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Minkowski inequality via Generalized $\kappa-$Conformable Fractional Integral Operators

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Abstract. In this paper, we define the left and right generalized κ-conformable fractional integral operators depending upon a parameter $\kappa > 0$. This operator obtain other operators in case of special selection the help of κ parameter. Furthermore, we obtain important generalizations of the reverse Minkowski inequality and novel results of some inequalities by using generalized κ-conformable fractional integrals.

Key Words and Phrases: Reverse minkowski inequality, generalized $\kappa-$fractional integral, generalized κ-conformable fractional integrals

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1. Introduction

In the recent times, inequalities have played a very substantial role in all fields of fractional integrals. A heart of the matter of this field has numerous fractional operators and also we can choose the most convenient operator for modeling problem. The authors introduced a lot of fractional integrals and derivatives and also they obtained their generalized fractional integrals and derivatives via different approaches $[1-7]$. Other authors developed generalized $\kappa-$fractional integrals. Mubeen and Habibullah studied $\kappa-$fractional integrals and their applications. Moreover, Set et al. demonstrated Gruss type inequalities for $\kappa-$fractional integrals $[9-10]$. Other authors studied Gruss inequality that establishes as a connection between the product of two functions and the product of the integrals [20]. Chinchane and Pachpatte have obtained new fractional inequalities and Minkowski fractional integral inequality by means of Hadamard fractional integrals [11]. In recent years, Mubeen et al. have obtained the Minkowski inequality generalizations through the instrumentality of generalized $\kappa-$fractional conformable integral [12]. Addition to these researches, Rashid et al. have stated and proved the new some important inequalities for generalized $\kappa-$fractional integral operators [19].

In this paper, we are going to demonstrate a new version for reverse Minkowski inequality with the generalized $\kappa-$conformable fractional integral operators and also obtain their particular consequences. We will show sufficient consequences like the reverse

Minkowski inequalities via the newly described generalized $\kappa-$conformable fractional integrals. Finally, we will obtain some inequalities by using the newly described generalized $\kappa-$conformable fractional integrals.

Now, primary definitions and theorems related to described generalized $\kappa-$conformable fractional integrals are being given as the following.

Definition 1. [5 − 8] [13] *A function δ_1 is said to be in $L_{d,r}[0,\infty)$ space if*

$$L_{d,r}[0,\infty)=\left\{\delta_1:\|\delta_1\|_{L_{d,r}[0,\infty)}=\left(\int_0^\infty|\delta_1(\tau)|^d\tau^r d\tau\right)^{\frac{1}{d}}<\infty,\ 1\le d<\infty,\ r\ge 0\right\} \tag{1.1}$$

for $r=0$,

$$L_d[0,\infty)=\left\{\delta_1:\|\delta_1\|_{L_d[0,\infty)}=\left(\int_0^\infty|\delta_1(\tau)|^d d\tau\right)^{\frac{1}{d}}<\infty,\ 1\le d<\infty\right\}. \tag{1.2}$$

Definition 2. [20] *Let $\delta_1\in L_1[0,\infty)$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. The space $X_\gamma^d(0,\infty)\,(1\le d<\infty)$ of those real-valued Lebesgue measurable functions δ_1 on $[0,\infty)$ for which*

$$\|\delta_1\|_{X_\gamma^d}=\left(\int_0^\infty|\delta_1(\tau)|^d\gamma'(\tau)\,d\tau\right)^{\frac{1}{d}}<\infty,\ 1\le d<\infty \tag{1.3}$$

and for the case $d=\infty$

$$\|\delta_1\|_{X_\gamma^\infty}=ess\ \sup_{0\le\tau<\infty}[\gamma'(\tau)\,\delta_1(\tau)].$$

In particular, if we take $\gamma(\tau)=\tau$ $(1\le d<\infty)$ the space $X_\gamma^d(0,\infty)$, we have the $L_d[0,\infty)$-space. Furthermore, if we take $\gamma(\tau)=\frac{\tau^{r+1}}{r+1}$ $(1\le d<\infty,\ r\ge 0)$ the space $X_\gamma^d(0,\infty)$, we have the $L_{d,r}[0,\infty)$-space.

In this circumstance, we will state the generalized κ-conformable fractional integral operator in the following form.

Definition 3. *Let $\delta_1\in X_\gamma^d(0,\infty)$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. Then the left and right sided generalized κ-conformable fractional integral operators of a function δ_1 in the sense of another function γ of order $\beta>0$ for $\alpha>0$ are stated as:*

$$\left({}^{\gamma}T_{\phi^+,\tau}^{\beta,\kappa}\delta_1\right)(\tau)=\frac{1}{\kappa\Gamma_\kappa(\beta)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)\delta_1(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}},\ \phi<\tau \tag{1.4}$$

and

$$\left({}^{\gamma}T_{\rho^-,\tau}^{\beta,\kappa}\delta_1\right)(\tau)=\frac{1}{\kappa\Gamma_\kappa(\beta)}\int_\tau^\rho\left[\frac{(\gamma(\rho)-\gamma(\tau))^\alpha-(\gamma(\rho)-\gamma(\pi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)\delta_1(\pi)d\pi}{(\gamma(\rho)-\gamma(\pi))^{1-\alpha}},\ \tau<\rho \tag{1.5}$$

where $\beta\in\mathbb{C}$, $\Re(\beta)>0$ and $\Gamma_\kappa(\beta)=\int_0^\infty\pi^{\beta-1}e^{-\frac{\pi^\kappa}{\kappa}}d\pi$ is the κ-Gamma function in [14].

Remark 1. *We will give some important consequences of generalized κ-conformable fractional integrals*

1. *Letting $\kappa = 1$ and $\alpha = 1$, we have generalized RL fractional integral operators* [5].
2. *Letting $\gamma(\tau) = \tau$ and $\alpha = 1$, we have κ-fractional integral operators* [9].
3. *Letting $\gamma(\tau) = \tau$, $\alpha = 1$ and $\kappa = 1$, we have RL fractional integral operators.*
4. *Letting $\gamma(\tau) = \log \tau$, $\alpha = 1$ and $\kappa = 1$, we have Hadamard fractional integral operators* [5 − 8].
5. *Letting $\gamma(\tau) = \frac{\tau^n}{n}\,(n > 0)$, $\alpha = 1$ and $\kappa = 1$, we have Katugampola fractional integral operators* [15].
6. *Letting $\gamma(\tau) = \frac{(\tau-\phi)^n}{n}\,(n > 0)$, $\alpha = 1$ and $\kappa = 1$, we have conformable fractional integral operators* [1].
7. *Letting $\gamma(\tau) = \frac{\tau^{u+v}}{u+v}$, $\alpha = 1$ and $\kappa = 1$, we have generalized conformable fractional integral operators* [16].

Definition 4. *Let $\delta_1 \in X_\gamma^d(0,\infty)$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0) = 0$. Then the one-sided generalized $\kappa-$conformable fractional integral operators of a function δ_1 in the sense of another function γ of order $\beta > 0$ are stated as:*

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1\right)(\tau) = \frac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau \left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)\delta_1(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}},\ \pi > 0 \tag{1.6}$$

where Γ_κ is the $\kappa-$Gamma function.

In here, we will give theorems which we use later in this article some theorems ın [17] and [18].

Theorem 1. [17] *Let δ_1, δ_2 be two positive functions on $[0,\infty)$ for $s \geq 1$. If $0 < \varsigma \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq \Omega$ and $\tau \in [v_1, v_2]$, then*

$$\begin{aligned}&\left(\int_{v_1}^{v_2}\delta_1^s(\tau)\,d\tau\right)^{\frac{1}{s}} + \left(\int_{v_1}^{v_2}\delta_2^s(\tau)\,d\tau\right)^{\frac{1}{s}}\\ &\qquad\leq \frac{1+\Omega(\varsigma+2)}{(\varsigma+1)(\Omega+1)}\left(\left(\int_{v_1}^{v_2}(\delta_1+\delta_2)^s(\tau)\,d\tau\right)^{\frac{1}{s}}\right).\end{aligned} \tag{1.7}$$

Theorem 2. [17] *Let δ_1, δ_2 be two positive functions on $[0,\infty)$ for $s \geq 1$. If $0 < \varsigma \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq \Omega$ and $\tau \in [v_1, v_2]$, then*

$$\begin{aligned}&\left(\int_{v_1}^{v_2}\delta_1^s(\tau)\,d\tau\right)^{\frac{2}{s}} + \left(\int_{v_1}^{v_2}\delta_2^s(\tau)\,d\tau\right)^{\frac{2}{s}}\\ &\qquad\geq \left(\frac{(1+\Omega)(\varsigma+1)}{\Omega}-2\right)\left(\int_{v_1}^{v_2}\delta_1^s(\tau)\,d\tau\right)^{\frac{1}{s}}\left(\int_{v_1}^{v_2}\delta_2^s(\tau)\,d\tau\right)^{\frac{1}{s}}.\end{aligned} \tag{1.8}$$

Theorem 3. [18] *Let δ_1, δ_2 be two positive functions on $[0,\infty)$, for $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, $s \geq 1$. If $0 < \varsigma \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq \Omega$, $\pi \in [v_1, \tau]$ with $\tau > 0$, $T^\beta_{v_1^+}\delta_1^s(\tau) < \infty$ and $T^\beta_{v_1^+}\delta_2^s(\tau) < \infty$,* *then,*

$$\begin{aligned}&\left(T^{\beta}_{v_1^+}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left(T^{\beta}_{v_1^+}\delta_2^s(\tau)\right)^{\frac{1}{s}}\\&\qquad\le\tfrac{1+\Omega(\varsigma+2)}{(\varsigma+1)(\Omega+1)}\left(T^{\beta}_{v_1^+}(\delta_1+\delta_2)^s(\tau)\right)^{\frac{1}{s}}.\end{aligned}\tag{1.9}$$

Theorem 4. [18] *Let* δ_1,δ_2 *be two positive functions on* $[0,\infty)$, *for* $\beta\in\mathbb{C}$, $\Re(\beta)>0$, $s\ge1$. *If* $0<\varsigma\le\frac{\delta_1(\pi)}{\delta_2(\pi)}\le\Omega$, $\pi\in[v_1,\tau]$ *with* $\tau>0$, $T^{\beta}_{v_1^+}\delta_1^s(\tau)<\infty$ *and* $T^{\beta}_{v_1^+}\delta_2^s(\tau)<\infty$, *then,*

$$\begin{aligned}&\left(T^{\beta}_{v_1^+}\delta_1^s(\tau)\right)^{\frac{2}{s}}+\left(T^{\beta}_{v_1^+}\delta_2^s(\tau)\right)^{\frac{2}{s}}\\&\qquad\ge\left(\tfrac{(1+\Omega)(\varsigma+2)}{\Omega}-2\right)\left(T^{\beta}_{v_1^+}\delta_1^s(\tau)\right)^{\frac{1}{s}}\left(T^{\beta}_{v_1^+}\delta_2^s(\tau)\right)^{\frac{1}{s}}.\end{aligned}\tag{1.10}$$

In Rashid et al. defined generalized $\kappa-$fractional integral operators for parameter $\kappa>0$. Moreover, they proved reverse Minkowski inequalities via generalized $\kappa-$fractional integral operators. Furthermore, they proved new theorems by using certain associated inequalities the help of generalized $\kappa-$fractional integrals. They have obtained *Theorem* 4, *theorem* 6 and *theorem* 7 by using the following *definition* 5 in [19].

Definition 5. *Let* $\delta_1\in X^d_\gamma(0,\infty)$. γ *be an increasing and positive monotone function on* $[0,\infty)$ *and also* γ' *be continuos derivative on* $[0,\infty)$ *and* $\gamma(0)=0$. *Then the left and right sided generalized* κ*-fractional integral operators of a function* δ_1 *in the sense of another function* γ *of order* $\beta>0$ *for* $\alpha>0$ *are stated as:*

$$\left({}^{\gamma}T^{\beta,\kappa}_{\phi^+,\tau}\delta_1\right)(\tau)=\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_\phi^\tau(\gamma(\tau)-\gamma(\pi))^{\frac{\beta}{\kappa}-1}.\gamma'(\pi)\,\delta_1(\pi)\,d\pi,\ \phi<\tau\tag{1.11}$$

and

$$\left({}^{\gamma}T^{\beta,\kappa}_{\rho^-,\tau}\delta_1\right)(\tau)=\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_\tau^\rho(\gamma(\pi)-\gamma(\tau))^{\frac{\beta}{\kappa}-1}.\gamma'(\pi)\,\delta_1(\pi)\,d\pi,\ \tau<\rho\tag{1.12}$$

where $\beta\in\mathbb{C}$, $\Re(\beta)>0$ *and* $\Gamma_\kappa(\beta)=\int_0^\infty\pi^{\beta-1}e^{-\frac{\pi^\kappa}{\kappa}}d\pi$ *is the* κ*-Gamma function in* [14].

Theorem 5. *Let two positive functions* δ_1,δ_2 *be defined on* $[0,\infty)$ *for* $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$ *and* $s\ge1$. γ *be an increasing and positive monotone function on* $[0,\infty)$ *and also* γ' *be continuous derivative on* $[0,\infty)$ *and* $\gamma(0)=0$. *If* $0<m\le\frac{\delta_1(\pi)}{\delta_2(\pi)}\le n$, $n,m\in\mathbb{R}^+$*and for all* $\pi\in[0,\tau]$, $\tau>0$, *then*

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}\le\theta_1.\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1+\delta_2)^s(\tau)\right)^{\frac{1}{s}}\tag{1.13}$$

with $\theta_1=\frac{n(1+m)+1+n}{(1+n)(1+m)}$, ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ *and* ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

Theorem 6. *Let two positive functions* δ_1,δ_2 *be defined on* $[0,\infty)$ *for* $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$ *and* $s\ge1$. γ *be an increasing and positive monotone function on* $[0,\infty)$ *and also*

γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$, $n,m\in\mathbb{R}^+$ and for all $\pi\in[0,\tau]$, $\tau>0$, then

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{2}{s}}+\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{2}{s}}\geq\theta_2\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \tag{1.14}$$

with $\theta_2=\frac{(1+n)(1+m)}{n}-2$, ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ and ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

Theorem 7. *Let two positive functions δ_1,δ_2 be defined on $[0,\infty)$ for $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$, $s,r\geq1$ and $\frac{1}{s}+\frac{1}{r}=1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$, $n,m\in\mathbb{R}^+$ and for all $\pi\in[0,\tau]$, $\tau>0$, then*

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2(\tau)\right)^{\frac{1}{r}}\leq\left(\frac{n}{m}\right)^{\frac{1}{sr}}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^{\frac{1}{s}}(\tau)\,\delta_2^{\frac{1}{r}}(\tau)\right) \tag{1.15}$$

with ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ and ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

2. REVERSE MINKOWSKI INEQUALITY VIA GENERALİZED κ-*CONFORMABLE* FRACTIONAL INTEGRALS

Under this heading, initially, we will show sufficiant consequences like the reverse Minkowski inequalities for generalized κ-Conformable fractional integral.

Theorem 8. *Let two positive functions δ_1,δ_2 be defined on $[0,\infty)$ for $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$ and $s\geq1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$, $n,m\in\mathbb{R}^+$ and for all $\pi\in[0,\tau]$, $\tau>0$, then*

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}\leq\theta_1.\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)+\delta_2(\tau)\right)^s\right)^{\frac{1}{s}} \tag{2.1}$$

with $\theta_1=\frac{n(1+m)+1+n}{(1+n)(1+m)}$, ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ and ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

Proof. By using $\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$ and $0\leq\pi\leq\tau$, we can say that

$$\begin{aligned}\tfrac{\delta_1(\pi)}{\delta_2(\pi)}\leq n\quad &\Longrightarrow \delta_1(\pi)\leq n\delta_2(\pi)\\ &\delta_1(\pi)\leq n\delta_2(\pi)+n\delta_1(\pi)-n\delta_1(\pi)\\ &\delta_1(\pi)(1+n)\leq n(\delta_1(\pi)+\delta_2(\pi))\\ &\delta_1^s(\pi)(1+n)^s\leq n^s(\delta_1(\pi)+\delta_2(\pi))^s.\end{aligned} \tag{2.2}$$

On the other side, by multiplying both sides of (2.2) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\frac{1}{\kappa\Gamma_{\kappa}(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned} &\frac{(1+n)^s}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)\delta_1^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\ &\quad\le\frac{n^s}{\kappa\Gamma_\kappa\left(\beta\right)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)(\delta_1(\pi)+\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{2.3}$$

In here, it can be written as the following inequality by using (1.6)

$$\begin{aligned} &(1+n)^s\,.^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_1^s\left(\tau\right)\le n^s.^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1\left(\tau\right)+\delta_2\left(\tau\right)\right)^s \\ &\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_1^s\left(\tau\right)\right)^{\frac{1}{s}}\le\left(\tfrac{n}{1+n}\right)\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1\left(\tau\right)+\delta_2\left(\tau\right)\right)^s\right)^{\frac{1}{s}}. \end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned} m\le\tfrac{\delta_1(\pi)}{\delta_2(\pi)}\quad &\Longrightarrow m\delta_2\left(\pi\right)\le\delta_1\left(\pi\right) \\ &\delta_2\left(\pi\right)\le\tfrac{1}{m}\delta_1\left(\pi\right) \\ &\delta_2\left(\pi\right)\le\tfrac{1}{m}\delta_1\left(\pi\right)+\tfrac{1}{m}\delta_2\left(\pi\right)-\tfrac{1}{m}\delta_2\left(\pi\right) \\ &\delta_2\left(\pi\right)\left(1+\tfrac{1}{m}\right)\le\tfrac{1}{m}\left(\delta_1\left(\pi\right)+\delta_2\left(\pi\right)\right) \\ &\delta_2^s\left(\pi\right)\left(1+\tfrac{1}{m}\right)^s\le\left(\tfrac{1}{m}\right)^s\left(\delta_1\left(\pi\right)+\delta_2\left(\pi\right)\right)^s. \end{aligned} \tag{2.5}$$

On the other part, by multiplying both sides of (2.5) by

$$\left[\frac{\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^\alpha-\left(\gamma\left(\pi\right)-\gamma\left(\phi\right)\right)^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'\left(\pi\right)}{\left(\gamma\left(\pi\right)-\gamma\left(\phi\right)\right)^{1-\alpha}}.\frac{1}{\kappa\Gamma_\kappa\left(\beta\right)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned} &\frac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)\delta_2^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\ &\quad\le\frac{\left(\frac{1}{1+m}\right)^s}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)(\delta_1(\pi)+\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{2.6}$$

In here, it can be written as the following inequality by using (1.6)

$$\begin{aligned} &{}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_2^s\left(\tau\right)\le\left(\tfrac{1}{1+m}\right)^s.^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1\left(\tau\right)+\delta_2\left(\tau\right)\right)^s \\ &\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_2^s\left(\tau\right)\right)^{\frac{1}{s}}\le\left(\tfrac{1}{1+m}\right)\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1\left(\tau\right)+\delta_2\left(\tau\right)\right)^s\right)^{\frac{1}{s}} \end{aligned} \tag{2.7}$$

Accordingly, we can write from (2.4) and (2.7)

$$\begin{aligned} &\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_1^s\left(\tau\right)\right)^{\frac{1}{s}}+\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_2^s\left(\tau\right)\right)^{\frac{1}{s}}\le\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1\left(\tau\right)+\delta_2\left(\tau\right)\right)^s\right)^{\frac{1}{s}}\left(\tfrac{n}{1+n}+\tfrac{1}{1+m}\right) \\ &\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_1^s\left(\tau\right)\right)^{\frac{1}{s}}+\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_2^s\left(\tau\right)\right)^{\frac{1}{s}}\le\theta_1\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1\left(\tau\right)+\delta_2\left(\tau\right)\right)^s\right)^{\frac{1}{s}}. \end{aligned} \tag{2.8}$$

In here, $\theta_1=\frac{n(1+m)+1+n}{(1+n)(1+m)}$. □

Theorem 9. *Let two positive functions δ_1, δ_2 be defined on $[0, \infty)$ for $\kappa > 0$, $\beta \in \mathbb{C}$, $\Re(\beta) > 0$ and $s \geq 1$. γ be an increasing and positive monotone function on $[0, \infty)$ and also γ' be continuous derivative on $[0, \infty)$ and $\gamma(0) = 0$. If $0 < m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n$, $n, m \in \mathbb{R}^+$ and for all $\pi \in [0, \tau]$, $\tau > 0$, then*

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{2}{s}} + \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{2}{s}} \geq \theta_2 \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}} \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \tag{2.9}$$

with $\theta_2 = \frac{(1+n)(1+m)}{n} - 2$, ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau) < \infty$ and ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau) < \infty$.

Proof. If we can write (2.4) and (2.7) inequalities,

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}} \leq \left(\frac{n}{1+n}\right)\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau) + \delta_2(\tau)\right)^s\right)^{\frac{1}{s}},$$

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \leq \left(\frac{1}{1+m}\right)\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau) + \delta_2(\tau)\right)^s\right)^{\frac{1}{s}}$$

we have. By multiplying (2.4) and (2.7) inequalities, we obtain as the following form,

$$\frac{(1+n)(1+m)}{n}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \leq \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\left(\delta_1(\tau) + \delta_2(\tau)\right)^s\right)\right)^{\frac{2}{s}}. \tag{2.10}$$

By the Minkowski inequality, we describe

$$\begin{aligned}&\frac{(1+n)(1+m)}{n}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \\ &\leq \left[\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}} + \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}\right]^2.\end{aligned} \tag{2.11}$$

In here, we clearly have the following inequality

$$\begin{aligned}&\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{2}{s}} + \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{2}{s}} \\ &\geq \left(\frac{(1+n)(1+m)}{n} - 2\right)\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}.\end{aligned} \tag{2.12}$$

□

3. Certain associated inequalities via the generalized κ-conformable fractional integral operator

In this part, we have obtained some inequalities by means of generalized $\kappa-$conformable fractional integrals. Furthermore we have proved their consequences. In addition, all functions are integrable in the Riemann sense.

Theorem 10. *Let two positive functions* δ_1, δ_2 *be defined on* $[0,\infty)$ *for* $\kappa > 0$, $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, $s, r \geq 1$ *and* $\frac{1}{s} + \frac{1}{r} = 1$. γ *be an increasing and positive monotone function on* $[0,\infty)$ *and also* γ' *be continuous derivative on* $[0,\infty)$ *and* $\gamma(0) = 0$. *If* $0 < m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n$, $n, m \in \mathbb{R}^+$ *and for all* $\pi \in [0,\tau]$, $\tau > 0$, *then*

$$\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_2(\tau)\right)^{\frac{1}{r}} \leq \left(\frac{n}{m}\right)^{\frac{1}{sr}}\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1^{\frac{1}{s}}(\tau)\,\delta_2^{\frac{1}{r}}(\tau)\right) \tag{3.1}$$

with ${}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1^s(\tau) < \infty$ *and* ${}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_2^s(\tau) < \infty$.

Proof. For $0 < m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n$, $m, n \in \mathbb{R}^+$, $\pi \in [0,\tau]$ and $\frac{1}{s} + \frac{1}{r} = 1$, we have

$$\begin{aligned}\frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n \quad &\Longrightarrow \delta_1(\pi) \leq n\delta_2(\pi)\\ &\delta_1^{\frac{1}{r}}(\pi) \leq n^{\frac{1}{r}}\delta_2^{\frac{1}{r}}(\pi)\\ &\delta_1^{\frac{1}{s}}(\pi)\,\delta_1^{\frac{1}{r}}(\pi) \leq n^{\frac{1}{r}}\delta_1^{\frac{1}{s}}(\pi)\,\delta_2^{\frac{1}{r}}(\pi)\\ &\delta_1(\pi) \leq n^{\frac{1}{r}}\delta_1^{\frac{1}{s}}(\pi)\,\delta_2^{\frac{1}{r}}(\pi).\end{aligned} \tag{3.2}$$

Additionally,by multiplying both sides of (3.2) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\cdot\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}&\frac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)\delta_1(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\ &\leq\frac{(n)^{\frac{1}{r}}}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)\delta_1^{\frac{1}{s}}(\pi).\delta_2^{\frac{1}{r}}(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\end{aligned} \tag{3.3}$$

In here, it can be written as the following inequality by using (1.6)

$$\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1(\tau)\right)^{\frac{1}{s}} \leq n^{\frac{1}{sr}}\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1^{\frac{1}{s}}(\tau)\,\delta_2^{\frac{1}{r}}(\tau)\right)^{\frac{1}{s}}. \tag{3.4}$$

Likewise, we get

$$\begin{aligned}m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \quad &\Longrightarrow m\delta_2(\pi) \leq \delta_1(\pi)\\ &\delta_2(\pi) \leq m^{-1}\delta_1(\pi)\\ &\delta_2^{\frac{1}{s}}(\pi) \leq m^{\frac{-1}{s}}\delta_1^{\frac{1}{s}}(\pi)\\ &\delta_2^{\frac{1}{r}}(\pi)\,\delta_2^{\frac{1}{s}}(\pi) \leq m^{-\frac{1}{s}}\delta_2^{\frac{1}{r}}(\pi)\,\delta_1^{\frac{1}{s}}(\pi)\\ &\delta_2(\pi) \leq m^{-\frac{1}{s}}\delta_2^{\frac{1}{r}}(\pi)\,\delta_1^{\frac{1}{s}}(\pi).\end{aligned} \tag{3.5}$$

On the other part, by multiplying both sides of (3.5) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\cdot\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}&\frac{1}{\kappa\Gamma_{\kappa}(\beta)}\int_{0}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)\delta_{2}(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\&\leq\frac{(m)^{-\frac{1}{s}}}{\kappa\Gamma_{\kappa}(\beta)}\int_{0}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)\delta_{1}^{\frac{1}{s}}(\pi).\delta_{2}^{\frac{1}{r}}(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\end{aligned}\tag{3.6}$$

In here, it can be written as the following inequality by using (1.6)

$$\left({}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{2}(\tau)\right)^{\frac{1}{r}}\leq m^{-\frac{1}{sr}}\left({}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{1}^{\frac{1}{s}}(\tau)\,\delta_{2}^{\frac{1}{r}}(\tau)\right)^{\frac{1}{r}}\tag{3.7}$$

In here we have multiply (3.4) and (3.7) for $\frac{1}{s}+\frac{1}{r}=1$. We obtain the following as

$$\left({}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{1}(\tau)\right)^{\frac{1}{s}}\left({}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{2}(\tau)\right)^{\frac{1}{r}}\leq\left(\frac{n}{m}\right)^{\frac{1}{sr}}\left({}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{1}^{\frac{1}{s}}(\tau)\,\delta_{2}^{\frac{1}{r}}(\tau)\right).\tag{3.8}$$

Proof is done. □

Theorem 11. *Let two positive functions δ_1,δ_2 be defined on $[0,\infty)$ for $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$, $s,r\geq 1$ and $\frac{1}{s}+\frac{1}{r}=1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$, $n,m\in\mathbb{R}^{+}$and for all $\pi\in[0,\tau]$, $\tau>0$, then*

$$\begin{aligned}{}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}(\delta_{1}(\tau).\delta_{2}(\tau))&\leq\theta_{3}.{}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}(\delta_{1}^{s}(\tau)+\delta_{2}^{s}(\tau))\\&+\theta_{4}.{}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}(\delta_{1}^{r}(\tau)+\delta_{2}^{r}(\tau))\end{aligned}\tag{3.9}$$

with $\theta_3=\frac{2^{s-1}}{s}\left(\frac{n}{n+1}\right)^{s}$, $\theta_4=\frac{2^{r-1}}{r}\left(\frac{1}{m+1}\right)^{r}$, ${}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{1}^{s}(\tau)<\infty$ and ${}^{\gamma}T_{0^{+},\tau}^{\beta,\kappa}\delta_{2}^{s}(\tau)<\infty$.

Proof. We can write (2.2) inequality which we obtain in the *theorem* 8 for $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$,

$$(n+1)^{s}\,\delta_{1}^{s}(\pi)\leq\quad n^{s}\,(\delta_{1}(\pi)+\delta_{2}(\pi))^{s}.\tag{3.10}$$

By multiplying both sides of (3.10) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\frac{1}{\kappa\Gamma_{\kappa}(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}&\frac{(n+1)^{s}}{\kappa\Gamma_{\kappa}(\beta)}\int_{0}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)\delta_{1}^{s}(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\&\leq\frac{(n)^{s}}{\kappa\Gamma_{\kappa}(\beta)}\int_{0}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)(\delta_{1}(\pi)+\delta_{2}(\pi))^{s}d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\end{aligned}\tag{3.11}$$

It can be written by using (1.6)

$$\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1^s(\tau)\right) \leq \left(\frac{n}{n+1}\right)^s \left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}(\delta_1(\tau)+\delta_2(\tau))^s\right). \tag{3.12}$$

Likewisely, We can write (2.5) inequality which we obtain in the *theorem* 8 for $0 < m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n$,

$$(m+1)^r \delta_2^r(\pi) \leq \quad (\delta_1(\pi)+\delta_2(\pi))^r. \tag{3.13}$$

Additionlly, by multiplying both sides of (3.13) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha - (\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \cdot \frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}
&\frac{(m+1)^r}{\kappa\Gamma_\kappa(\beta)} \int_0^\tau \left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)\delta_2^r(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\
&\quad \leq \frac{1}{\kappa\Gamma_\kappa(\beta)} \int_0^\tau \left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)(\delta_1(\pi)+\delta_2(\pi))^r d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.
\end{aligned} \tag{3.14}$$

It can be written as the following inequalities by using (1.6)

$$\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_2^r(\tau)\right) \leq \frac{1}{(m+1)^r}\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}(\delta_1(\tau)+\delta_2(\tau))^r\right) \tag{3.15}$$

by using Young's inequality,

$$\delta_1(\pi).\delta_2(\pi) \leq \frac{\delta_1^s(\pi)}{s} + \frac{\delta_2^r(\pi)}{r}, \quad \text{for } \frac{1}{s}+\frac{1}{r}=1. \tag{3.16}$$

Otherwise, by multiplying both sides of (3.16) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha - (\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \cdot \frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$${}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}(\delta_1(\tau).\delta_2(\tau)) \leq \frac{{}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1^s(\tau)}{s} + \frac{{}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_2^r(\tau)}{r}. \tag{3.17}$$

We have the following by using (3.12), (3.15) and (3.17).

$$\begin{aligned}
{}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}(\delta_1(\tau).\delta_2(\tau)) &\leq \frac{{}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_1^s(\tau)}{s} + \frac{{}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}\delta_2^r(\tau)}{r} \\
&\leq \frac{1}{s}\left(\frac{n}{n+1}\right)^s\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}(\delta_1(\tau)+\delta_2(\tau))^s\right) \\
&\quad + \frac{1}{r}\left(\frac{1}{m+1}\right)^r\left({}^{\gamma}T_{0^+,\tau}^{\beta,\kappa}(\delta_1(\tau)+\delta_2(\tau))^r\right).
\end{aligned} \tag{3.18}$$

In here by using $(u+v)^z \le 2^{z-1}(u^z+v^z)$, $z>1$ and $u,v>0$, we have

$$\begin{aligned} {}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)+\delta_2(\tau)\right)^s &\le 2^{s-1}.{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1^s(\tau)+\delta_2^s(\tau)\right), \\ {}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)+\delta_2(\tau)\right)^r &\le 2^{r-1}.{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1^r(\tau)+\delta_2^r(\tau)\right). \end{aligned} \tag{3.19}$$

We obtain the following inequalities by using (3.18) and (3.19),

$$\begin{aligned} {}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1(\tau)\delta_2(\tau) &\le \theta_3.{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1^s(\tau)+\delta_2^s(\tau)\right) \\ &\quad +\theta_4.{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1^r(\tau)+\delta_2^r(\tau)\right). \end{aligned} \tag{3.20}$$

Proof is done with $\theta_3 = \frac{2^{s-1}}{s}\left(\frac{n}{n+1}\right)^s$ and $\theta_4 = \frac{2^{r-1}}{r}\left(\frac{1}{m+1}\right)^r$. □

Theorem 12. *Let two positive functions δ_1, δ_2 be defined on $[0,\infty)$ for $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$, $s,r\ge 1$ and $\frac{1}{s}+\frac{1}{r}=1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\le\frac{\delta_1(\pi)}{\delta_2(\pi)}\le n$, $n,m\in\mathbb{R}^+$and for all $\pi\in[0,\tau]$, $\tau>0$, then*

$$\begin{aligned} &\tfrac{n+1}{n-p}\left[{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}} \\ &\quad\le \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}} + \left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \\ &\quad\le \tfrac{m+1}{m-p}\left[{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}. \end{aligned} \tag{3.21}$$

with ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ and ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

Proof. For $0<m\le\frac{\delta_1(\pi)}{\delta_2(\pi)}\le n$,

$$\begin{aligned} &m-p \le \tfrac{\delta_1(\pi)}{\delta_2(\pi)} - p \le n-p \\ &m-p \le \tfrac{\delta_1(\pi)-p\delta_2(\pi)}{\delta_2(\pi)} \le n-p \\ &\tfrac{1}{n-p} \le \tfrac{\delta_2(\pi)}{\delta_1(\pi)-p\delta_2(\pi)} \le \tfrac{1}{m-p}, \end{aligned} \tag{3.22}$$

and

$$\tfrac{(\delta_1(\pi)-p\delta_2(\pi))^s}{(n-p)^s} \le \delta_2^s(\pi) \le \tfrac{(\delta_1(\pi)-p\delta_2(\pi))^s}{(m-p)^s}. \tag{3.23}$$

Likely,

$$\begin{aligned} &\tfrac{1}{n} \le \tfrac{\delta_2(\pi)}{\delta_1(\pi)} \le \tfrac{1}{m} \\ &1-\tfrac{p}{m} \le 1-\tfrac{p\delta_2(\pi)}{\delta_1(\pi)} \le 1-\tfrac{p}{n} \\ &\tfrac{m-p}{m} \le \tfrac{\delta_1(\pi)-p\delta_2(\pi)}{\delta_1(\pi)} \le \tfrac{n-p}{n} \\ &\tfrac{n}{n-p} \le \tfrac{\delta_1(\pi)}{\delta_1(\pi)-p\delta_2(\pi)} \le \tfrac{m}{m-p}, \end{aligned} \tag{3.24}$$

and

$$\frac{n^s}{(n-p)^s}\left(\delta_1(\pi)-p\delta_2(\pi)\right)^s \leq \delta_1^s(\pi) \leq \frac{m^s}{(m-p)^s}\left(\delta_1(\pi)-p\delta_2(\pi)\right)^s. \tag{3.25}$$

Otherwise, by multiplying both sides of (3.23) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\cdot\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we can say that

$$\begin{aligned}
&\frac{1}{(n-p)^s\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)(\delta_1(\pi)-p\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\
&\leq \frac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)\delta_2^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\
&\leq \frac{1}{(m-p)^s\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)(\delta_1(\pi)-p\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.
\end{aligned} \tag{3.26}$$

We obtain from (3.23) by using (1.6)

$$\begin{aligned}
&\left[{}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}\frac{1}{n-p}\\
&\leq \left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_2^s(\tau)\right)^{\frac{1}{s}}\\
&\leq \left[{}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}\frac{1}{m-p}.
\end{aligned} \tag{3.27}$$

In the same way, if we have multiply both sides of (3.25) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\cdot\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}
&\frac{n^s}{(n-p)^s\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)(\delta_1(\pi)-p\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\
&\leq \frac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)\delta_1^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\
&\leq \frac{m^s}{(m-p)^s\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)(\delta_1(\pi)-p\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}
\end{aligned} \tag{3.28}$$

We have from (3.25) by using (1.6)

$$\begin{aligned}
&\left[{}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}\frac{n}{n-p}\\
&\leq \left({}^\gamma T_{0^+,\tau}^{\beta,\kappa}\delta_1^s(\tau)\right)^{\frac{1}{s}}\\
&\leq \left[{}^\gamma T_{0^+,\tau}^{\beta,\kappa}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}\frac{m}{m-p}.
\end{aligned} \tag{3.29}$$

On the other side, if we gather side by side both of (3.27) and (3.29) , we have

$$\begin{aligned}\tfrac{n+1}{n-p}&\left[{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}\\&\le\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}\\&\le\tfrac{m+1}{m-p}\left[{}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)-p\delta_2(\tau)\right)^s\right]^{\frac{1}{s}}.\end{aligned}\tag{3.30}$$

Proof is done. □

Theorem 13. *Let two positive functions δ_1,δ_2 be defined on $[0,\infty)$ for $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$, $s,r\ge1$ and $\frac{1}{s}+\frac{1}{r}=1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\le\delta_1(\pi)\le k$ and $0<n\le\delta_2(\pi)\le c$, $n,m\in\mathbb{R}^+$ and for all $\pi\in[0,\tau]$, $\tau>0$, then*

$$\begin{aligned}&\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}\\&\qquad\le\theta_5\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\left(\delta_1(\tau)+\delta_2(\tau)\right)^s\right)^{\frac{1}{s}}.\end{aligned}\tag{3.31}$$

In here we have inequalities with $\theta_5=\frac{c(k+n)+k(c+m)}{(c+m)(k+n)}$, ${}^{\gamma}\pi^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ and ${}^{\gamma}\pi^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

Proof. For $0<m\le\delta_1(\pi)\le k$ and $0<n\le\delta_2(\pi)\le c$, we obtain

$$\begin{aligned}&\tfrac{1}{c}\le\tfrac{1}{\delta_2(\pi)}\le\tfrac{1}{n}\\&\tfrac{m}{c}\le\tfrac{\delta_1(\pi)}{\delta_2(\pi)}\le\tfrac{k}{n}\\&\tfrac{m+c}{c}\le\tfrac{\delta_1(\pi)+\delta_2(\pi)}{\delta_2(\pi)}\le\tfrac{k+n}{n}\\&\delta_2^s(\pi)\le\left(\tfrac{c}{c+m}\right)^s\left(\delta_1(\pi)+\delta_2(\pi)\right)^s.\end{aligned}\tag{3.32}$$

Similarly,

$$\begin{aligned}&\tfrac{n}{k}\le\tfrac{\delta_2(\pi)}{\delta_1(\pi)}\le\tfrac{c}{m}\\&\tfrac{n+k}{k}\le\tfrac{\delta_1(\pi)+\delta_2(\pi)}{\delta_1(\pi)}\le\tfrac{m+c}{m}\\&\delta_1^s(\pi)\le\left(\tfrac{k}{k+n}\right)^s\left(\delta_1(\pi)+\delta_2(\pi)\right)^s.\end{aligned}\tag{3.33}$$

On the other part, by multiplying both sides of (3.32) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\cdot\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}&\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\tfrac{\gamma'(\pi)\delta_2^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\&\quad\le\tfrac{c^s}{(c+m)^s\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\tfrac{\gamma'(\pi)(\delta_1(\pi)+\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\end{aligned}\tag{3.34}$$

We can written as the following by using (1.6)

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \leq \tfrac{c}{c+m}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau)+\delta_2(\tau))^s\right)^{\frac{1}{s}}. \tag{3.35}$$

On the other hand, by multiplying both sides of (3.33) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\cdot\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned}&\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\tfrac{\gamma'(\pi)\delta_1^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}\\ &\leq \tfrac{k^s}{(k+n)^s\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}\cdot\tfrac{\gamma'(\pi)(\delta_1(\pi)+\delta_2(\pi))^s d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\end{aligned} \tag{3.36}$$

We can write as the following by using (1.6)

$$\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}} \leq \tfrac{k}{k+n}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau)+\delta_2(\tau))^s\right)^{\frac{1}{s}}. \tag{3.37}$$

Furthermore, if we gather side by side both of (3.35) and (3.37) ,we have

$$\begin{aligned}&\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}}\\ &\qquad\leq\theta_5\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau)+\delta_2(\tau))^s\right)^{\frac{1}{s}}\end{aligned} \tag{3.38}$$

Proof is done with $\theta_5=\frac{c(k+n)+k(c+m)}{(c+m)(k+n)}$. □

Theorem 14. *Let two positive functions δ_1,δ_2 be defined on $[0,\infty)$ for $\kappa>0$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$, and $s\geq 1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0)=0$. If $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$, $n,m\in\mathbb{R}^+$ and for all $\pi\in[0,\tau]$, $\tau>0$, then*

$$\begin{aligned}\tfrac{1}{n}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1(\tau)\delta_2(\tau)\right) &\leq \tfrac{1}{(m+1)(n+1)}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau)+\delta_2(\tau))^2\right)\\ &\leq\tfrac{1}{m}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1(\tau)\delta_2(\tau)\right).\end{aligned} \tag{3.39}$$

with ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)<\infty$ and ${}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)<\infty$.

Proof. By using $0<m\leq\frac{\delta_1(\pi)}{\delta_2(\pi)}\leq n$, we can write the following inequalities,

$$\begin{aligned}&m+1\leq\tfrac{\delta_1(\pi)}{\delta_2(\pi)}+1\leq n+1\\ &(m+1)\delta_2(\pi)\leq\delta_1(\pi)+\delta_2(\pi)\leq(n+1)\delta_2(\pi)\end{aligned} \tag{3.40}$$

and

$$\begin{aligned} 1+\tfrac{1}{n} &\leq 1+\tfrac{\delta_2(\pi)}{\delta_1(\pi)} \leq 1+\tfrac{1}{m} \\ \left(\tfrac{n+1}{n}\right)\delta_1(\pi) &\leq \delta_1(\pi)+\delta_2(\pi) \leq \left(\tfrac{m+1}{m}\right)\delta_1(\pi). \end{aligned} \tag{3.41}$$

If we multiply side by side both of (3.41) and (3.42) , we have

$$\begin{aligned} &\tfrac{(m+1)(n+1)}{n}\delta_1(\pi)\,\delta_2(\pi) \leq (\delta_1(\pi)+\delta_2(\pi))^2 \leq \tfrac{(m+1)(n+1)}{m}\delta_1(\pi)\,\delta_2(\pi) \\ &\tfrac{\delta_1(\pi)\delta_2(\pi)}{n} \leq \tfrac{(\delta_1(\pi)+\delta_2(\pi))^2}{(m+1)(n+1)} \leq \tfrac{\delta_1(\pi)\delta_2(\pi)}{m}. \end{aligned} \tag{3.42}$$

On the other side, by multiplying both sides of (3.43) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}.\frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned} &\left(\tfrac{1}{n}\right)\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\tfrac{\gamma'(\pi).\delta_1(\pi).\delta_2(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\ &\quad\leq \left(\tfrac{1}{(m+1)(n+1)}\right)\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\tfrac{\gamma'(\pi).(\delta_1(\pi)+\delta_2(\pi))^2d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\ &\quad\leq \left(\tfrac{1}{m}\right)\tfrac{1}{\kappa\Gamma_\kappa(\beta)}\int_0^\tau\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\pi)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\frac{\beta}{\kappa}-1}.\tfrac{\gamma'(\pi).(\delta_1(\pi).\delta_2(\pi))d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{3.43}$$

we can write by using (1.6)

$$\begin{aligned} &\tfrac{1}{n}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau).\delta_2(\tau))\right) \\ &\qquad\leq \tfrac{1}{(m+1)(n+1)}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau)+\delta_2(\tau))^2\right) \\ &\qquad\leq \tfrac{1}{m}\left({}^{\gamma}T^{\beta,\kappa}_{0^+,\tau}(\delta_1(\tau).\delta_2(\tau))\right). \end{aligned} \tag{3.44}$$

Proof is done. □

Theorem 15. *Let two positive functions δ_1, δ_2 be defined on $[0,\infty)$ for $\kappa > 0$, $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, and $s \geq 1$. γ be an increasing and positive monotone function on $[0,\infty)$ and also γ' be continuous derivative on $[0,\infty)$ and $\gamma(0) = 0$. If $0 < m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n$, $n, m \in \mathbb{R}^+$and for all $\pi \in [0,\tau]$, $\tau > 0$, then*

$$\begin{aligned} &\left({}^{h}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau)\right)^{\frac{1}{s}}+\left({}^{h}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau)\right)^{\frac{1}{s}} \\ &\qquad\leq 2\left({}^{h}T^{\beta,\kappa}_{0^+,\tau}H^s(\delta_1(\tau),\delta_2(\tau))\right)^{\frac{1}{s}}, \end{aligned} \tag{3.45}$$

where $(H(\delta_1(\pi),\delta_2(\pi))) = \max\left\{n\left(\frac{n}{m}+1\right)\delta_1(\pi)-n\delta_2(\pi), \frac{(n+m)\delta_2(\pi)-\delta_1(\pi)}{m}\right\}$ with ${}^{h}T^{\beta,\kappa}_{0^+,\tau}\delta_1^s(\tau) < \infty$ and ${}^{h}T^{\beta,\kappa}_{0^+,\tau}\delta_2^s(\tau) < \infty$.

Proof. By using $0 < m \leq \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n$ and $0 \leq \pi \leq \tau$, we have

$$0 < m \leq n + m - \frac{\delta_1(\pi)}{\delta_2(\pi)} \quad \text{and} \quad n + m - \frac{\delta_1(\pi)}{\delta_2(\pi)} \leq n.$$

In addition, we write

$$\begin{aligned} & m\delta_2(\pi) \leq (n+m)\,\delta_1(\pi) - \delta_2(\pi) \\ & \delta_2(\pi) \leq \frac{(n+m)\delta_2(\pi) - \delta_1(\pi)}{m} \leq H(\delta_1(\pi), \delta_2(\pi)). \end{aligned} \tag{3.46}$$

In here, we can determine for $H(\delta_1(\pi), \delta_2(\pi))$

$$(H(\delta_1(\pi), \delta_2(\pi))) = \max\left\{ n\left(\frac{n}{m} + 1\right)\delta_1(\pi) - n\delta_2(\pi), \frac{(n+m)\delta_2(\pi) - \delta_1(\pi)}{m} \right\}. \tag{3.47}$$

Similarly

$$\frac{1}{n} \leq \frac{1}{n} + \frac{1}{m} - \frac{\delta_2(\pi)}{\delta_1(\pi)} \quad \text{and} \quad \frac{1}{n} + \frac{1}{m} - \frac{\delta_2(\pi)}{\delta_1(\pi)} \leq \frac{1}{m} \tag{3.48}$$

for this inequalities, we have

$$\begin{aligned} & \frac{\delta_1(\pi)}{n} \leq \left(\frac{1}{n} + \frac{1}{m}\right)\delta_1(\pi) - \delta_2(\pi) \\ & \delta_1(\pi) \leq \frac{n(n+m)\delta_1(\pi) - n^2 m\delta_2(\pi)}{n.m} \\ & \delta_1(\pi) \leq \left(\frac{n}{m} + 1\right)\delta_1(\pi) - n\delta_2(\pi) \\ & \delta_1(\pi) \leq n\left[\left(\frac{n}{m} + 1\right)\delta_1(\pi) - n\delta_2(\pi)\right] \\ & \delta_1(\pi) \leq H(\delta_1(\pi), \delta_2(\pi)). \end{aligned} \tag{3.49}$$

We can write from (3.47) and (3.50)

$$\delta_1^s(\pi) \leq H^s(\delta_1(\pi), \delta_2(\pi)), \tag{3.50}$$

and

$$\delta_2^s(\pi) \leq H^s(\delta_1(\pi), \delta_2(\pi)). \tag{3.51}$$

Further, by multiplying both sides of (3.51) by

$$\left[\frac{(\gamma(\tau) - \gamma(\phi))^\alpha - (\gamma(\pi) - \gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa} - 1} \cdot \frac{\gamma'(\pi)}{(\gamma(\pi) - \gamma(\phi))^{1-\alpha}} \cdot \frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain as the following

$$\begin{aligned} & \frac{1}{\kappa\Gamma_\kappa(\beta)} \int_0^\tau \left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha - (\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)\delta_1^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\ & \leq \frac{1}{\kappa\Gamma_\kappa(\beta)} \int_0^\tau \left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha - (\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)H^s(\delta_1(\pi),\delta_2(\pi))d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{3.52}$$

We have by using (1.6)

$$\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} \delta_1^s(\tau)\right)^{\frac{1}{s}} \leq \left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} H^s(\delta_1(\tau), \delta_2(\tau))\right)^{\frac{1}{s}}. \tag{3.53}$$

Likewise, by applying same method for the following inequality

$$\delta_2^s(\pi) \leq H^s(\delta_1(\pi), \delta_2(\pi)). \tag{3.54}$$

Beside, by multiplying both sides of (3.54) by

$$\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \frac{\gamma'(\pi)}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \cdot \frac{1}{\kappa\Gamma_\kappa(\beta)}$$

and integrate from 0 to τ, we obtain

$$\begin{aligned} &\tfrac{1}{\kappa\Gamma_\kappa(\beta)} \int_0^\tau \left[\tfrac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \tfrac{\gamma'(\pi)\delta_2^s(\pi)d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}} \\ &\leq \tfrac{1}{\kappa\Gamma_\kappa(\beta)} \int_0^\tau \left[\tfrac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\pi)-\gamma(\phi))^\alpha}{\alpha}\right]^{\frac{\beta}{\kappa}-1} \cdot \tfrac{\gamma'(\pi)H^s(\delta_1(\pi),\delta_2(\pi))d\pi}{(\gamma(\pi)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{3.55}$$

We have by using (1.6)

$$\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} \delta_2^s(\tau)\right)^{\frac{1}{s}} \leq \left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} H^s(\delta_1(\tau), \delta_2(\tau))\right)^{\frac{1}{s}}. \tag{3.56}$$

If we gather side by side both of (3.54) and (3.57) ,we obtain

$$\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} \delta_1^s(\tau)\right)^{\frac{1}{s}} + \left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} \delta_2^s(\tau)\right)^{\frac{1}{s}} \quad \leq 2\left({}^\gamma T_{0^+,\tau}^{\beta,\kappa} H^s(\delta_1(\tau), \delta_2(\tau))\right)^{\frac{1}{s}} \tag{3.57}$$

Proof is done. □

4. Conclusion

In this study, we have defined the left and right generalized $\kappa-$conformable fractional integral operators depend on a parameter $\kappa > 0$. Furthermore, we have demonstrated their the classical consequences under some special elections. In section 3, we have demonstrated reverse Minkowski inequality for generalized $\kappa-$conformable fractional integrals. Meanwhile, we have expressed and proved certain associated inequalities for generalized $\kappa-$conformable integrals in section 4.

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