_n(t) - \bar{\varphi}_n]$, and taking this into account in (21), we have

$$p_n(t) = -\frac{1}{\gamma} B_n^* K_n(t)[\bar{u}_n(t) - \bar{\varphi}_n]. \tag{25}$$

Thus, if the matrix $K_n(t)$ is determined from the equation (23) with the condition (24), then the control parameter is found in the form of feedback by the formula (25).

Let's rewrite the formula (25) with coordinates, in order to define the control parameter for the original problem:

$$\begin{aligned} p_n(t) = -\tfrac{1}{\gamma} \begin{pmatrix} 0 & \tfrac{1}{\beta} \end{pmatrix} \begin{pmatrix} k_{0n} & k_{1n} \\ k_{1n} & k_{2n} \end{pmatrix} \begin{pmatrix} u_{1n}(t) - \varphi_n^0 \\ u_{2n}(t) - \varphi_n \end{pmatrix} = -\tfrac{1}{\beta\gamma} k_{1n}(t)(u_{1n}(t) - \varphi_n^0) = \\ = \tfrac{1}{\beta\gamma} k_{2n}(t)(u_{2n}(t) - \varphi_n^1) \end{aligned} \tag{26}$$

Denote

$$K_i(t, x, s) = \sum_{n=1}^{\infty} k_{1n}(t) X_n(x) X_n(s), \; i = 1, 2.$$

Then, from (26) we have

$$p(t,x)=\int_0^l K_1(t,x,s)u(t,x,s)ds+\int_0^l K_2(t,x,s)u_t(x,s)ds+q(t,x), \qquad (27)$$

where

$$q(t,x)=-\int_0^l (K_1(t,x,s)\varphi^0(s)+K_2(t,x,s)\varphi^1(s)ds.$$

So, for the original problem too, the control parameter is defined in the form of synthesis or in the form of feedback.

References

[1] M.E. Kerimova. Asymptotic of the solution of a boundary value problem stated in an infinite half strip for a non-classical type singularly perturbed differential equation. Journal of Contemporary Applied Mathematics, 2021, v.11, No 2, p.61-70.

[2] Ya.E.Megraliyev, A.I.Ismailov. Obratnya krayevaya zadacha dlya psevdo parabolicheskogo uravneniya tretyevo poryadka s neklassicheskimi krayevimi usloviyami. Journal of Contemporary Applied Mathematics. 2018, v.8, No 2, pp. 83-97.

[3] R.S. Mamedov, S.Y. Gasimov. Solving control problem with minimum energy for the process of heat transfer in a medium with temperature viscosity. International Conference on Actual Problems of Applied Mathematics, 22-26 May 2018, Nalchik, Russia. (*in Russian*)

[4] Mamedov R.S., Gasimov S.Y. Approximate solution of optimal problem for cooling process with minimum energy. Baku Mathematical Journal, 2022, Vol. 1, No.1, p. 106-110

[5] Yegorov A.I. The foundations of control theory. M.: FIZMATLIT, 2004. – 504 p. (*in Russian*)

[6] K.R. Aida-zade , V.A. Hashimov. A problem of feedback of the moving sources to heat the plate. Proceedings of the 7^{th} International Conference on Control and Optimization with Industrial Applications, Baku, 2020. Volume 1, pp.47-49.

[7] A.I.Yeqorov. Uravneniye Rikkati. M.: FIZMATLIT, 2001. -320 S.

[8] M.S. Almamedov. Ob indekse defekte sistemi odnogo parametrichesogo matrichnogo uravneniya tipa Shturm-Liuvillya. Journal of Contemporary Applied Mathematics. 2019, v.9, No 2, pp. 32-47.

R.S.Mamedov
Azerbaijan State University of oil and Industry, Baku, Azerbaijan
E-mail: rasadmammedov@gmail.com

Sh.I.Kuliyeva
Azerbaijan State University of oil and Industry, Baku, Azerbaijan
E-mail: shafaquluzade@gmail.com

N. Bagirli
Azerbaijan State University of oil and Industry, Baku, Azerbaijan
E-mail: bagirli1980@mail.ru

Received 12 May 2023
Accepted 25 September 2023