Journal of Contemporary Applied Mathematics
V. 13, No 2, 2023, December
ISSN 2222-5498
DOI 10.5281/zenodo.8416992

Generalized discrete Kneser's Theorems

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Abstract. As is known, qualitative analysis of the solutions of difference equations has a growing interest in the last three decade. In thes qualitative analysis, Kneser's Theorem has the large location. In this work, we generalize discrete Kneser's Theorem using generalized difference operators $\Delta_a = E - aI$ and $\Delta_a = E^a - I$ where E is forward shift operator and I is identity operator in order to assist in qualitative analyzing of difference equations involving generalized difference operators and give some new results. Also we give some examples to illustrate the results.

Key Words and Phrases: Discrete Kneser's Theorems; Oscillation; Difference equations; Generalized difference operators; Qualitative analysis

2010 Mathematics Subject Classifications: Primary 39A21, 39A10

1. Introduction

We know that just by observing, some precise and accurate results to produce for real-life problems which depanding on time-variables or other parameters is not possible. The only way it would be possible to use mathematics. By this way when modeling the real life problems reveal difference equations or differential equations. The majority of these equations are nonlinear equations. As is known there is no general method for obtaining analytical solutions of nonlinear difference equations and differential equations. Therefore trying to learn about the behavior (especially oscillation, stability, asymptotic behavior etc.) of the solution is more important than to obtain its analytical solutions. Qualitative analysis of the difference equations and differential equations has a growing interest in the last thirty years. In particular, examination of the behavior of difference equations involving generalized difference operator has become one of the most attractive areas in the last decade. For some examples see references [1, 2, 3, 4]. Kneser's Theorem **[11, 12]** (discrete or continuous) has great importance to obtain concerning results with the behavior of the solutions of high order linear or nonlinear difference equations and differential equations [4, 5, 6, 7, 8, 9, 10]. Therefore, we generalize the Discrete Kneser's Theorem in order to assist in qualitative analyzing of difference equations involving generalized difference operators.

2. Main results

Lemma 2.1. *Let $a \in \mathbb{R}$, u_k and v_k any two function defined on $\mathbb{Z}$ and define generalized difference operator Δ_a as $\Delta_a y_k = y_{k+1} - a y_k$. Then we have*

$$\Delta_a(u_k.v_k) = v_{k+1}\Delta_a u_k + u_k \Delta_a v_k + (a-1)u_k v_{k+1}$$

and

$$\Delta_a(\frac{u_k}{v_k}) = \frac{v_k \Delta_a u_k - u_k \Delta_a v_k - a(a-1)u_k v_k}{v_k v_{k+1}},\ v_k \neq 0.$$

Proof. It is very easy to prove, therefore we omit it in here. □

Lemma 2.2. *Let $a \in \mathbb{R}^+$, u_n and v_n any two function defined on $\mathbb{Z}$ and define generalized difference operator Δ_a as $\Delta_a y_n = y_{n+a} - y_n$. Then we have*

$$\Delta_a(u_n.v_n) = v_{n+a}\Delta_a u_n + u_n \Delta_a v_n$$

and

$$\Delta_a(\frac{u_n}{v_n}) = \frac{v_n \Delta_a u_n - u_n \Delta_a v_n}{v_n v_{n+a}},\ v_n \neq 0.$$

Proof. The proof is easy. Therefore we omit it in here. □

Lemma 2.3. *Let $a \in \mathbb{Z}^+$ and define generalized difference operator Δ_a as in Lemma 2.2. Then*

$$\sum_{i=n_1}^{n-a} \Delta_a y_i = (I + E^{-1} + \cdots + E^{-a+1})y_n - (I + E + \cdots + E^{a-1})y_{n_1},$$

where the operators E^s and E^{-s} are forward and backward shift operators respectively and defined as $E^{\pm s} y_n = y_{n \pm s}$, $s \in \mathbb{N}$, and $E^1 = E$.

Proof. It is very easy to prove, therefore we omit it in here. □

Lemma 2.4. *Let $1 \leq a \in \mathbb{Z}^+$, $1 \leq m \leq n-1$ and z_k be defined on $\mathbb{N}_{k_0}$. Define $\Delta_a z_k = z_{k+1} - a z_k$ and $\Delta_a^m = \Delta_a(\Delta_a^{m-1})$. Then;*

i. $\liminf_{k\to\infty} \Delta_a^m z_k > 0$ *implies that* $\lim_{k\to\infty} \Delta_a^i z_k = \infty$,

ii. $\limsup_{k\to\infty} \Delta_a^m z_k < 0$ *implies that* $\lim_{k\to\infty} \Delta_a^i z_k = -\infty$.

Proof. (i) If $\liminf_{k\to\infty} \Delta_a^m z_k > 0$ then we can find a sufficiently large $k_1 \in \mathbb{N}_{k_0} = \{k_0, k_0 + 1, ...\}$, $k_0 \in \mathbb{N}$, such that $\Delta_a^m z_k \geq c > 0$ for all $k \in \mathbb{N}_{k_1}$. Summing up $\Delta_a^m z_k$ from k_1 to $k-1$ we have

$$\sum_{l=k_1}^{k-1} \Delta_a^{m-1} z_{l+1} = a \sum_{l=k_1}^{k-1} \Delta_a^{m-1} z_l + \sum_{l=k_1}^{k-1} \Delta_a^m z_l. \quad (1)$$

Since $\Delta_a^m z_k = \Delta_a^{m-1} z_{k+1} - a\Delta_a^{m-1} z_k > 0$ and $\Delta_a^{m-1} z_{k+1} > a\Delta_a^{m-1} z_k$, from (1) we have

$$\begin{aligned}
\Delta_a^{m-1} z_k \sum_{l=k_1}^{k-1} (\frac{1}{a})^l &> \sum_{l=k_1}^{k-1} \Delta_a^{m-1} z_{l+1} \\
&= a \sum_{l=k_1}^{k-1} \Delta_a^{m-1} z_l + \sum_{l=k_1}^{k-1} \Delta_a^m z_l \\
&> a\Delta_a^{m-1} z_{k_1} \sum_{l=k_1}^{k-1} a^l + c(k-k_1)
\end{aligned}$$

that is,

$$\frac{a^k-1}{(a-1)a^{k-1}} \Delta_a^{m-1} z_k > a\frac{a^k-1}{a-1} \Delta_a^{m-1} z_{k_1} + c(k-k_1)$$

or

$$\Delta_a^{m-1} z_k > a^k \Delta_a^{m-1} z_{k_1} + c(\frac{a-1}{a})(\frac{1}{1-\frac{1}{a^k}})(k-k_1) \to \infty$$

as $k \to \infty$. Hence it follows that $\lim_{k\to\infty} \Delta_a^{m-1} z_k = \infty$. The rest of the proof can be made by induction. The case (ii) can be made by the same way. □

Lemma 2.5. *Let* $1 \le a \in \mathbb{Z}^+$, $1 \le m \le n-1$ *and* z_k *be defined on* $\mathbb{N}_{k_0}$. *Define* $\Delta_a z_k = z_{k+a} - z_k$ *and* $\Delta_a^m = \Delta_a(\Delta_a^{m-1})$ *Then;*

i. $\liminf_{k\to\infty} \Delta_a^m z_k > 0$ *implies that* $\lim_{k\to\infty} \Delta_a^i z_k = \infty$,

ii. $\limsup_{k\to\infty} \Delta_a^m z_k < 0$ *implies that* $\lim_{k\to\infty} \Delta_a^i z_k = -\infty$.

Proof. *(i)* If $\liminf_{k\to\infty} \Delta_a^m z_k > 0$ then we can find a sufficiently large $k_1 \in \mathbb{N}_{k_0}$ such that $\Delta_a^m z_k \ge c > 0$ for all $k \in \mathbb{N}_{k_1}$. Summing up $\Delta_a^m z_k$ from k_1 to $k-a$ by Lemma 2.3 we have

$$\begin{aligned}
\sum_{k_1}^{k-a} \Delta_a^m z_k &= \sum_{k_1}^{k-a} \Delta_a(\Delta_a^{m-1}) z_k \qquad (2) \\
&= \sum_{k_1}^{k-a} (E^a - I)(\Delta_a^{m-1}) z_k \\
&= \left(\Delta_a^{m-1}\right)(I + E^{-1} + \cdots + E^{-a+1}) z_k \\
&- \left(\Delta_a^{m-1}\right)(I + E + \cdots + E^{a-1}) z_{k_1} \\
&\ge c(k-k_1).
\end{aligned}$$

Threfore from (2) we can write

$$\left(\Delta_a^{m-1}\right)(I + E^{-1} + \cdots + E^{-a+1}) z_k \ge \left(\Delta_a^{m-1}\right)(I + E + \cdots + E^{a-1}) z_{k_1} + c(k-k_1) \to \infty$$

as $k \to \infty$. Hence it follows that $\lim_{k\to\infty} \Delta_a^{m-1} z_k = \infty$. The rest of the proof can be made by the mathematical induction way. The case (ii) can be made by the same way. □

Lemma 2.6. *Let $a \in \mathbb{R}^+$, $1 \le m \le n-1$ and z_k be defined on $\mathbb{R}_a = \{0, a, 2a, ...\}$. Define $\Delta_a z_k = z_{k+a} - z_k$ and $\Delta_a^m = \Delta_a(\Delta_a^{m-1})$ Then;*

i. $\liminf_{k\to\infty} \Delta_a^m z_k > 0$ *implies that* $\lim_{k\to\infty} \Delta_a^i z_k = \infty$,

ii. $\limsup_{k\to\infty} \Delta_a^m z_k < 0$ *implies that* $\lim_{k\to\infty} \Delta_a^i z_k = -\infty$.

Proof. $\liminf_{k\to\infty} \Delta_a^m z_k > 0$ implies that there exists a large $k_1 \in \mathbb{R}_a$ such that $\Delta_a^m z_k \ge c > 0$ for all $k \in \mathbb{R}_{k_1}$. Summing up $\Delta_a^m z_k$ we have (from k_1 to $k-a$)

$$\sum_{j=k_1}^{k-a} \Delta_a^m z_j = \sum_{j=k_1}^{k-a} \Delta_a(\Delta_a^{m-1}) z_j = (\Delta_a^{m-1}) z_{ka} - (\Delta_a^{m-1}) z_{ak_1} \ge c(k-k_1). \tag{3}$$

Threfore from (3) we can write

$$(\Delta_a^{m-1}) z_k \ge (\Delta_a^{m-1}) z_{k_1} + c(k-k_1) \to \infty$$

as $k \to \infty$. Hence it follows that $\lim_{k\to\infty} \Delta_a^{m-1} z_k = \infty$. The rest of the proof can be made by the mathematical induction way. The case (ii) can be made by the same way. □

Firstly we give the following generalized discrete Kneser's Theorem by defining generalized forward difference operator Δ_a as $\Delta_a z_k = z_{k+1} - a z_k$.

Theorem 2.7. *(First Generalized Discrete Kneser's Theorem). Let $1 \le a \in \mathbb{R}^+$, z_k be defined on $\mathbb{N}_{k_0}$, and $z_k > 0$ with $\Delta_a^n z_k$ of not identically zero and constant sign on $\mathbb{N}_{k_0}$ .Hence, exists an integer m, $0 \le m \le n$ with $n+m$ even for $\Delta_a^n z_k \ge 0$ or odd for $\Delta_a^n z_k \le 0$ and such that*

(i) $m \ge 1$ implies that $\Delta_a^i z_k > 0$, $k \in \mathbb{N}_{k_0}$, $1 \le i \le m-1$,

(ii) $m \le n-1$ implies that $(-1)^{m+i} \Delta_a^i z_k > 0$, $k \in \mathbb{N}_{k_0}$, $m \le i \le n-1$.

Proof. We need to consider two cases:

Case 1. $\Delta_a^n z_k \le 0$ on $\mathbb{N}_{k_0}$. First of all, we will prove that $\Delta_a^{n-1} z_k > 0$ on $\mathbb{N}_{k_0}$. On the contrary, suppose that we can find a $k_1 \ge k_0$ in $\mathbb{N}_{k_0}$ such that $\Delta_a^{n-1} z_{k_1} \le 0$. Since $\Delta_a^n z_k = \Delta_a^{n-1} z_{k+1} - a\Delta_a^{n-1} z_k \le 0$ on $\mathbb{N}_{k_0}$, we have $\Delta_a^{n-1} z_{k_1+1} \le a\Delta_a^{n-1} z_{k_1} \le 0$. Therefore we can write

$$\cdots \le \Delta_a^{n-1} z_{k_1+s} \le a\Delta_a^{n-1} z_{k_1+s-1} \le \cdots \le a^{s-1}\Delta_a^{n-1} z_{k_1+1} \le a^s \Delta_a^{n-1} z_{k_1} \le 0, \tag{4}$$

$s \in \mathbb{N}$. Considering $\Delta_a^{n-1} z_k$ is not identically constant, from (4) by Lemma 2.4 we find $\lim_{k\to\infty} \Delta_a^i z_k = -\infty$ and thus we have $\lim_{k\to\infty} z_k = -\infty$ which is a contradiction to $z_k > 0$. Hence, $\Delta_a^{n-1} z_k > 0$ on $\mathbb{N}_{k_0}$ and we can find a sufficiently small integer m such that $0 \le m \le n-1$ with $n+m$ odd and

$$(-1)^{m+i} \Delta_a^i z_k > 0 \text{ on } \mathbb{N}_{k_0}, \;\; m \le i \le n-1. \tag{5}$$

Now let $m > 1$ and

$$\Delta_a^{m-1} z_k < 0 \text{ on } \mathbb{N}_{k_0}, \tag{6}$$

then again from Lemma 2.4 it follows that

$$\Delta_a^{m-2} z_k > 0 \text{ on } \mathbb{N}_{k_0}. \tag{7}$$

From inequalities (5)-(7), we have

$$(-1)^{(m-2)+i} \Delta_a^i z_k > 0 \text{ on } \mathbb{N}_{k_0}, \ \ m-2 \le i \le n-1$$

which is a contradiction to the defination of m. Therefore, (6) fails and $\Delta_a^{m-1} z_k \ge 0$ on $\mathbb{N}_{k_0}$. Considering (4) and (6), thus we have $\lim_{k\to\infty} \Delta_a^{m-1} z_k > 0$. If $m > 2$, we obtain from Lemma 2.4 that $\lim_{k\to\infty} \Delta_a^i z_k = \infty$, $1 \le i \le m-1$. Hence, $\Delta_a^i z_k > 0$ for all sufficiently large $k \in \mathbb{N}_{k_0}$, $1 \le i \le m-1$.

Case 2.$\Delta_a^n z_k \ge 0$ on $\mathbb{N}_{k_0}$. Assume that there exists a $k_3 \in \mathbb{N}_{k_2}$ such that $\Delta_a^{n-1} z_{k_3} \ge 0$, then since $\Delta_a^{n-1} z_{k+1} \ge a\Delta_a^{n-1} z_k$ and not identically constant, there exists a $k_4 \in \mathbb{N}_{k_3}$ such that $\Delta_a^{n-1} z_k > 0$ for all $k \in \mathbb{N}_{k_4}$. Hence, $\lim_{k\to\infty} \Delta_a^{n-1} z_k > 0$ and from Lemma 2.4 $\lim_{k\to\infty} \Delta_a^i z_k = \infty$, $1 \le i \le n-2$ and therefore $\Delta_a^i z_k > 0$ for all sufficiently large k in $\mathbb{N}_{k_0}$, $1 \le i \le n-1$. The proof of theorem is completed for $m = n$. In the case of $\Delta_a^{n-1} z_k < 0$ for all $k \in \mathbb{N}_{k_0}$, we obtain from Lemma 2.4 that $\Delta_a^{n-2} z_k > 0$ for all $k \in \mathbb{N}_{k_0}$. The rest of the proof can be made by the same way in Case 1. □

Theorem 2.8. *Let $z_k > 0 (< 0)$ be defined on $\mathbb{N}_{k_0}$, $a \in \mathbb{R}^-$ and $\Delta_a^n z_k$ is not identically zero and with constant sign on $\mathbb{N}_{k_0}$. Then*

$$\Delta_a^i z_k > 0 (< 0), \quad 0 \le i \le n.$$

Proof. Assume that $z_k > 0$ on $\mathbb{N}_{k_1}$ for $k_1 \ge k_0$, without loss of generality (When $z_k < 0$ can be proved in similar manner). Now, we will prove that $\Delta_a^i z_k > 0$ on $\mathbb{N}_{k_0}$, $1 \le i \le n$ (one can prove by the similar way to the case of $\Delta_a^i z_k < 0$ on $\mathbb{N}_{k_0}$, $1 \le i \le n$).We assume the contrary, then we can find a $k_1 \ge k_0$ in $\mathbb{N}_{k_0}$ and any i, $1 \le i \le n$, such that $\Delta_a^i z_k > 0$ but $\Delta_a^{i+1} z_k \le 0$ on $\mathbb{N}_{k_1}$. Then we have

$$\Delta_a^i z_k \Delta_a^{i+1} z_k \le -c < 0, \ \ c \in \mathbb{R}^+, \tag{8}$$

on $\mathbb{N}_{k_1}$. Summing up (8) from k_1 to $k-1$ we obtain

$$\sum_{l=k_1}^{k-1} \Delta_a^i z_l \Delta_a^{i+1} z_l \le -c(k-k_1). \tag{9}$$

Get $\Delta_a^i z_l = u_l > 0$ and $\Delta_a^{i+1} z_l = \Delta_a u_l \le 0$. By Lemma 2.1 we have

$$\sum_{l=k_1}^{k-1} u_l \Delta_a u_l = \sum_{l=k_1}^{k-1} \Delta_a (u_l.u_l) - \sum_{l=k_1}^{k-1} u_{l+1} \Delta_a u_l - \sum_{l=k_1}^{k-1} (a-1) u_l u_{l+1} \tag{10}$$

$$= u_{k-1} u_{k-1} - a u_{k_1} v_{k_1} + (1-a) \sum_{l=k_1}^{k-1} u_{l+1} u_{l+1}$$

$$- \sum_{l=k_1}^{k-1} u_{l+1}\Delta_a u_l - \sum_{l=k_1}^{k-1}(a-1)u_l u_{l+1}$$
$$= u_{k-1}^2 - au_{k_1}^2 + \sum_{l=k_1}^{k-1} u_{l+1}(u_l - au_{l+1}).$$

From (9) and (10) we obtain

$$\sum_{l=k_1}^{k-1} u_l\Delta_a u_l = u_{k-1}^2 - au_{k_1}^2 + \sum_{l=k_1}^{k-1} u_{l+1}(u_l - au_{l+1}) \le -c(k-k_1)$$

which contradicts with the fact that $u_{k-1}^2 - au_{k_1}^2 + \sum_{l=k_1}^{k-1}(u_l u_{l+1} - au_{l+1}^2) > 0$. So, our assumption $\Delta_a^{i+1} z_k \le 0$ fails and $\Delta_a^{i+1} z_k > 0$ on $\mathbb{N}_{k_1}$. Hence we reach $\Delta_a^i z_k > 0$ on $\mathbb{N}_{k_1}$, $0 \le i \le n$. Proof is complete.

Also the proof can be made by mathematical induction method. Since $a < 0$, $\Delta_a z_k = z_{k+1} - az_k > 0$ on $\mathbb{N}_{k_1}$ for $i = 1$, $\Delta_a^2 z_k = \Delta_a(\Delta_a z_k) = \Delta_a z_{k+1} - a\Delta_a z_k > 0$ on $\mathbb{N}_{k_1}$ for $i = 2$, and so continued we obtain $\Delta_a^i z_k > 0$ on $\mathbb{N}_{k_1}$ for $i = n$. □

Secondly we give generalized the following discrete Kneser's Theorems by defining generalized forward difference operator Δ_a as $\Delta_a z_k = z_{k+a} - z_k$.

Theorem 2.9. *(Second Generalized Discrete Kneser's Theorem). Let $a \in \mathbb{Z}^+$, $z_k > 0$ be defined on $\mathbb{N}_{k_0}$, and $\Delta_a^n z_k$ is not identically zero and with constant sign on $\mathbb{N}_{k_0}$. Then, exists an integer m, $0 \le m \le n$ with $n+m$ odd for $\Delta_a^n z_k \le 0$ or $n+m$ even for $\Delta_a^n z_k \ge 0$ and such that*

(i) $m \ge 1$ implies that $\Delta_a^i z_k > 0$, $k \in \mathbb{N}_{k_0}$, $1 \le i \le m-1$,

(ii)$m \le n-1$ implies that $(-1)^{m+i}\Delta_a^i z_k > 0$, $k \in \mathbb{N}_{k_0}$, $m \le i \le n-1$.

Proof. We need to consider two cases:

Case 1. $\Delta_a^n z_k \le 0$ on $\mathbb{N}_{k_0}$. First of all, we will prove that $\Delta_a^{n-1} z_k > 0$ on $\mathbb{N}_{k_0}$. On the contrary, suppose that we can find a $k_1 \ge k_0$ in $\mathbb{N}_{k_0}$ such that $\Delta_a^{n-1} z_{k_1} \le 0$. Since $\Delta_a^n z_k = \Delta_a^{n-1} z_{k+a} \le \Delta_a^{n-1} z_k \le 0$, that is, $\Delta_a^{n-1} z_k$ is decreasing and not identically constant on $\mathbb{N}_{k_0}$, wecan find a $k_2 \in \mathbb{N}_{k_1}$ such that $\Delta_a^{n-1} z_k \le \Delta_a^{n-1} z_{k_2} < \Delta_a^{n-1} z_{k_1}$ for all $k \in \mathbb{N}_{k_2}$. But by the Lemma 2.5 we see that $\lim_{k\to\infty} \Delta_a^i z_k = -\infty$ and thus we have $\lim_{k\to\infty} z_k = -\infty$ which is a contradiction to $z_k > 0$. Hence, $\Delta_a^{n-1} z_k > 0$ on $\mathbb{N}_{k_0}$ and there exists a smallest integer m such that $0 \le m \le n-1$ with $n+m$ odd and

$$(-1)^{m+i}\Delta_a^i z_k > 0 \text{ on } \mathbb{N}_{k_0}, \;\; m \le i \le n-1. \tag{11}$$

Now let $m > 1$ and

$$\Delta_a^{m-1} z_k < 0 \text{ on } \mathbb{N}_{k_0}, \tag{12}$$

then again from Lemma 2.5 it follows that

$$\Delta_a^{m-2} z_k > 0 \text{ on } \mathbb{N}_{k_0}. \tag{13}$$

From inequalities (11)-(13), we have

$$(-1)^{(m-2)+i}\Delta_a^i z_k > 0 \text{ on } \mathbb{N}_{k_0}, \ \ m-2 \le i \le n-1$$

which is a contradiction to the defination of m. Therefore, (12) fails and $\Delta_a^{m-1} z_k \ge 0$ on $\mathbb{N}_{k_0}$. From (12), $\Delta_a^{m-1} z_k$ is increasing and $\lim_{k\to\infty} \Delta_a^{m-1} z_k > 0$. If $m > 2$, we obtain from Lemma 2.5 that $\lim_{k\to\infty} \Delta_a^i z_k = \infty$, $1 \le i \le m-1$. Hence, $\Delta_a^i z_k > 0$ for all sufficiently large $k \in \mathbb{N}_{k_0}$, $1 \le i \le m-1$.

Case 2.$\Delta_a^n z_k \ge 0$ on $\mathbb{N}_{k_0}$. Assume that there exists a $k_3 \in \mathbb{N}_{k_2}$ such that $\Delta_a^{n-1} z_{k_3} \ge 0$, then since $\Delta_a^{n-1} z_{k+1} \ge a\Delta_a^{n-1} z_k$ and not identically constant, there exists some $k_4 \in \mathbb{N}_{k_3}$ such that $\Delta_a^{n-1} z_k > 0$ for all $k \in \mathbb{N}_{k_4}$. Thus, $\lim_{k\to\infty} \Delta_a^{n-1} z_k > 0$ and from Lemma 5 $\lim_{k\to\infty} \Delta_a^i z_k = \infty$, $1 \le i \le n-2$ and hence $\Delta_a^i z_k > 0$ for all large k in $\mathbb{N}_{k_0}$, $1 \le i \le n-1$. The proof of theorem is completed for $m = n$. In the case $\Delta_a^{n-1} z_k < 0$ for all $k \in \mathbb{N}_{k_0}$, we find from Lemma 2.5 that $\Delta_a^{n-2} z_k > 0$ for all $k \in \mathbb{N}_{k_0}$. The rest of the proof can be made by the same way in Case 1. □

Theorem 2.10. *Let $a \in \mathbb{R}^+$, $z_k > 0$ be defined on $\mathbb{R}_{k_0}$, and $\Delta_a^n z_k$ is not identically zero and with constant sign on $\mathbb{R}_{k_0}$. Then, exists an integer m, $0 \le m \le n$ with $n+m$ odd for $\Delta_a^n z_k \le 0$ or $n+m$ even for $\Delta_a^n z_k \ge 0$ and such that $m \le n-1$ implies that $(-1)^{m+i}\Delta_a^i z_k > 0$, $k \in \mathbb{R}_{k_0}$, $m \le i \le n-1$, $m \ge 1$ implies that $\Delta_a^i z_k > 0$, $k \in \mathbb{R}_{k_0}$, $1 \le i \le m-1$.*

Proof. We need to consider two cases:

Case 1. $\Delta_a^n z_k \le 0$ on $\mathbb{R}_{k_0}$. First of all, we will prove that $\Delta_a^{n-1} z_k > 0$ on $\mathbb{R}_{k_0}$. On the contrary, suppose that we can find a $k_1 \ge k_0$ in $\mathbb{R}_{k_0}$ such that $\Delta_a^{n-1} z_{k_1} \le 0$ on $\mathbb{R}_{k_1}$. Since $\Delta_a^n z_k = \Delta_a^{n-1} z_{k+a} \le \Delta_a^{n-1} z_k \le 0$, that is, $\Delta_a^{n-1} z_k$ is decreasing and not identically constant on $\mathbb{R}_{k_1}$, there exists $k_2 \in \mathbb{R}_{k_1}$ such that $\Delta_a^{n-1} z_k \le \Delta_a^{n-1} z_{k_2} < \Delta_a^{n-1} z_{k_1}$ for all $k \in \mathbb{R}_{k_2}$. But by the Lemma 2.6 we obtain $\lim_{k\to\infty} \Delta_a^i z_k = -\infty$ and thus we have $\lim_{k\to\infty} z_k = -\infty$ which is a contradiction to $z_k > 0$. Hence, $\Delta_a^{n-1} z_k > 0$ on $\mathbb{R}_{k_0}$ and there exists a smallest integer m, $0 \le m \le n-1$ with $n+m$ odd and

$$(-1)^{m+i}\Delta_a^i z_k > 0 \text{ on } \mathbb{R}_{k_0}, \ \ m \le i \le n-1. \tag{14}$$

Next let $m > 1$ and

$$\Delta_a^{m-1} z_k < 0 \text{ on } \mathbb{R}_{k_0}, \tag{15}$$

then again from Lemma 2.6 it follows that

$$\Delta_a^{m-2} z_k > 0 \text{ on } \mathbb{R}_{k_0}. \tag{16}$$

From inequalities (14)-(16), we have

$$(-1)^{(m-2)+i}\Delta_a^i z_k > 0 \text{ on } \mathbb{R}_{k_0}, \ \ m-2 \le i \le n-1$$

which is a contradiction to the defination of m. Therefore, (15) fails and $\Delta_a^{m-1} z_k \ge 0$ on $\mathbb{R}_{k_0}$. From (15), $\Delta_a^{m-1} z_k$ is increasing and $\lim_{k\to\infty} \Delta_a^{m-1} z_k > 0$. If $m > 2$, we obtain from

Lemma 2.6 that $\lim_{k\to\infty} \Delta_a^i z_k = \infty$ $1 \le i \le m-1$. Hence, $\Delta_a^i z_k > 0$ for all sufficiently large $k \in \mathbb{R}_{k_0}$, $1 \le i \le m-1$.

Case 2.$\Delta_a^n z_k \ge 0$ on $\mathbb{R}_{k_0}$. Let $k_3 \in \mathbb{R}_{k_2}$ be such that $\Delta_a^{n-1} z_k \ge 0$, then since $\Delta_a^{n-1} z_{k+1} \ge a\Delta_a^{n-1} z_k$ and not identically constant, there exists some $k_4 \in \mathbb{R}_{k_3}$ such that $\Delta_a^{n-1} z_k > 0$ for all $k \in \mathbb{R}_{k_4}$. Thus, $\lim_{k\to\infty} \Delta_a^{n-1} z_k > 0$ and from Lemma 2.6 $\lim_{k\to\infty} \Delta_a^i z_k = \infty$, $1 \le i \le n-2$ and so $\Delta_a^i z_k > 0$ for all large k in $\mathbb{R}_{k_0}$, $1 \le i \le n-1$. The proof of theorem is completed for $m = n$. In the case of $\Delta_a^{n-1} z_k < 0$ for all $k \in \mathbb{R}_{k_0}$, we obtain from Lemma 2.6 that $\Delta_a^{n-2} z_k > 0$ for all $k \in \mathbb{R}_{k_0}$. The rest of the proof can be made by the same way in Case 1. □

Following examples $(1-4)$ are for the operator $\Delta_a = E - aI$ where $a \in \mathbb{N}$, examples $(5-7)$ are for the operator $\Delta_a = E^a - I$ where $a \in \mathbb{N}$ and examples $(8,9)$ are for the operator $\Delta_a = E^a - I$ where $a \in \mathbb{R}$:

Example 2.11. *Let $a = 3$. Consider the function $z_k = \frac{k(k-1)a^k}{2a^2} - k^2 > 0$. z_k be defined on $\mathbb{N}_2$, and $\Delta_a^n z_k$ is not identically zero and with constant sign on $\mathbb{N}_2$. Hence all the conditions of theorem 2.7 are satisfied. Really for $m = 3$, $\Delta_a z_k > 0$, $\Delta_a^2 z_k > 0$, and $(-1)^{i+3}\Delta_a^i z_k > 0$, $m = 3 \le i \le n-1$.*

Example 2.12. *Let $a = 2$. Consider the functions $z_k = k > 0$ or $z_k = \frac{1}{k} > 0$, $k \in \mathbb{N}$. Then $(-1)^i \Delta_a^i z_k > 0$ for large $k \in \mathbb{N}$, $0 \le i \le n-1$. Hence all the conditions of Theorem 2.7 are satisfied.*

Example 2.13. *Let $1 < a < e$. Consider the function $z_k = e^k$. $\Delta_a^i z_k > 0$ for all $k \in \mathbb{N}$, $0 \le i \le n$. Really $\Delta_a z_k = \Delta_a e^k = e^{k+1} - ae^k = e^k(e-a) > 0$, $\Delta_a^2 e^k = (e-a)^2 e^k > 0$, ..., $\Delta_a^i e^k = (e-a)^i e^k > 0$.*

Example 2.14. *Let $a < 0$. Consider the function $z_k = k^2 k \in \mathbb{N}$. Then $\Delta_a z_k = \Delta_a k^2 = (k+1)^2 - (-3)k^2 = 4k^2 + 2k + 1 > 0$, $\Delta_a^2 k^2 = \Delta_a(4k^2 + 2k + 1) = 16k^2 + 16k + 6 > 0$, ..., $\Delta_a^i k^2 > 0$, $1 \le i \le n$. Hence all the conditions of Theorem 2.8 are satisfied.*

Example 2.15. *Let $a = 2$. Consider the function $z_k = k^3 - \frac{1}{k} > 0$, $k \in \mathbb{Z}^+$. $\Delta_a^n z_k$ is not identically zero and with constant sign on $\mathbb{Z}^+$. Then, exists an integer m, $0 \le m \le n$ with $n + m$ odd for $\Delta_a^n z_k \le 0$ and such that $m \le n-1$ implies that $(-1)^{m+i}\Delta_a^i z_k > 0$, $k \in \mathbb{Z}^+$, $m \le i \le n-1$, $m \ge 1$ implies that $\Delta_a^i z_k > 0$, $k \in \mathbb{Z}^+$, $1 \le i \le m-1$. Really for $m = 3$, $\Delta_a z_k > 0$, $\Delta_a^2 z_k > 0$, $\Delta_a^3 z_k > 0$, and $(-1)^{i+3}\Delta_a^i z_k > 0$, $m = 3 \le i \le n-1$. Hence all the conditions of Theorem 2.9 are satisfied.*

Example 2.16. *Let $a = 3$. Consider the function $z_k = k^3 + \frac{1}{k} > 0$, $k \in \mathbb{Z}^+$. $\Delta_a^n z_k$ is not identically zero and with constant sign on $\mathbb{Z}^+$. Then, exists an integer m, $0 \le m \le n$ with $n+m$ even for $\Delta_a^n z_k \ge 0$ and such that $m \le n-1$ implies that $(-1)^{m+i}\Delta_a^i z_k > 0$, $k \in \mathbb{Z}^+$, $m \le i \le n-1$, $m \ge 1$ implies that $\Delta_a^i z_k > 0$, $k \in \mathbb{Z}^+$, $1 \le i \le m-1$. Really for $m = 4$, $\Delta_a z_k > 0$, $\Delta_a^2 z_k > 0$, $\Delta_a^3 z_k > 0$, $\Delta_a^4 z_k > 0$ and $(-1)^{i+4}\Delta_a^i z_k > 0$, $m = 4 \le i \le n-1$. Hence all the conditions of Theorem 2.9 are satisfied.*

Example 2.17. *Let* $a = 4$. *Consider the function* $z_k = 2^k$, $k \in \mathbb{Z}^+$. $m \geq 1$ *implies* $\Delta_a^i z_k > 0$ *for all large* $k \in \mathbb{Z}^+$, $1 = m \leq i \leq n$. *Hence all the conditions of theorem 2.9 are satisfied.*

Example 2.18. *Let* $a = 2\sqrt{2}$. *Consider the function* $z_k = k^4 + \frac{1}{k}$, $k \in \mathbb{R}^+$. $z_k > 0$ *is not identically zero and with* $\Delta_a^n z_k$ *of constant sign on* $\mathbb{R}^+$. *Hence all the conditions of theorem 2.10 are satisfied. Really for* $m = 5$, $\Delta_a z_k > 0$, $\Delta_a^2 z_k > 0$, $\Delta_a^3 z_k > 0$, $\Delta_a^4 z_k > 0$, $\Delta_a^5 z_k > 0$ *and* $(-1)^{i+5}\Delta_a^i z_k > 0$, $m = 5 \leq i \leq n-1$.

Example 2.19. *Let* $a = \frac{3}{4}$. *Consider the function* $z_k = 3^k$, $k \in \mathbb{R}^+$. $m \geq 1$ *implies* $\Delta_a^i z_k > 0$ *for all large* $k \in \mathbb{R}^+$, $1 = m \leq i \leq n$. *Hence all the conditions of theorem 2.10 are satisfied.*

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Received 11 June 2023
Accepted 05 October 2023