Journal of Contemporary Applied Mathematics
V. 14, No 1, 2024, July
ISSN 2222-5498
DOI 10.5281/zenodo.10465123

Transformation Operators for the Schrödinger Equation with a Linearly Increasing Potential

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Abstract. The Schrödinger equation with a linearly increasing potential is considered. Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.

Key Words and Phrases: Schrödinger equation, transformation operator, Airy functions, Heaviside function.

2010 Mathematics Subject Classifications: Primary 34A55, 34B20, 34L05

1. Introduction and main result

In many aspects of the theory of inverse problems of spectral analysis, an important role is played by so-called transformation operators (see [1]-[10] and the references therein). These operators arose from the general ideas of the theory of generalized shift operators created by Delsarte [1]. For arbitrary Sturm–Liouville equations, transformation operators were constructed by Povzner [2]. Marchenko [3] used transformation operators for studying inverse spectral problems and the asymptotic behavior of the spectral function of the singular Sturm–Liouville operator. Levin [4] introduced transformation operators of a new form that preserve the asymptotic expansions of solutions at infinity. Marchenko [3] used them to solve the inverse problem of scattering theory. Similar problems for the Schrodinger equation with unbounded potentials were considered in [5]-[9].

We consider the differential equation

$$-y'' + \theta(x)\, xy + q(x)\, y = \lambda y,\ -\infty < x < \infty,\quad \lambda \in C, \tag{1}$$

where $\theta(x)$ is Heaviside function, i.e.

$$\theta(x) = \begin{cases} 1, x \geq 0, \\ 0, x < 0 \end{cases}$$

and the real potential $q(x)$ satisfies the conditions

$$q(x) \in C(-\infty, +\infty)\,, \int\limits_{-\infty}^{\infty} |xq(x)|\, dx < \infty. \tag{2}$$

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In the present paper, using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. The results obtained can be used to solve inverse spectral problems for an equation (1). Note that for $x \geq 0$, equation (1) turns into the one-dimensional Stark equation. Some questions of the spectral theory of the one-dimensional Stark equation were studied in [10]- [12].

2. The transformation operators

In what follows, we deal with special functions satisfying the Airy equation

$$-y'' + zy = 0.$$

It is well known (e.g., see [13]) that this equation has two linearly independent solutions $Ai\,(z)$ and $Bi\,(z)$ with the initial conditions

$$Ai\,(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)},\ Ai'\,(0) = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)},$$

$$Bi\,(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)},\ Bi'\,(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}.$$

where $\Gamma\,(\cdot)$ is Euler's Gamma function. The Wronskian $\{Ai\,(z)\,, Bi\,(z)\}$ of these functions satisfies

$$\{Ai\,(z)\,, Bi\,(z)\} = Ai\,(z)\,Bi'\,(z) - Ai'\,(z)\,Bi\,(z) = \pi^{-1}.$$

Both functions are entire functions of order $\frac{3}{2}$ and type $\frac{2}{3}$. We have (see [13]) asymptotic equalities for $|z| \to \infty$

$$\mathrm{A}i\,(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}\left[1 + O\left(\zeta^{-1}\right)\right],\ |\arg z| < \pi$$

$$\mathrm{A}i\,(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin\left(\zeta + \frac{\pi}{4}\right)\left[1 + O\left(\zeta^{-1}\right)\right],\ |\arg z| < \frac{2\pi}{3},$$

$$Bi\,(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\zeta}\left[1 + O\left(\zeta^{-1}\right)\right],\ |\arg z| < \frac{\pi}{3},$$

$$Bi\,(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \cos\left(\zeta + \frac{\pi}{4}\right)\left[1 + O\left(\zeta^{-1}\right)\right],\ |\arg z| < \frac{2\pi}{3}.$$

where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$. In what follows we will need special solutions of the unperturbed equation

$$-y'' + \theta\,(x)\,xy = \lambda y,\ -\infty < x < \infty,\quad \lambda \in C. \tag{3}$$

Let us denote by G the complex λ- plane with a cut along the positive semi-axis. Let us consider a function $\sqrt{\lambda}$ in the plane G, choosing a regular branch of the radical such that $\sqrt{\lambda + i0} > 0$ for $\lambda > 0$.

Theorem 2.1. *For any λ from the complex plane, equation (3) has solutions $\psi_{\pm}\,(x, \lambda)$ in the form*

$$\psi_+(x,\lambda)=\begin{cases} Ai\,(x-\lambda)\,, x\ge 0, \\ \frac{1}{2}\left[Ai\,(-\lambda)+\frac{1}{i\sqrt{\lambda}}Ai'\,(-\lambda)\right]e^{i\sqrt{\lambda}x}+ \\ +\frac{1}{2}\left[Ai\,(-\lambda)-\frac{1}{i\sqrt{\lambda}}Ai'\,(-\lambda)\right]e^{-i\sqrt{\lambda}x}, x<0, \end{cases} \tag{4}$$

$$\psi_-(x,\lambda)=\begin{cases} \pi\left[Bi'\,(-\lambda)-i\sqrt{\lambda}Bi\,(-\lambda)\right]Ai\,(x-\lambda)+ \\ +\pi\left[Bi'\,(-\lambda)\,i\sqrt{\lambda}Ai\,(-\lambda)-Ai'\,(-\lambda)\right]Bi\,(x-\lambda)\,, x\ge 0, \\ e^{-i\sqrt{\lambda}x}, x<0. \end{cases} \tag{5}$$

Proof. Obviously, when $x \ge 0$ one of the solutions of equation (3) is function $Ai\,(x-\lambda)$. On the other hand, for $x \le 0$ any solution of equation (3) can be represented as

$$\alpha e^{i\sqrt{\lambda}x}+\beta\, e^{-i\sqrt{\lambda}x}.$$

If we glue these solutions at a point$x=0$, we get

$$\begin{cases} \alpha+\beta\ = Ai\,(-\lambda)\,, \\ i\sqrt{\lambda}\alpha-i\sqrt{\lambda}\,\beta = Ai'\,(-\lambda)\,. \end{cases}$$

Solving the last system of equations for α, β we find that

$$\alpha=\frac{1}{2}\left[Ai\,(-\lambda)+\frac{1}{i\sqrt{\lambda}}Ai'\,(-\lambda)\right], \beta=\frac{1}{2}\left[Ai\,(-\lambda)-\frac{1}{i\sqrt{\lambda}}Ai'\,(-\lambda)\right].$$

Thus, formula (4) is established. Formula (5) is derived similarly. □

The theorem is proved.

Obviously, for any $\lambda \in G$ the function $\psi_+(x,\lambda)$ belongs to $L_2(0,+\infty)$ and the function $\psi_-(x,\lambda)$ belongs to$L_2(-\infty,0)$.

We shall use the following notation

$$\sigma_\pm(x)=\pm\int_x^{\pm\infty}\left|\theta\,(t)\,t-\frac{1\pm 1}{2}t+q\,(t)\right|dt.$$

In the following theorem the representation of solution from the equation (1) is found by means of transformation operator.

Theorem 2.2. *If the potential $q\,(x)$satisfies conditions (2), then for any λ from the plane G equation (1) has a solution $f_+(x,\lambda)$ that can be represented in the form*

$$f_+(x,\lambda)=\psi_+(x,\lambda)+\int_x^\infty K_+(x,t)\ \psi_+(t,\lambda)\,dt, \tag{6}$$

where kernel $K_+(x,t)$ is continuous function and satisfies relations

$$K_+(x,t)=O\left(\sigma_+\left(\frac{x+t}{2}\right)\right), x+t\to\infty, \tag{7}$$

$$K_+(x,x)=\frac{1}{2}\int_x^\infty\left[\theta\,(t)\,t-t+q\ (t)\right]dt. \tag{8}$$

Proof. We rewrite the perturbed equation (1) in the form

$$-y'' + xy + Q(x)\,y = \lambda y, \;\; -\infty < x < \infty. \tag{9}$$

where $Q(x) = [\theta(x) - 1]\,x + q(x)$. Obviously, the $Q(x)$ function for all $x > a, a > -\infty$ satisfies the condition

$$\int_a^\infty |xQ(x)|\,dx < \infty. \tag{10}$$

Let $f_+(x,\lambda)$ be solution of equation (9) with the asymptotic behavior$f_+(x,\lambda) = f_0(x,\lambda)(1 + o(1)),\, x \to \infty$, where $f_0(x,\lambda) = Ai(x - \lambda)$. Subject to the conditions (10), such solution exist, is determined uniquely by its asymptotic behavior (see [9]). With the aid of operator transformations, we have the representation

$$f_+(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K(x,t)\;\; f_0(t,\lambda)\,dt, \tag{11}$$

Moreover, the kernel $K(x,t)$ is a continuous function and satisfies the following relations

$$K(x,t) = O\left(\sigma_+\left(\frac{x+t}{2}\right)\right), x + t \to \infty, \tag{12}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty Q\;(t)\;dt. \tag{13}$$

In addition, rewriting the unperturbed equation (3) in the form

$$-y'' + xy + Q_0(x)\,y = \lambda y, \;\; -\infty < x < \infty.$$

where $Q_0(x) = [\theta(x) - 1]\,x$, we obtain

$$\psi_+(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K_0(x,t)\;\; f_0(t,\lambda)\,dt.$$

Moreover, in this case, $K_0(x,t)$ satisfies the identity$K_0(x,t) \equiv, x \geq 0$. From the well-known properties of the transformation operators it follows that (see [3]) the function $f_0(x,\lambda)$ also admits the representation

$$f_0(x,\lambda) = \;\psi_+(x,\lambda) + \int_x^\infty \tilde{K}_0(x,t)\;\;\psi_+(t,\lambda)\,dt\,, \tag{14}$$

where the kernels $K_0(x,t)\,, \tilde{K}_0(x,t)$ are connected by the equality

$$K_0(x,t) + \tilde{K}_0(x,t) + \int_x^t \tilde{K}_0(x,u)\;\;K_0(u,t)\,du = 0. \tag{15}$$

Substituting the expression (14) from the $f_0(x,\lambda)$ in (11), we get

$$f_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^\infty K(x,t)\left[\psi_+(t,\lambda) + \int_t^\infty \tilde{K}_0(t,u)\,\psi_+(u,\lambda)\,du\right]\;dt =$$

$$= \psi_+(x,\lambda) + \int_x^\infty K(x,t)\,\psi_+(t,\lambda)\,dt + \int_x^\infty K(x,t)\int_t^\infty \tilde{K}_0(t,u)\,\psi_+(u,\lambda)\,dudt =$$

$$= \psi_+(x,\lambda) + \int_x^\infty K(x,t)\,\psi_+(t,\lambda)\,dt + \int_x^\infty \left(\int_x^t K(x,u)\,\tilde{K}_0(u,t)\,du\right)\psi_+(t,\lambda)\,dt.$$

Setting

$$K_+(x,t) = K(x,t) + \int_x^t K(x,u)\;\; \tilde{K}_0(u,t)\,du, \tag{16}$$

one can recast the last relation in the form

$$f_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^\infty K_+(x,t)\;\;\psi_+(t,\lambda)\,dt.$$

Formula (15) is a straightforward consequence of (12), (15), (16). Taking $t = x$ in the equality (16), we find that $K_+(x,t) = K(x,t)$. Whence, by virtue of (13), formula (8) follows. □

The theorem is proved.

The following theorem is proved in a similar way.

Theorem 2.3. *If the potential* (x) *satisfies condition* (2), *then, for all values of* λ *of the plane* G, *equation* (1) *has a solution* $f_-(x,\lambda)$ *representable as*

$$f_-(x,\lambda) = \psi_-(x,\lambda) + \int_{-\infty}^x K_-(x,t)\;\;\psi_-(t,\lambda)\,dt.$$

where the kernel $K_-(x,t)$ *is continuous function and satisfy the following conditions*

$$K_-(x,t) = O\left(\sigma_-\left(\frac{x+t}{2}\right)\right), x+t \to -\infty, K_-(x,x) = \frac{1}{2}\int_{-\infty}^x \left[\theta(t)\,t + q\;(t)\right]\,dt.$$

References

[1] J. Delsarte ,*Sur une extension de la formule de Taylor*, J.Math. Pures Appl.1938, 17, 213–231 .

[2] A. Ya.Povzner , *On differential equations of Sturm–Liouville type on a half-axis*, Mat. Sb. 1948, 23 no. 1, 3–52.

[3] V. A. Marchenko, *Sturm–Liouville Operators and Their Applications,* Naukova Dumka, Kiev, (1977) [in Russian].

[4] B. Ya. Levin, *Transformation of Fourier and Laplace type by means of solutions of a differential equation of second order*, Dokl. Akad. Nauk SSSR 1956, 106 no.2, 187–190.

[5] M. G. Gasymov and B. A. Mustafaev, *Inverse scattering problem for an anharmonic equation on the semiaxis,* Dokl. Akad. Nauk SSSR 1976, 228, no.1, 11–14.

[6] G.M. Masmaliev, A.Kh. Khanmamedov, *Transformation Operators for Perturbed Harmonic Oscillators, Mathematical Notes*, 2019, 105, no.5, 728–733.

[7] Li.Yishen, *One special inverse problem of the second-order differential equation for the whole real axis,* Chinese Ann. Math. 1981, 2 no.2, 147–155.

[8] A.Kh. Khanmamedov, D.HOrudjov, *Triangular Representation of the Solution to the Schrodinger Equation with an Additional Linear Potential,* Azerbaijan Journal of Mathematics, 2023, 13, no 2, 100-109.

[9] D.H.Orudjov, *On the transformation operator for the Schrodinger equation with an additional increasing potential,* Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 2022, 48, no 2, 311–322.

[10] A. M. Savchuk, A. A.Shkalikov, *Spectral Properties of the Complex Airy Operator on the Half-Line , Functional Analysis and Its Applications,* 2017, 51, no.1, 66–79 .

[11] E.L.Korotyaev, *Resonances for 1D Stark operators*, Journal Spectral Theory, 2017, 7, no.3, 633-658.

[12] E.L. Korotyaev, *Asymptotics of resonances for 1D Stark operators*, Lett. Math. Phys., 2018, 118, no.5, 1307-1322 .

[13] M.Abramowitz and I.Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* Dover Publications, New York, 1972.

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Received 11 June 2023
Accepted 21 December 2023