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Control Problem with Minimum Energy for Vibrations of AN Elastic ROD

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Abstract. Vibrations of elastic rod with a constant cross section are studied assuming that the rod has an axis of symmetry. Moreover, if it is affected by the distributed external loads, then the small free vibrations are described by the fourth order partial differential equation with various boundary conditions. The problem of controllability is solved using external forces, which play the role of control parameter.
By the Fourier method, we restate the given optimal control problem in an infinite-dimensional phase space. As a result, we obtain an optimal control problem in a functional space. Using the given condition at the finite moment of time, we reduce this problem to the problem of moments. This allows us to find the control parameter in analytical form.

Key Words and Phrases: minimum energy control, vibrations of elastic rod, Fourier method, problem of moments.

2010 Mathematics Subject Classifications: 49K20, 49J20, 93D15

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1. Introduction

1. **Problem statement.** Consider the following boundary value problem for the equation of transverse vibrations of a rod [1]:

$$a^2\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=p(t,\,x),0<t<T,0<x<1, \tag{1}$$

$$u(0,\,x)=\varphi_0(x),u_t(0,\,x)=\varphi_1(x),0\le x\le l, \tag{2}$$

$$u(t,\,0)=0,u_{xx}(t,\,0)=0,0<t<T, \tag{3}$$

$$u(t,\,1)=0,u_{xx}(t,\,1)=0,0<t<T, \tag{4}$$

where $\varphi_0\in W_2^2(0,\,1)$, $\varphi_1\in L_2(0,\,1)$ are the given functions, and the function $p(t,\,x)\in L_2(Q)$, $Q=\{0<t<T,\ 0<x<1\}$ is a control parameter subject to definition.

Under the given conditions, every admissible control defines a unique generalized solution $u(t,\, x) \in W_2^{1,2}(Q)$, which, for every function $\$(t,\, x) \in W_2^{1,2}(Q)$ such that $\$(T,\, x) = 0$, $\$_x(t,\, 0) = 0$ and $\$_x(t,\, 1) = 0$, satisfies the integral identity [2], [3]:

$$\int_0^T \int_0^1 \left[-\frac{\partial u(t,\, x)}{\partial t} \cdot \frac{\partial \$}{\partial t} + a^2 \frac{\partial^2 u(t,\, x)}{\partial x^2} \cdot \frac{\partial^2 \$}{\partial x^2} \right] dxdt,$$

$$-\int_0^1 \varphi_1(x)\$(0,\, x)dx = \int_0^T \int_0^1 p(t,\, x)\$(t,\, x)dxdt, \tag{5}$$

and the satisfaction of the condition $u(0,\, x) = \varphi_0(x)$ is understood in the sense of

$$\lim_{t \to 0+} \int_0^T (u(0,\, x) - (x))^2\, dx = 0.$$

The optimal control problem is to find a control parameter such that the corresponding solution $u(t,\, x)$ of the problem (1)-(4) satisfies the condition

$$u(T,\, x) = \psi(x) \tag{6}$$

and the functional

$$I[p] = \int_0^T \int_0^1 p^2(t,\, x)dxdt \tag{7}$$

takes the smallest possible value.

2. Statement of problem in infinite-dimensional phase space

Consider in $L_2(0,\, 1)$ the orthonormal system of functions $X_n(x) = \frac{1}{\omega_n} \sin \lambda_n x$, $n = 1,\, 2,\, ...$, where $\lambda_n = \pi n$, $n = 1,\, 2,\, ...$ are the eigenvalues of the boundary value problem

$$\frac{d^4 X}{dx^4} - \lambda^4 X = 0, \quad 0 < x < 1, \tag{8}$$

$$X(0) = X''(0) = 0, \quad X(1) = X''(1) = 0, \tag{9}$$

and $\omega_n^2 = \frac{2\lambda_n - \sin 2\lambda_n}{4\lambda_n}$.

Then, we can seek for the solution of the boundary value problem (1)-(4) in the form of Fourier series

$$u(t,\, x) = \sum_{n=1}^{\infty} u_n(t) X_n(x), u_n(t) = \int_0^1 u(t,\, x) X_n(x) dx. \tag{10}$$

Multiply both sides of the equation (1) by $X_n(x)$ and then integrate over x from 0 to 1:

$$a^2 \int_0^1 \frac{\partial^2 y}{\partial t^2} X_n(x)dx + \int_0^1 u(t,\, x) X_n^4(x)dx +$$

$$+\left(\frac{\partial^3 u}{\partial x^3}X_n(x)-\frac{\partial^2 u}{\partial x^2}X_n'(x)+\frac{\partial u}{\partial x}X_n''(x)-u\cdot X_n'''(x)\right)\Big|_0^1=\int_0^1 p(t,\,x)X_n(x)dx.$$

By (8) and the boundary conditions (3), (4) and (9), we have

$$\frac{d^2}{dt}\int_0^1 u(t,\,x)X_n(x)dx+\lambda_n^4\int_0^1 u(t,\,x)X_n(x)dx=\int_0^1 p(t,\,x)X_n(x)dx,$$

or

$$\ddot{u}_n(t)+a^2\lambda_n^4 u_n(t)=p_n(t), \tag{11}$$

where $p_n(t)$'s are the Fourier coefficients of the function $p(t,\,x)$.

From the initial conditions (2) it follows that the equation (11) is solved with the initial conditions

$$u_n(0)=\varphi_{0n}, u_n(0)=\varphi_{1n}, n=1,\,2,\,.... \tag{12}$$

In view of the orthonormality of the system of functions $X_n(x)$, the functional (7) becomes

$$I[\bar{p}]=\sum_{n=1}^{\infty}I_n[p_n], I_n[p_n]=\int_0^T p_n^2(t)dt. \tag{13}$$

So the problem is reduced to determining the control parameters $p_n(t)\in t_2(0,\,T)$, $n=1,\,2,\,\ldots$, such that the corresponding solutions of the problem (11)-(12) satisfy the conditions

$$u_n(T)=\psi_n,\quad n=1,\,2,\,\ldots, \tag{14}$$

and the functional (13) takes the smallest possible value, where ψ_n's are the Fourier coefficients of the function $\psi(x)$.

As the Fourier coefficients $u_n(t)$ are defined by (11)-(12) independently of each other, it suffices to consider the minimization problem for the functional $I_n[p_n]$ for any n.

3. Applying the *l*-problem of moments. By Cauchy formula, the solution of the equation (11) with the initial conditions (12) can be represented in the form [4]

$$u_n(t)=\varphi_{0n}\cos a^2\lambda_n^2 t+\frac{1}{a^2\lambda_n^2}\varphi_{1n}\sin a^2\lambda_n^2 t+\frac{a}{\lambda_n}\int_0^t \sin a\lambda_n(t-T)p_n(\tau)d\tau. \tag{15}$$

Then the condition (14) can be rewritten as

$$\varphi_{0n}\cos a^2\lambda_n^2 T+\frac{1}{a^2\lambda_n^2 T}\varphi_{1n}\sin a^2\lambda_n^2 T+\frac{a}{\lambda_n}\int_0^T \sin a\lambda_n(T-t)p_n(t)d\tau=\psi_n.$$

Hence it follows that the control parameter satisfies the integral equation

$$\frac{a}{\lambda_n}\int_0^T \sin a\lambda_n(T-t)p_n(t)d\tau=c_n, \tag{16}$$

where
$$c_n = \psi_n - \varphi_{0n} \cos a^2\lambda_n^2 T - \frac{1}{a^2\lambda_n^2 T}\varphi_{1n} \sin a^2\lambda_n^2 T.$$

So the following statement is true:

Lemma. For the solution $u_n(t)$ of the problem (11)-(12) to satisfy the condition (14), it is necessary and sufficient that the control $p_n(t)$ satisfies the moment relations (16) for $n = 1, 2, ...$.

To determine the optimal parameters of $p_n(t)$ in the space $L_2(0, T)$, let's consider the one-dimensional space H_n of elements $q_n(t)$ defined by the formula [5]

$$q_n(t) = \alpha_n e_n(t), e_n(t) = \frac{a}{\lambda_n} \sin a\lambda_n(T - t), \tag{17}$$

where α_n's are real coefficients subject to definition.

By Levi's theorem, every element $p_n(t) \in L_2(0, T)$ can be uniquely represented in the following form [6]:
$$p_n(t) = q_n(t) + g_n(t), q_n(t) \in H_n, g_n(t) \perp H_n,$$

and
$$\|p_n\|^2_{L_2(0,T)} = \|q_n\|^2_{L_2(0,T)} + \|g_n\|^2_{L_2(0,T)}.$$

In the case under consideration,

$$< g_n(t), q_n(t) >= \int_0^T g_n(t)q_n(t)dt = \frac{a\alpha_n}{\lambda_n}\int_0 g_n(t) \sin a\lambda_n(T - t)dt = 0,$$

and, consequently, the term $g_n(t)$ does not affect the solution of the equation (16). Nonetheless, it has a non-zero norm.

So the following theorem is true:

Theorem 1. For the control $p_n(t)$ to satisfy the moment relations (16), it is necessary and sufficient that these relations are satisfied by the projection $q_n(t)$ of this control onto the subspace H_n.

So, if the considered optimal control problem has a solution, then it belongs to H_n, i.e. it can be represented as

$$p_n(t) = \frac{a\alpha_n}{\lambda_n} \sin a\lambda_n(t - T). \tag{18}$$

To determine the unknown coefficients α_n, we substitute (18) into (16):

$$\frac{a}{\lambda_n}\int_0^T \frac{a\alpha_n}{\lambda_n} \sin^2 a\lambda_n(T - t)dt = c_n.$$

Hence, after elementary transformations we get

$$\alpha_n = \frac{4c_n\lambda_n^3}{a2aT\lambda_n - \sin 2aT\lambda_n}.$$

And, consequently,

$$p_n(t) = \frac{a\alpha_n}{\lambda_n} \sin a\lambda_n (T-t) = \frac{4c_n \lambda_n^3}{2aT\lambda_n - \sin 2aT\lambda_n} \sin a\lambda_n (T-t). \tag{19}$$

The corresponding value of the functional I_n is

$$I_n = \int_0^T p_n^2(t)dt = \frac{8c_n^2 \lambda_n^3}{2a^2 T\lambda_n - a \sin 2aT\lambda_n}.$$

By (13), we get

$$I[\bar{p}] = \sum_{n=1}^{\infty} I_n[p_n] = \sum_{n=1}^{\infty} \frac{8c_n^2 \lambda_n^3}{2a^2 T\lambda_n - a \sin 2aT\lambda_n}. \tag{20}$$

Obviously, the series in (20) converges if the condition

$$\sum_{n=1}^{\infty} I_n[p_n] = c_n^2 \lambda_n^2 < \infty \tag{21}$$

holds, where the coefficients A_ns are defined by (17).

So we have got the validity of the following theorem:

Theorem 2. Let the controlled process be described by the boundary value problem (1)-(4), where the admissible controls are the arbitrary functions $p = p(t, x) \in L_2(Q)$. Also, let the condition (21) hold, where the coefficients A_n are defined by the formula (17), and ψ_n, φ_{0n} and φ_{1n} are the Fourier coefficients of the functions $\psi(x)$, $\varphi_0(x)$, $\varphi_1(x)$, respectively. Then the control problem with minimum energy has a unique solution and this solution can be represented in the form of (19), and the corresponding minimal value of the functional (7) is calculated by (20).

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