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On Some Applications of the New Extended Hypergeometric Function

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Abstract. The fundamental goal of this article is to investigate applications of the new extended hypergeometric function that contained two Fox-Wright functions in its kernel to generating function. Moreover, the new extended Riemann-Liouville fractional derivative operator with some of its properties is also studied.

Key Words and Phrases: beta function; wright function; generating function.
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1. Introduction

In 1994, Chaudhry and Zubair [1] established the following first extension of the gamma function that considered exponential kernel of the form $\exp\left(-\Pi t^{-1}\right)$

$$\Gamma_{\Pi}(x) = \int_0^\infty t^{x-1} \exp\left(-t - \frac{\Pi}{t}\right) dt, \tag{1}$$

where $Re(\Pi) > 0$, and $Re(x) > 0$.
Then later, using the concept in (1), Chaudhry et al., [2] investigated the following extended beta function:

$$B_{\Pi}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{\Pi}{t(1-t)}\right) dt, \tag{2}$$

where $Re(\Pi) > 0$, $\min\{Re(x), Re(y)\} > 0$.
They also (Chaudhry et al., [2]) studied the properties of the extended Euler beta function in (2) such as functional relations, recurrence relations, summation formulas, integral formulas, Mellin transform, application to statistical distribution and connection with some known special functions such as error, MacDonald, and Whittaker functions.
Then in 2004, Chaudhry et al., [3] used the extended beta function in (2) to establish the following extended Gauss and confluent hypergeometric functions:

$$F_{\Pi}(\lambda, \mu; \nu; z) = \frac{1}{B(\mu, \nu-\mu)} \int_0^1 \frac{t^{\mu-1}(1-t)^{\nu-\mu-1}}{(1-tz)^{\lambda}} \exp\left(-\frac{\Pi}{t(1-t)}\right) dt, \tag{3}$$

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where $Re(\Pi) > 0$, $Re(\lambda) > 0$, $Re(\nu) > Re(\mu) > 0$, and $|arg(1-z)| < \pi$, and

$$\Phi_{\Pi}(\mu;\nu;z) = \frac{1}{B(\mu,\nu-\mu)} \int_0^1 t^{\mu-1}(1-t)^{\nu-\mu-1} \exp\left(tz - \frac{\Pi}{t(1-t)}\right) dt, \tag{4}$$

where $Re(\Pi) > 0$, $Re(\nu) > Re(\mu) > 0$, and $|arg(1-z)| < \pi$.
Ozarslan and Ozergin [4] used the extended beta function in (2) to studied the generalized extended Riemann-Liouville fractional derivative operator given as:

$$D_z^{\hbar,\Pi}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(\hbar)} \int_0^z f(t)(z-t)^{-\hbar-1} \exp\left(-\frac{z^2\Pi}{t(z-t)}\right) dt & (Re(\hbar) < 0, Re(\Pi) > 0) \\ \frac{d^\varrho}{dz^\varrho} D_z^{\hbar-\varrho,\Pi}\{f(z)\} & (\varrho - 1 < Re(\hbar) < \varrho,\ \varrho \in \mathbb{N}) \end{cases} \tag{5}$$

The following new extended beta function with two Fox-Wright function as its regularizer is studied by Kaurangini et al., [5]:

$$\begin{aligned} {}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(x,y) &= {}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}\left[\begin{array}{c|c|c} (p_w,P_w)_{1,m} & (h_u,H_u)_{1,l} & x,y \\ (q_i,Q_i)_{1,g} & (b_j,B_j)_{1,s} & \end{array}\right] \\ &= \int_0^1 t^{x-1}(1-t)^{y-1} {}_r\Psi_g\left(-\frac{\Pi}{t^\Delta}\right) {}_l\Psi_s\left(-\frac{\aleph}{(1-t)^\Theta}\right) dt, \end{aligned} \tag{6}$$

where $\min\{Re(\Pi), Re(\aleph)\} > 0$, $\min\{Re(\Delta), Re(\Theta)\} > 0$, $\min\{Re(x), Re(y)\} > 0$, and ${}_m\Psi_g$ is the generalized Wright function defined for $z \in \mathbb{C}$, complex p_w, q_ι and $P_w, Q_\iota \in \mathbb{R}$ $(w = 1,2,3\ldots,m; i = 1,2,3,\ldots,g)$ by the series in Kilbas and Srivastava [6]

$$ {}_m\Psi_g(z) = {}_m\Psi_g\left[\begin{array}{c|c} (p_w,P_w)_{1,m} & z \\ (q_i,Q_i)_{1,g} & \end{array}\right] = \sum_{r=0}^{\infty} \frac{\prod_{w=1}^{m}\Gamma(p_w + rP_w)}{\prod_{\iota}^{g}\Gamma(q_\iota + rQ_\iota)} \frac{z^r}{r!} \tag{7}$$

They also (Kaurangini et al., [5]) introduced the following new extended Gauss and confluent hypergeometric functions:

$$ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu;\nu;z) = {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left[\begin{array}{c|c|c} (p_w,P_w)_{1,m} & (h_u,H_u)_{1,l} & \lambda,\mu;\nu;z \\ (q_i,Q_i)_{1,g} & (b_j,B_j)_{1,s} & \end{array}\right] = \sum_{r=0}^{\infty}(\lambda)_r \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\mu+r,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^r}{r!}, \tag{8}$$

where $Re(\nu) > Re(\mu) > 0$ and $|z| < 1$; and

$$ {}^{\Psi}\Phi_{\Pi,\aleph}^{\Delta,\Theta}(\mu;\nu;z) = {}^{\Psi}\Phi_{\Pi,\aleph}^{\Delta,\Theta}\left[\begin{array}{c|c|c} (p_w,P_w)_{1,m} & (h_u,H_u)_{1,l} & \mu;\nu;z \\ (q_i,Q_i)_{1,g} & (b_j,B_j)_{1,s} & \end{array}\right] = \sum_{r=0}^{\infty} \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\mu+r,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^r}{r!}, \tag{9}$$

where $Re(\nu) > Re(\mu) > 0$.
Definition 1: Pohlen defined the following Hadamard convolution (product) for the two

power series $h(z) = a_r z^r$ $(|z| < R_h)$ and $k(z) = b_r z^r$ $(|z| < R_k)$, where R_h and R_k are the radii of convergence is expressed in [7] as

$$(h * k)(z) = \sum_{r=0}^{\infty} a_r b_r z^r = (k * h)(z) \quad (R_h R_k \leq R) \tag{10}$$

Recently, many researchers for example, Chand et al.,[8] , Centikaya et al., [9], Abubakar et al., [10], Menon et al., [11] and Ozarslan and Ustaoglu [12] used generalized beta and hypergeometric functions to studied generating functions and extended fractional and integral operators. In this work, new generalized special functions that consist of the two Fox-Wright functions as their regularizer in (6), (8) and (9) are to be use to study generating functions and extended Riemann-Liouville fractional integral operator.

2. Applications of extended hypergeometric function to generating functions

Applications of the new extended Gauss hypergeometric functions to generating functions is obtained in this section.

Theorem 2: The following equation is true

$$\sum_{n=0}^{\infty} (\nu)_n \ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu+n, \phi; \varphi; z\right) \frac{t^n}{n!} = (1-t)^{-\nu} \ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu, \phi; \varphi; \frac{z}{1-t}\right) \quad (|t| < 1) \tag{11}$$

Proof: Let the left-hand side of (11) be G_1, then

$$G_1 = \sum_{n=0}^{\infty} (\nu)_n \left\{ \sum_{r=0}^{\infty} (\nu+n)_r \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+r, \varphi-\phi)}{B(\phi, \varphi-\phi)} \frac{z^r}{r!} \right\} \frac{t^n}{n!} \tag{12}$$

Using the pocchammer symbol identity in Chand et al., [13]

$$(\nu)_n (\nu+n)_r = (\nu)_{r+n} = (\nu)_r (\nu+r)_n, \tag{13}$$

in (12), leads to the following

$$G_1 = \sum_{r=0}^{\infty} (\nu)_r \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+r, \varphi-\phi)}{B(\phi, \varphi-\phi)} \left\{ \sum_{n=0}^{\infty} (\nu+r)_n \frac{t^n}{n!} \right\} \frac{z^r}{r!} \tag{14}$$

Applying the following generalized binomial theorem in Chand et al, [13]

$$\sum_{n=0}^{\infty} (\Delta)_n \frac{(tz)^n}{n!} = (1-tz)^{-\Delta} \qquad (|t| < 1), \tag{15}$$

to (14), leads to the following

$$G_1 = (1-t)^{-\nu} \sum_{r=0}^{\infty} (\nu)_r \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+r, \varphi-\phi)}{B(\phi, \varphi-\phi) r!} \left(\frac{z}{1-t} \right)^r \tag{16}$$

Using the new generalized Gauss hypergeoemetric function in (8) to (16), then (11) is obtained.

Theorem 3: The following formul is also valid

$$\sum_{n=0}^{\infty}\binom{\nu+n-1}{n}\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu+n,\phi;\varphi;z\right)t^{n}=(1-t)^{-\nu}\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu,\phi;\varphi;\frac{z}{1-t}\right)\quad(|t|<1) \tag{17}$$

Proof: Let the left-hand side of (17) be G_2, then

$$G_2=\sum_{n=0}^{\infty}\binom{\nu+n-1}{n}\left\{\sum_{r=0}^{\infty}(\nu+n)_r\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+r,\varphi-\phi)}{B(\phi,\varphi-\phi)}\frac{z^r}{r!}\right\}t^n \tag{18}$$

Using (13) to (18) and re-arranging leads us to

$$G_2=\sum_{n=0}^{\infty}(\nu)_n\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)}\left\{\sum_{k=0}^{\infty}\binom{\upsilon+n+k-1}{k}t^k\right\}\frac{z^n}{n!} \tag{19}$$

Using the fact that in Chand et al, [13]

$$\sum_{k=0}^{\infty}\binom{\nu+n+k-1}{k}t^k=(1-t)^{-(\nu+n)}\qquad(|t|<0),$$

in (19)

$$G_2=(1-t)^{-\nu}\sum_{n=0}^{\infty}(\nu)_n\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)n!}\left(\frac{z}{1-t}\right)^n \tag{20}$$

Applying the new generalized Gauss hypergeometric function in (20) the needful result in (17) is obtained.

Lemma 4: The n-derivative of the function $f(u)=u^{-c-k\zeta}$ (see Jain et al., [14]) is given by

$$f^n(u)=(-1)^n u^{-c-k\zeta-n}\frac{\Gamma(c+k\zeta+n)}{\Gamma(c+k\zeta)}, \tag{21}$$

where $c,\zeta\in\mathbb{C}$, $k\in\mathbb{N}$, and $n\in\mathbb{N}_0$.

Theorem 5:

$$(1+t)^{-c}\,{}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu,\phi;\varphi;\frac{z}{1+t}\right)=\sum_{n=0}^{\infty}(-1)^n(c)_n\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu,\phi;\varphi;z\right)*\ {}_2F_1\left(c+n,1;c;z\right)\frac{t^n}{n!}, \tag{22}$$

where $c,z\in\mathbb{C};|t|<1$.

Proof: For convenience, substituting $1+t$ by u and setting the left-hand side of (27) to be $f(u)$, the the extended Gauss hypergeometric function in (8) leads to

$$f(u)=\sum_{r=0}^{\infty}(\nu)_r\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+r,\varphi-\phi)}{B(\phi,\varphi-\phi)}\frac{z^r}{r!}u^{-(c+r)} \tag{23}$$

Differentiating (23) n-times with respect to u and using (21), gives

$$f^{(n)}(u)=(-1)^n u^{-(c+n)}(c)_n\sum_{r=0}^{\infty}(\nu)_r\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+r,\varphi-\phi)}{B(\phi,\varphi-\phi)r!}\frac{(c+n)_r(1)_r}{(c)_r r!}\left(\frac{z}{u}\right)^r \tag{24}$$

Applying the Hadamard convolution (product) in (10) to (24), yields

$$f^{(n)}(u)=(-1)^n u^{-(c+n)}(c)_n\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu,\phi;\varphi;\frac{z}{u}\right)*\ {}_2F_1\left(c+n,1;c;\frac{z}{u}\right) \tag{25}$$

On expanding $f(u+t)$ as the Talor series, give the following

$$\begin{aligned}f(u+t)=(u+t)^{-c}\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu,\phi;\varphi;\frac{z}{u+t}\right)&=\sum_{n=0}^{\infty}f^{(n)}(u)\frac{t^n}{n!}\\ =\sum_{n=0}^{\infty}(-1)^n u^{-(c+n)}(c)_n\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}&\left(\nu,\phi;\varphi;\frac{z}{u}\right)*\ {}_2F_1\left(c+n,1;c;\frac{z}{u}\right)\frac{t^n}{n!}\end{aligned} \tag{26}$$

Putting $u=1$ in (26) the needful result in (27) is obtained.
Corollary 6:

$$(1+t)^{-c}\ {}^{\Psi}\Phi_{\Pi,\aleph}^{\Delta,\Theta}\left(\phi;\varphi;\frac{z}{1+t}\right)=\sum_{n=0}^{\infty}(-1)^n(c)_n\ {}^{\Psi}\Phi_{\Pi,\aleph}^{\Delta,\Theta}(\phi;\varphi;z)*\ {}_2F_1(c+n,1;c;z)\frac{t^n}{n!}, \tag{27}$$

where $c,z\in\mathbb{C};|t|<1$.

3. The extended Appell and Lauricella hypergeometric functions

The new extended Appell and Lauricella hypergeometric functions are defined in section with their integral representations
Definition 7: The new extended Appell $F_1(;)$ is

$$\begin{aligned}{}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu,\nu;\omega;x,y)&={}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}\left[\begin{array}{c|c|c}(p_w,P_w)_{1,m}&(h_u,H_u)_{1,l}&\\ &&\lambda,\mu,\nu;\omega;x,y\\ (q_i,Q_i)_{1,g}&(b_j,B_j)_{1,s}&\end{array}\right]\\ &=\sum_{r,n=0}^{\infty}(\mu)_r(\nu)_n\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\lambda+r+n,\omega-\lambda)}{B(\lambda,\omega-\lambda)}\frac{x^r}{r!}\frac{y^n}{n!},\end{aligned} \tag{28}$$

where $\lambda,\mu,\nu,\omega\in\mathbb{C}$, $Re(\omega)>Re(\lambda)>0$ and $\max\{|x|,|y|\}<1$.
Definition 8: The new extended Appell $F_2(;)$ is defined by

$${}^{\Psi}F_{2;\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu,\nu;\omega,\delta;x,y)={}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}\left[\begin{array}{c|c|c}(p_w,P_w)_{1,m}&(h_u,H_u)_{1,l}&\\ &&\lambda,\mu,\nu;\omega,\delta;x,y\\ (q_i,Q_i)_{1,g}&(b_j,B_j)_{1,s}&\end{array}\right]$$

$$= \sum_{r,n=0}^{\infty} (\lambda)_{r+n} \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\mu+r,\omega-\mu)}{B(\mu,\omega-\mu)} \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\nu+n,\delta-\nu)}{B(\nu,\delta-\nu)} \frac{x^r}{r!}\frac{y^n}{n!}, \tag{29}$$

where $\lambda,\mu,\nu,\omega \in \mathbb{C}$, $Re(\omega) > Re(\mu) > 0$, $Re(\delta) > Re(\nu) > 0$ and $|x|+|y| < 1$.

Definition 9: The new extended Lauricella $F_D^3(;)$ is given by

$$ {}^{\Psi}F_{D;\Pi,\aleph}^{3;\Delta,\Theta}(\lambda,\mu,\nu,\omega;\delta;x,y,z) = {}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}\left[\begin{array}{c|c|c} (p_w,P_w)_{1,m} & (h_u,H_u)_{1,l} & \\ & & \lambda,\mu,\nu,\omega;\delta;x,y,z \\ (q_i,Q_i)_{1,g} & (b_j,B_j)_{1,s} & \end{array}\right]$$

$$= \sum_{r,n,s=0}^{\infty} (\mu)_r(\nu)_n(\delta)_s \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\lambda+r+n,\delta-\lambda)}{B(\lambda,\delta-\lambda)} \frac{x^r}{r!}\frac{y^n}{n!}\frac{z^s}{s!}, \tag{30}$$

where $\lambda,\mu,\nu,\omega,\delta \in \mathbb{C}$, $Re(\delta) > Re(\lambda) > 0$ and $\max\{|x|,|y|,|z|\} < 1$.

Theorem 10:

$${}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu,\nu;\omega;x,y)$$
$$= \frac{1}{B(\lambda,\omega-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\omega-\lambda-1}(1-tx)^{-\mu}(1-ty)^{-\nu} {}_r\Psi_g\left(-\frac{\Pi}{t^{\Delta}}\right) {}_l\Psi_s\left(-\frac{\aleph}{(1-t)^{\Theta}}\right) dt \tag{31}$$

Proof Using (28) and (6) into the left-hand side of (31), we have

$${}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu,\nu;\omega;x,y)$$
$$= \sum_{r,n=0}^{\infty} (\mu)_r(\nu)_n \left\{\frac{1}{B(\lambda,\omega-\lambda)} \int_0^1 t^{\lambda+r+n-1}(1-t)^{\omega-\lambda-1} {}_r\Psi_g\left(-\frac{\Pi}{t^{\Delta}}\right) {}_l\Psi_s\left(-\frac{\aleph}{(1-t)^{\Theta}}\right) dt\right\} \frac{x^r}{r!}\frac{y^n}{n!} \tag{32}$$

Changing the order of summation and integration in (32), we have

$${}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu,\nu;\omega;x,y)$$
$$= \frac{1}{B(\lambda,\delta-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\delta-\lambda-1} \left\{\sum_{r=0}^{\infty}(\mu)_r\frac{(tx)^r}{r!}\right\} \left\{\sum_{n=0}^{\infty}(\nu)_n\frac{(ty)^n}{n!}\right\} {}_r\Psi_g\left(-\frac{\Pi}{t^{\Delta}}\right) {}_l\Psi_s\left(-\frac{\aleph}{(1-t)^{\Theta}}\right) dt \tag{33}$$

Applying generalized binomial theorem to (33) the desired result in (31) is obtained.

Theorem 11:

$${}^{\Psi}F_{2;\Pi,\aleph}^{\Delta,\Theta}(\lambda,\mu,\nu;\omega,\delta;x,y)$$
$$= \frac{1}{B(\mu,\omega-\mu)B(\nu,\delta-\nu)} \int_0^1 t^{\mu-1}(1-t)^{\omega-\mu-1}p^{\nu-1}(1-p)^{\delta-\nu-1}(1-tx-ty)^{-\lambda}$$

$$\times\, {}_r\Psi_g\left(-\frac{\Pi}{t^{\Delta}}\right){}_l\Psi_s\left(-\frac{\aleph}{(1-t)^{\Theta}}\right){}_r\Psi_g\left(-\frac{\Pi}{p^{\Delta}}\right){}_l\Psi_s\left(-\frac{\aleph}{(1-p)^{\Theta}}\right)dt \tag{34}$$

Proof Using same procedure as in (31) and the fact that in Ozarslan and Ozergin [4]

$$\sum_{M=0}^{\infty} f(M)\frac{(x+y)^M}{M!} = \sum_{r,n=0}^{\infty} f(r+n)\frac{x^r}{r!}\frac{y^n}{n!},$$

he required the result in (34) is obtained.

Theorem 12:

$$\begin{aligned}&{}^{\Psi}F_{D;\Pi,\aleph}^{3;\Delta,\Theta}(\lambda,\mu,\nu,\omega;\delta;x,y,z)\\ &= \frac{1}{B(\lambda,\omega-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\omega-\lambda-1}(1-tx)^{-\mu}(1-ty)^{-\nu}(1-tz)^{-\delta}{}_r\Psi_g\left(-\frac{\Pi}{t^{\Delta}}\right){}_l\Psi_s\left(-\frac{\aleph}{(1-t)^{\Theta}}\right)dt\end{aligned} \tag{35}$$

Proof: The proof of (35) follows directly from (31).

4. New extended Riemann-Liouville fractional derivative operator

Definition 13: The following generalized Riemann-Liouville fractional derivative operator is defined by

$${}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{f(z)\} = \frac{1}{\Gamma(-\hbar)}\int_0^z f(t)(z-t)^{-\hbar-1}{}_r\Psi_g\left(-\frac{z^{\Delta}\Pi}{t^{\Delta}}\right){}_l\Psi_s\left(-\frac{z^{\Theta}\aleph}{(z-t)^{\Theta}}\right)dt \quad (Re(\hbar)<0), \tag{36}$$

and for $\varrho-1 < Re(\hbar) < \varrho$, $\varrho\in\mathbb{N}$, we have

$$\begin{aligned}{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{f(z)\} &= \frac{d^{\varrho}}{dz^{\varrho}}\,{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar-\varrho;\Delta,\Theta}\{f(z)\}\\ &= \frac{d^{\varrho}}{dz^{\varrho}}\left\{\frac{1}{\Gamma(\varrho-\hbar)}\int_0^z f(t)(z-t)^{\varrho-\hbar-1}{}_r\Psi_g\left(-\frac{z^{\Delta}\Pi}{t^{\Delta}}\right){}_l\Psi_s\left(-\frac{z^{\Theta}\aleph}{(z-t)^{\Theta}}\right)dt\right\},\end{aligned} \tag{37}$$

the path of integration is value for 0 to z in complex t-plane.

Theorem 14:

$${}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{z^{\phi}\} = \frac{z^{\phi-\hbar}}{\Gamma(-\hbar)}\ {}^{\Psi}B_{\Delta,\Theta}^{\Pi,\aleph}(\phi+1,-\hbar), \tag{38}$$

where $\phi,\hbar\in\mathbb{C}$, $Re(\phi)>-1$, and $Re(\hbar)<0$.

Proof: Using (36), we obtained

$${}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{z^{\phi}\} = \frac{1}{\Gamma(-\hbar)}\int_0^z t^{\phi}(z-t)^{-\hbar-1}{}_r\Psi_g\left(-\frac{z^{\Delta}\Pi}{t^{\Delta}}\right){}_l\Psi_s\left(-\frac{z^{\Theta}\aleph}{(z-t)^{\Theta}}\right)dt \quad (Re(\aleph)<0) \tag{39}$$

Putting $t = yz$ in (39) leads us to

$$ {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\} = \frac{z^{\phi-\hbar}}{\Gamma(-\hbar)} \int_0^1 y^{\phi+1-1}(1-y)^{-\hbar-1} {}_r\Psi_g\left(-\frac{\Pi}{y^{\Delta}}\right) {}_l\Psi_s\left(-\frac{\aleph}{(1-y)^{\Theta}}\right) dy \tag{40} $$

Applying (6) to (40) the desired result in (38) is obtained.
Corollary 15: If $f(z)$ is an analytic function in the disc $|z| < \varpi$ and it has power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$ {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi-1}f(z)\} = \frac{z^{\phi-\hbar-1}}{\Gamma(-\hbar)} \sum_{k=0}^{\infty} a_k \ {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+k, -\hbar) z^k, \tag{41} $$

where $\phi, \aleph \in \mathbb{C}$, $Re(\phi) > 0$, and $Re(\aleph) < 0$.
Theorem 16:

$$ {}^{\Psi}D^{\phi-\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi-1}(1-z)^{-\nu}\} = z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)} \ {}^{\Psi}F^{\Delta,\Theta}_{\Pi,\aleph}(\nu,\phi;\hbar;z), \tag{42} $$

where $\nu, \phi, \hbar \in \mathbb{C}$, $\min\{Re(\nu), Re(\phi)\} > 0$ $Re(\hbar) < 0$, and $|z| < 1$.
Proof: Using (36) we obtained

$$ {}^{\Psi}D^{\phi-\hbar;\Delta,\Theta,}_{z;\Pi,\aleph}\{z^{\phi-1}(1-z)^{-\nu}\} = \sum_{r=0}^{\infty} \frac{(\nu)_r}{r!} \ {}^{\Psi}D^{\Delta,\Theta,\phi-\hbar}_{z;\Pi,\aleph}\{z^{\phi+r-1}\} \tag{43} $$

Applying (38) to (43), yield

$$ \begin{aligned} {}^{\Psi}D^{\phi-\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi-1}(1-z)^{-\nu}\} &= \frac{z^{\hbar-1}}{\Gamma(\hbar-\phi)} \sum_{r=0}^{\infty} \frac{(\nu)_r}{r!} \ {}^{\Psi}B^{\Pi,\aleph}_{\Delta,\Theta}(\phi+r,\hbar-\phi) z^r \\ &= z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)} \sum_{r=0}^{\infty} (\nu)_r \frac{{}^{\Psi}B^{\Pi,\aleph}_{\Delta,\Theta}(\phi+r,\hbar-\phi)}{B(\Gamma(\phi,\hbar-\phi))} \frac{z^r}{r!} \\ &= z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)} \ {}^{\Psi}F^{\Delta,\Theta}_{\Pi,\aleph}(\upsilon,\phi;\hbar;z) \end{aligned} $$

Corollary 17:

$$ {}^{\Psi}D^{\phi-\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi-1}(1-az)^{-\nu}(1-bz)^{-\varphi}\} = z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)} \ {}^{\Psi}F^{\Delta,\Theta}_{1;\Pi,\aleph}(\phi,\nu,\varphi;\hbar;az,bz), \tag{44} $$

where $\nu, \phi, \hbar \in \mathbb{C}$, $\min\{Re(\nu), Re(\phi)\} > 0$ $Re(\hbar) < 0$, and $\min\{|az|, |bz|\} < 1$.
Corollary 18:

$$ {}^{\Psi}D^{\phi-\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi-1}(1-az)^{-\nu}(1-bz)^{-\varphi}(1-cz)^{-\omega}\} = z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)} \ {}^{\Psi}F^{3;\Delta,\Theta}_{D;\Pi,\aleph}(\phi,\nu,\varphi;\omega;\hbar;az,bz,cz), \tag{45} $$

where $\nu, \phi, \hbar \in \mathbb{C}$, $\min\{Re(\nu), Re(\phi)\} > 0$ $Re(\hbar) < 0$, and $\min\{|az|, |bz|, |bz|\} < 1$.
Theorem 19:

$$ {}^{\Psi}D_{z;\Pi,\aleph}^{\phi-\hbar;\Delta,\Theta}\left\{z^{\phi-1}(1-z)^{-\nu}\ {}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\nu,\omega;\varphi;\frac{y}{1-z}\right)\right\} = z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)}\ {}^{\Psi}F_{2;\Pi,\aleph}^{\Delta,\Theta}(\nu,\omega,\phi;\varphi,\hbar;y,z), \tag{46} $$

where $\nu, \phi, \hbar \in \mathbb{C}$, $\min\{Re(\nu), Re(\phi)\} > 0$ $Re(\hbar) < 0$, and $|z| < 1$.
Proof: Let Q be the left-hand side of (46), using (36) and (8), we obtained

$$ Q = \sum_{r=0}^{\infty}(\nu)_r \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\omega+r,\varphi-\omega)}{B(\omega,\varphi-\omega)}\frac{y^r}{r!}\,{}^{\Psi}D_{z;\Pi,\aleph}^{\Delta,\Theta,\phi-\hbar}\left\{z^{\phi-1}(1-z)^{-(v+r)}\right\} \tag{47} $$

Applying (42) to (47) leads us to

$$ Q = z^{\hbar-1}\frac{\Gamma(\phi)}{\Gamma(\hbar)}\sum_{r,n=0}^{\infty}(\nu)_r(\nu+r)_n \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\omega+r,\varphi-\omega)}{B(\omega,\varphi-\omega)}\frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+n,\hbar-\phi)}{B(\phi,\hbar-\phi)}\frac{y^r}{r!}\frac{z^n}{n!} \tag{48} $$

Using (13) and (29) to the above equation, the needful result in (46) is obtained.

5. Integral transforms of the new extended Riemann-Liouville fractional integral operator

Theorem 20: (Beta transform)

$$ \mathcal{B}\left\{{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{z^{\phi}\};\Omega,\Lambda\right\} = \frac{{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+1,-\hbar)}{\Gamma(-\hbar)}B(\Omega+\phi-\hbar,\Lambda), \tag{49} $$

where $\phi, \hbar \in \mathbb{C}$, $Re(\phi) > -1$, and $Re(\hbar) < 0$.
Proof Using the definition of beta transform in Mubeen and Ali [15], one can obtain

$$ \mathcal{B}\left\{{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{z^{\phi}\};\Omega,\Lambda\right\} = \int_0^1 z^{\Omega-1}(1-t)^{\Lambda-1}\,{}^{\Psi}D_{z;\Pi,\aleph}^{\Delta,\Theta;\hbar}\{z^{\phi}\}dz \tag{50} $$

Simplifying (50) using (38), gives

$$ \mathcal{B}\left\{{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{z^{\phi}\};\Omega,\Lambda\right\} = {}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(\phi+1,-\hbar)\int_0^1 z^{\Omega+\phi-\hbar-1}(1-t)^{\Lambda-1}dz \tag{51} $$

Upon simplying (51), the required result in (49) is obtained.
Theorem 21: (Mellin transform)

$$ \mathcal{M}\left\{{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{z^{\phi}\}\right\}(r,s) = \frac{{}^{\Psi}\Gamma(r)\,{}^{\Psi}\Gamma(s)}{\Gamma(-\hbar)}B(\phi+\Pi r+1,\aleph s-\hbar)z^{\phi-\hbar}, \tag{52} $$

where $\phi, \hbar \in \mathbb{C}$, $Re(\phi) > -1$, $Re(\hbar) < 0$, $Re(s) > 0$, $Re(r) > 0$.
Proof By definition of the Mellin transform from (Debnath and Bhatta [16]) we obtain

$$ \mathcal{M}\left\{{}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;;\Delta,\Theta}\{z^{\phi}\}\right\}(r,s) = \int_0^1\int_0^1 \Delta^{r-1}\Theta^{s-1}\,{}^{\Psi}D_{z;\Pi,\aleph}^{\Delta,\Theta;\hbar}\{z^{\phi}\}d\Delta d\Theta \tag{53} $$

Simplifying (54) using (38), gives

$$\mathcal{M}\left\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\}\right\}(r,s)=\frac{z^{\phi-\hbar}}{\Gamma(-\hbar)}\int_0^1\int_0^1 \Delta^{r-1}\Theta^{s-1}\ {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)d\Delta d\Theta \tag{54}$$

Considering the result from Kaurangini et al., [5]

$$\int_0^1\int_0^1 \Delta^{r-1}\Theta^{s-1\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(x,y)d\Delta d\Theta={}^{\Psi}\Gamma(r)^{\Psi}\Gamma(s)B(x+\Pi r,y+\aleph s),$$

to (54) the desired result in (52) is obtained.
Corollary 22:

$$\mathcal{M}\left\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{(1-z)^{\lambda}\}\right\}(r,s)=\frac{{}^{\Psi}\Gamma(r)^{\Psi}\Gamma(s)}{\Gamma(-\hbar)}B(\Pi r+1,\aleph s-\hbar)\,{}_2F_1(\lambda,1+\Pi r;\Pi r+\aleph s+1-\hbar;z),$$

where $\phi,\hbar\in\mathbb{C}$, $Re(\phi)>-1$, $Re(\hbar)<0$, $Re(s)>0$, $Re(r)>0$.
Theorem 23: (SUM transform)

$$S_a\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\}\}(s)=\frac{{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)}{\Gamma(-\hbar)}\frac{\Gamma(\phi-\hbar+1)}{s^r[slog(a)]^{\phi-\hbar+1}}, \tag{55}$$

where $r\in\mathbb{N}$, $a\in(0,\infty)\setminus\{1\}$, $\rho_1\le s\le\rho_2$, $\rho_1,\rho_2>0$ also $\phi,\hbar\in\mathbb{C}$, $Re(\phi)>-1$, and $Re(\hbar)<0$.
Proof The definition of SUM transform from Hasan et al., [17], we have

$$S_a\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\}\}(s)=\frac{1}{s^r}\int_0^1 a^{-zt}\ {}^{\Psi}D^{\Delta,\Theta;\hbar}_{z;\Pi,\aleph}\{z^{\phi}\}dz \tag{56}$$

Simplifying (54) using (38), gives

$$S_a\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\}\}(s)=\frac{{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)}{\Gamma(-\hbar)}\left\{\frac{1}{s^r}\int_0^1 a^{-zt}z^{\phi-\hbar}dz\right\} \tag{57}$$

Further simplification of (57) gives the required result in (55).
Theorem 24: (Laplace transform)

$$\mathcal{L}\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\}\}(s)=\frac{{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)}{\Gamma(-\hbar)}\frac{\Gamma(\phi-\hbar+1)}{s^{\phi-\hbar+1}}, \tag{58}$$

where $\phi,\hbar\in\mathbb{C}$, $Re(\phi)>-1$, and $Re(\hbar)<0$.
Proof Setting $r=0$ and $a=e$ in (55) the desired resultin (58) is obtained.
Theorem 25: (Whittaker transform)

$$\int_0^{\infty} z^{\alpha-1}\exp\left(-\frac{\Lambda z}{2}\right)W_{p,q}(\Lambda z)\ {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^{\phi}\}dz$$

$$= \frac{{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)}{\Gamma(-\hbar)\Lambda^{\alpha+\phi-\hbar}} \frac{\Gamma\left(\frac{1}{2}+q+\alpha+\phi-\hbar\right)\Gamma\left(\frac{1}{2}-q+\alpha+\phi-\hbar\right)}{\Gamma\left(1-p+\alpha+\phi-\hbar\right)} \tag{59}$$

where $\phi, \hbar \in \mathbb{C}$, $Re(\phi) > -1$, $Re(\hbar) < 0$.
Proof Let left-hand sides of (59) be W, using (38) and changing the order of summation and integration , leads to

$$W = \frac{{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)}{\Gamma(-\hbar)} \int_0^\infty z^{\alpha+\phi-\hbar-1} \exp\left(-\frac{\Lambda z}{2}\right) W_{p,q}(\Lambda z) dz \tag{60}$$

Substituting $u = \Lambda z$ in (60) and applying the result [18]

$$\int_0^\infty z^{\Lambda-1} \exp\left(-\frac{z}{2}\right) W_{p,q}(z) dz = \frac{\Gamma\left(\frac{1}{2}+q+\Lambda\right)\Gamma\left(\frac{1}{2}-q+\Lambda\right)}{\Gamma\left(1-p\right)} \tag{61}$$

where $Re(q \pm \Delta) > -\frac{1}{2}$ and $W_{p,q}(;)$ is Whittaker confluent hypergeometric function, the required result in (59) is acquired.
Theorem 26: (Verma transform)

$$\mathcal{V}\left\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^\phi\}\right\} = \frac{{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(\phi+1,-\hbar)}{\Gamma(-\hbar)s^{\phi-\hbar-1}} \frac{\Gamma\left(q+\Lambda+\phi-\hbar+1\right)\Gamma\left(q+\Lambda+\phi-\hbar+1\right)}{\Gamma\left(\frac{3}{2}-p+\Lambda+\phi-\hbar\right)} \tag{62}$$

where $\phi, \hbar \in \mathbb{C}$, $Re(\phi) > -1$, $Re(\hbar) < 0$.
Proof Using definition of the Verma transform in Al-Gonah and Mohammed [19], we obtained

$$\mathcal{V}\left\{{}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^\phi\}\right\} = \int_0^\infty (sz)^{\Lambda-1} \exp\left(-\frac{1}{2}sz\right) W_{p,q}(sz)\, {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{z^\phi\} dz \tag{63}$$

Following the same procedure to (63) as in (59), the result in (62) is obtained.

6. Applications of the new extended Riemann-Liouville to generating functions

Theorem 27:

$$\sum_{k=0}^{\infty} \frac{(\phi)_k}{k!}\, {}^{\Psi}F^{\Delta,\Theta}_{\Pi,\aleph}\left(\phi+k,\mu;\nu;x\right) t^k = (1-t)^{-\phi}\, {}^{\Psi}F^{\Delta,\Theta}_{\Pi,\aleph}\left(\phi,\mu;\nu;\frac{x}{1-t}\right) \tag{64}$$

where $Re(\lambda) > 0$, $Re(\nu) > Re(\nu) > 0$ and $|z| < \min\{1, |1-t|\}$.
Proof: Using the elementary identity in chand et al., [13]

$$[(1-t)-x]^{-\phi} = (1-t)^{-\phi}\left[1-\frac{x}{1-t}\right]^{-\phi} \tag{65}$$

Expanding the left-hand side of (65) for $|t| < |1-x|$, yields

$$\sum_{k=0}^{\infty} \frac{(\phi)_k}{k!}(1-x)^{\phi}\left(\frac{t}{1-x}\right)^{k} = (1-t)^{-\phi}\left[1-\frac{x}{1-t}\right]^{-\phi} \tag{66}$$

Multiplying both-sides of (66) by $z^{\mu-1}$, applying the generalized Riemann-Liouville fractional derivative operator ${}^{\Psi}D_{x;\Delta,\Theta}^{\mu-\nu;\Pi,\aleph}$ and simplified, yields the following

$$\sum_{k=0}^{\infty} \frac{(\phi)_k}{k!}\,{}^{\Psi}D_{x;\Pi,\aleph}^{\mu-\nu;\Delta,\Theta}\left\{x^{\mu-1}(1-x)^{-\phi-k}\right\}t^{k} = (1-t)^{-\phi}\,{}^{\Psi}D_{x;\Pi,\aleph}^{\mu-\nu;\Delta,\Theta}\left\{x^{\mu-1}\left[1-\frac{x}{1-t}\right]^{-\phi}\right\} \tag{67}$$

Applying the result established in (42) into (67), after some simplification, the needful results in (64) is obtained.

Theorem 28:

$$\sum_{k=0}^{\infty} \frac{(\nu)_k}{k!}\,{}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\mu-k,\lambda;\omega;z\right)t^{k} = (1-t)^{-\nu}\,{}^{\Psi}F_{1;\Pi,\aleph}^{\Delta,\Theta}\left(\lambda,\mu,\nu;\omega;z;-\frac{zt}{(1-t}\right) \tag{68}$$

where $Re(\omega) > Re(\lambda) > 0$, $Re(\mu) > 0$, $Re(\nu) > 0$ and $|t| < \frac{1}{1+|z|}$.

Proof: Using the elementary identity in Srivastava [20]

$$[1-(1-z)t]^{-\nu} = (1-t)^{-\nu}\left[1+\frac{zt}{1-t}\right]^{-\nu} \tag{69}$$

Expanding the left-hand side of (69) for $|t| < |1-z|$, gives

$$\sum_{k=0}^{\infty} \frac{(\nu)_k}{k!}(1-z)^{\nu}t^{\nu} = (1-t)^{-nu}\left[1-\frac{-zt}{1-t}\right]^{-\nu} \tag{70}$$

Multiplying both-sides of (70) by $z^{\lambda-1}(1-z)^{\mu}$, then applying the generalized Riemann-Liouville fractional derivative operator ${}^{\Psi}D_{x;\Pi,\aleph}^{\lambda-\omega;\Delta,\Theta}$ and simplified, leads us to the following

$$\sum_{k=0}^{\infty} \frac{(\nu)_k}{k!}\,{}^{\Psi}D_{x;\Pi,\aleph}^{\lambda-\omega;\Delta,\Theta}\left\{z^{\lambda-1}(1-z)^{-(k+\nu)}\right\}t^{k} = (1-t)^{-\nu}\,{}^{\Psi}D_{x;\Pi,\aleph}^{\lambda-\omega;\Delta,\Theta}\left\{z^{\lambda-1}(1-z)^{-\mu}\left[1-\frac{-zt}{1-t}\right]^{-\nu}\right\} \tag{71}$$

Applying the result established in (42) and (44) into (71), after some simplification, the desired results in (68) is obtained.

Theorem 29:

$$\sum_{k=0}^{\infty} \frac{(\upsilon)_k}{k!}\,{}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\omega,-r;\delta;z\right)\,{}^{\Psi}F_{\Pi,\aleph}^{\Delta,\Theta}\left(\upsilon+k,\phi;\varphi;z\right)t^{k} = (1-t)^{-\phi}\,{}^{\Psi}F_{2;\Pi,\aleph}^{\Delta,\Theta}\left(\phi,\mu,\omega;\nu,\delta;\frac{z}{1-t}\right) \tag{72}$$

Proof: Substituting t by $(1-y)t$ in (64), multiplying the resulting equation by $y^{\omega-1}$ and then applying the operator ${}^{\Psi}D_{y;\Pi,\aleph}^{\omega-\delta;\Delta,\Theta}$, leads us o

$$\begin{aligned} &{}^{\Psi}D_{y;\Pi,\aleph}^{\omega-\delta;\Delta,\Theta}\left\{\sum_{k=0}^{\infty}\frac{(\phi)_k}{k!}y^{\omega-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi+k,\mu;\nu;x\right)(1-y)^{\omega}t^k\right\} \\ &= {}^{\Psi}D_{y;\Pi,\aleph}^{\omega-\delta;\Delta,\Theta}\left\{(1-(1-y)t)^{-\lambda}\,y^{\omega-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x}{1-(1-y)t}\right)\right\} \end{aligned} \tag{73}$$

Changing the order of summation and interchanging the order of summation and integral operator for $|x|<1$, $|\frac{1-y}{1-x}t|<1$ and $|\frac{x}{1-t}|+|\frac{ty}{1-t}|<1$, we have

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{(\phi)_k}{k!}{}^{\Psi}D_{y;\Pi,\aleph}^{\omega-\delta;\Delta,\Theta}\left\{y^{\omega-1}(1-y)^{\omega}\right\}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi+k,\mu;\nu;x\right)t^k \\ &= (1-t)^{-\lambda}\ {}^{\Psi}D_{y;\Pi,\aleph}^{\omega-\delta;\Delta,\Theta}\left\{y^{\omega-1}\left(1-\frac{yt}{1-t}\right)^{-\lambda}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{\frac{x}{1-t}}{1-\frac{-ty}{1-t}}\right)\right\} \end{aligned} \tag{74}$$

Applying (42) and (45) to (74), the required the result in (72) is obtained.

7. The new extended Riemann-Liouville fractional integral operator of some special functions

Theorem 30:

$${}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{\exp(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)}\sum_{r=0}^{\infty}{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(r+1,-\hbar)\frac{z^r}{r!} \quad (z\in\mathbb{C}) \tag{75}$$

Proof Using (38) and (41) to the left-hand side of (75), give

$${}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{\exp(z)\} = \sum_{r=0}^{\infty}\frac{1}{r!}{}^{\Psi}D_{z;\Pi,\aleph}^{\Delta,\Theta;\hbar}\{z^r\}$$

$$\frac{z^{-\hbar}}{\Gamma(-\hbar)}\sum_{r=0}^{\infty}{}^{\Psi}B_{\Delta,\Theta}^{\Pi,\aleph}(r+1,-\hbar)\frac{z^r}{r!}.$$

Theorem 31:

$${}^{\Psi}D_{z;\Pi,\aleph}^{\hbar;\Delta,\Theta}\{F_{(\rho,\varrho)_m}^{(\tau)_m}(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)}\sum_{r=0}^{\infty}\frac{z^r}{\prod_{j=1}^{m}[\Gamma(\rho_j r+\varrho_j)]^{\tau_j}}{}^{\Psi}B_{\Pi,\aleph}^{\Delta,\Theta}(r+1,-\hbar)z^r, \tag{76}$$

where $\rho_j,\varrho_j,\tau_j>0$, $j=1,2,3,\dots,m$, $(\rho,\varrho)_m=(\rho_1,\varrho_1)\dots,(\rho_m,\varrho_m)$, $(\tau)_m=(\tau_1,\dots,\tau_m)$ and $F_{(\rho,\varrho)_m}^{(\tau)_m}(;)$ is the multi-index Le Roy function.

Proof Using definition of multi-index Le Roy function in Kiryakova, and Paneva-Konovska [21]

$$F^{(\tau)_m}_{(\rho,\varrho)_m}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\prod_{j=1}^{m}[\Gamma(\rho_j r + \varrho_j)]^{\tau_j}} \tag{77}$$

The desired result in (76) is obtained.

Corollary 32:

$${}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{E_{(\rho,\varrho)_m}(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)} \sum_{r=0}^{\infty} \frac{z^r}{\prod_{j=1}^{m}[\Gamma(\rho_j r + \varrho_j)]} {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar), \tag{78}$$

where $\rho_j, \varrho_j, \tau_j > 0$, $j = 1, 2, 3, \ldots, m$, $(\rho,\varrho)_m = (\rho_1, \varrho_1)\ldots,(\rho_m, \varrho_m)$, and $E_{(\rho,\varrho)_m}(;)$ is the multi-index Mittag-Leffler function Al-Bassam and Lunchko [22].

Corollary 33:

$${}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{F^{\tau}_{\rho,\varrho}(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)} \sum_{r=0}^{\infty} \frac{z^r}{[\Gamma(\rho r + \varrho)]^{\tau}} {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar), \tag{79}$$

where $z, \rho, \varrho, \tau \in \mathbb{C}$, $\tau > 0$ and $F^{\tau}_{\rho,\varrho}(;)$ is the Mittag-Leffler type Le Roy function [23].

Corollary 34:

$${}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{E_{\rho,\varrho}(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)} \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\rho r + \varrho)} {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar), \tag{80}$$

where $z, \rho, \varrho \in \mathbb{C}$, and $E_{\rho,\varrho}(;)$ is the Wiman function [24].

Corollary 35:

$${}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{E_{\rho}(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)} \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\rho r + 1)} {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar), \tag{81}$$

where $z, \rho, \varrho \in \mathbb{C}$, and $E_{\rho}(;)$ is the classical Mittag-Leffler function [24].

Corollary 36:

$${}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\{F_{\tau}(z)\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)} \sum_{r=0}^{\infty} \frac{z^r}{[\Gamma(r + 1)]^{\tau}} {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar), \tag{82}$$

where z, $\tau > 0$ and $F_{\tau}(;)$ is the Le Roy function in [24].

Theorem 37:

$${}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\left\{ {}_pF_q \left[\begin{array}{c|c} (\zeta_p) & \\ & z \\ (\kappa_q) & \end{array} \right] \right\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p}(\zeta_j)_r}{\prod_{j=1}^{q}(\kappa_j)_r} {}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar) \frac{z^r}{r!}, \tag{83}$$

where z, and ${}_pF_q(;)$ is generalized hypergeometric function in [].

Proof Using the definition of generalized hypergeometric function in [25] and following

the same process as in (76), equation (83) follows.
Corollary 38:

$$ {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\left\{ {}_2F_1\left[\begin{array}{c|c} \zeta_1,\zeta_2 & \\ & z \\ \kappa_1 & \end{array}\right]\right\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)}\sum_{r=0}^{\infty}\frac{(\zeta_1)_r(\zeta_2)_r}{(\kappa)_r}\,{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar)\frac{z^r}{r!}, \tag{84} $$

where z, and ${}_1F_1(;)$ is the classical Gauss hypergeometric function [25].
Corollary 39:

$$ {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\left\{ {}_1F_1\left[\begin{array}{c|c} \zeta_1 & \\ & z \\ \kappa_1 & \end{array}\right]\right\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)}\sum_{r=0}^{\infty}\frac{(\zeta_1)_r}{(\kappa)_r}\,{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar)\frac{z^r}{r!}, \tag{85} $$

where z, and ${}_1F_1(;)$ is the classical confluent hypergeometric function [25].
Theorem 40:

$$ {}^{\Psi}D^{\hbar;\Delta,\Theta}_{z;\Pi,\aleph}\left\{ {}_m\Psi_g(z)\right\} = \frac{z^{-\hbar}}{\Gamma(-\hbar)}\sum_{r=0}^{\infty}\frac{\prod_{w=1}^{m}\Gamma(p_w+rP_w)}{\prod_{\iota}^{g}\Gamma(q_\iota+rQ_\iota)}\,{}^{\Psi}B^{\Delta,\Theta}_{\Pi,\aleph}(r+1,-\hbar)\frac{z^r}{r!}, \tag{86} $$

where ${}_m\Psi_g(;)$ is the Fox-Wright function mention in (7).
Proof Using the definition of the Fox-Wright function mention in (7) and following the same process as in (76), equation (86) follows.

8. Conclusion

Generating function such as linear, bilinear, bilateral and mixed multilateral generating functions are used in fractional calculus, combinatorics, number theory, q-calculus and so on to study numbers, special functions and polynomials, interested reader may refer to several recent article on the subject, see for example [26, 27] and the references cited therein. In this article, some generating functions of the Gauss, Appell and Laurecilla hypergeometric functions are investigated. New extended Riemann-Liouville fractional integral operator using extended beta fuction in (6) is studied, also application of this fractional integral operator to some well-know special function such as Le Roy type functions, Mittag-Leffler type functions, hypergeometric function and Wright function is demonstrated.

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