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# Algebraic points of any degree on the affine curve $\mathcal{C} : y^2 = x^5 - 243$

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**Abstract.** We determine the set of algebraic points of any degree over $\mathbb{Q}$ on the affine curve $\mathcal{C} : y^2 = x^5 - 243$. This result generalize the previous respective results of Mulholland who described in [3] the set of $\mathbb{Q}$-rationals points and of Sow, Sarr and Sall who described in [4] the set of algebraic points of degree at most 5 over $\mathbb{Q}$ on this curve.

**Key Words and Phrases:** Plane curves ; degree of algebraic points; rational points; algebraic extensions; Jacobian.

**2010 Mathematics Subject Classifications**: Primary 14H50, 14H40, 11D68, 12FO5

## 1. Introduction

Let $\mathcal{C}$ be a smooth algebraic curve defined over $\mathbb{Q}$. Let $K$ be a numbers field. We denote by $\mathcal{C}(K)$ the set of rational points of $\mathcal{C}$ on $K$ and $\bigcup_{[K:\mathbb{Q}]\leq d} \mathcal{C}(K)$ the set of points of $\mathcal{C}$ defined over $K$ of degree at most $d$.
In this note we determine the set of algebraic points of any degree at most $d$ on the affine curve $\mathcal{C} : y^2 = x^5 - 243$.
This curve had been studied by Mulholland [3] who had determined the algebraic points of degree 1 over $\mathbb{Q}$. Then, the result obtained by Mulholland was extended to algebraic points of degree at most 5 over $\mathbb{Q}$ by Sow, Sarr and Sall [4].
The Mordell-Weil group $J_{\mathcal{C}}(\mathbb{Q})$ of rational points of the Jacobian is a finite set (refer to [3]). We denote by $P = (3, 0)$ et $\infty = (0, 1, 0)$.
In [3] Mulholland gave a description of the rational points of $\mathcal{C}$.
In [4] Sow, Sarr and Sall gave a description of the algebraic points of degree at most 5 on $\mathbb{Q}$ on this same curve. These descriptions are respectively stated as follows:

**Proposition 1.1 (Refer to [3]).**

*The rational points of the curve $\mathcal{C}$ are given by :*

$$\mathcal{C}(\mathbb{Q}) = \{P\ ,\ \infty\}.$$

 

**Proposition 1.2 (Refer to [4]).**

1. *The set of quadratic points on* $\mathcal{C}$ *is given by*

$$\mathcal{A}_0 = \left\{ \left(\alpha, \pm\sqrt{\alpha^5 - 243}\right),\ \alpha \in \mathbb{Q} \right\}.$$

2. *The set of cubic points on* $\mathcal{C}$ *is empty.*
3. *The set of quartic points on* $\mathcal{C}$ *is given by* $\mathcal{A}_1 \cup \mathcal{A}_2$ *with*

$$\mathcal{A}_1 = \left\{ \left(x, \pm\sqrt{x^5 - 243}\right) \mid x \in \overline{\mathbb{Q}},\ [\mathbb{Q}(x) : \mathbb{Q}] = 2 \right\},$$

$$\mathcal{A}_2 = \left\{ \begin{array}{l} (x, (x-3)(\lambda_1 + \lambda_2(x+3))) \mid \lambda_1, \lambda_2 \in \mathbb{Q} \textit{ and } x \textit{ root of} \\ \\ A(x) = x^4 + 3x^3 + 9x^2 + 27x + 81 - (x-3)(\lambda_1 + \lambda_2(x+1))^2 \end{array} \right\}.$$

4. *The set of quintic points on* $\mathcal{C}$ *is given by* $\mathcal{B}_1 \cup \mathcal{B}_2$ *with*

$$\mathcal{B}_1 = \left\{ \begin{array}{l} \left(x, \alpha_1 + \alpha_2 x + \alpha_3 x^2\right) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}^* \textit{ and } x \textit{ root of} \\ \\ B_1(x) = x^5 - \alpha_3^2 x^4 - 2\alpha_2\alpha_3 x^3 - (\alpha_2^2 + 2\alpha_1\alpha_2)x^2 - 2\alpha_1\alpha_2 x - (\alpha_1^2 + 243) \end{array} \right\},$$

$$\mathcal{B}_2 = \left\{ \begin{array}{l} \left(x, (x-3)\left[n_1 + n_2(x+3) + n_3(x^2+3x+9)\right]\right) \mid n_1, n_2, n_3 \in \mathbb{Q}^* \textit{ and } x \textit{ root of} \\ \\ B_2(x) = (x-3)\left(n_1 + n_2(x+3) + n_3(x^2+3x+9)\right)^2 - (x^4 + 3x^3 + 9x^2 + 27x + 81) \end{array} \right\}.$$

Our main tools are the Mordell-Weil group $J_{\mathcal{C}}(\mathbb{Q})$ of rational points on the Jacobian $J_{\mathcal{C}}$ of $\mathcal{C}$ (refer to [3]), the Abel-Jacobi theorem (refer to [2]), linear systems on the curve $\mathcal{C}$. Our main result is as follows:

**Theorem 1.3.** *The set of algebraic points of any degree at most d over* $\mathbb{Q}$ *on the curve* $\mathcal{C}$ *is given by :*

$$\bigcup_{[K:\mathbb{Q}]\leq d} \mathcal{C}(K) = \mathcal{H}_0 \cup \mathcal{H}_1 \textit{ or}$$

$$\mathcal{H}_0 = \left\{ \begin{array}{l} \left( x, -\dfrac{\sum\limits_{r \leq \frac{k}{2}} a_r x^r}{\sum\limits_{s \leq \frac{k-5}{2}} b_s x^s} \right) \mid a_r, b_s \in \mathbb{Q} \\ \\ \textit{and } x \textit{ the root of the equation } (\mathcal{E}_0) \end{array} \right\},$$

$$\mathcal{H}_1 = \left\{ \begin{array}{l} \left( x, -\dfrac{\sum\limits_{r\leq\frac{k+1}{2}} a_r x^r}{\sum\limits_{s\leq\frac{k-4}{2}} b_s x^s} \right) \mid a_r, b_s \in \mathbb{Q} \textit{ satisfying } \sum_{r\leq\frac{k+1}{2}} a_r(3)^r = 0 \\ \\ \textit{and } x \textit{ the root of the equation } (\mathcal{E}_1) \end{array} \right\}.$$

where $(\mathcal{E}_l)$ denote the following equation:

$$(\mathcal{E}_l) : \left( \sum_{r\leq\frac{k+l}{2}} a_r x^r \right)^2 = \left(x^5 - 243\right) \left( \sum_{s\leq\frac{k-5+l}{2}} b_s x^s \right)^2 .$$

## 2. Auxiliary results

For a divisor $D$ on $\mathcal{C}$ we denote by $\mathcal{L}(D)$ the $\overline{\mathbb{Q}}$-vector space of rational functions defined by

$$\mathcal{L}(D) = \left\{ f \in \overline{\mathbb{Q}}(\mathcal{C})^* \mid div\,(f) \geq -D \right\} \cup \{0\}$$

where $l(D)$ denote the $\overline{\mathbb{Q}}$-dimension of $\mathcal{L}(D)$.

We denote by $j(P)$ the class $[P-\infty]$ of $P-\infty$ that is to say that $j$ is the Jacobian diving $\mathcal{C} \to J_{\mathcal{C}}(\mathbb{Q})$.

Let $x$ and $y$ be two rational functions on $\mathcal{C}$ given by :

$$\begin{cases} x(X,Y,Z) = \frac{X}{Z} \\ \\ y(X,Y,Z) = \frac{Y}{Z}. \end{cases}$$

The projective equation of the curve $\mathcal{C}$ is :

$$Y^2 Z^3 = X^5 - 243 Z^5.$$

We denote by $\eta_1 = e^{i\frac{\Pi}{2}}$ and and let's put

$$A_k = \left(0, 9\sqrt{3}\,\eta_1^{2k+1}\right) \text{ for } k \in \{0,1\}.$$

We denote by $\eta_2 = e^{i\frac{2\Pi}{5}}$ and let's put

$$B_k = \left(3\eta_2^k, 0\right) \text{ for } k \in \{0,1,2,3,4\}.$$

We will denote by $\mathcal{M}\cdot\mathcal{C}$ the intersection cycle of an algebraic curve $\mathcal{M}$ and $\mathcal{C}$.

**Lemma 2.1.**

*(i)* $div(x-3) = 2P - 2\infty$,

*(ii)* $div(y) = B_0 + B_1 + B_2 + B_3 + B_4 - 5\infty$,

*(iii)* $div(x) = A_0 + A_1 - 2\infty$.

**Proof**.

This is a calculation of the type :

$$div(w - a) = (W - aZ = 0) \cdot \mathcal{C} - (Z = 0) \cdot \mathcal{C} \quad (*) \ ,$$

where $w$ is a variable (affine) which corresponds to $W$ (projective) and $a$ is a constant. It follows from $(*)$ that :

$(i)$ $div(x - 3) = (X - 3Z = 0) \cdot \mathcal{C} - (Z = 0) \cdot \mathcal{C}$ for $w = x$ and $a = 3$ in $(*)$.

For $(X - 3Z = 0) \cdot \mathcal{C}$ we have :

$$\begin{cases} X - 3Z = 0 \\ \\ Y^2Z^3 = X^5 - 243Z^5 \end{cases} \Rightarrow \begin{cases} X = 3Z \\ \\ Y^2Z^3 = 0 \end{cases}$$
$$\Rightarrow \begin{cases} X = 3Z \\ \\ Y^2 = 0 \ \text{ or } \ Z^3 = 0 \end{cases}$$

hence $Y = 0$ with multiplicity 2 or $Z = 0$ with multiplicity 3. Thus the intersection points of the curve of equation $X - 3Z = 0$ and $\mathcal{C}$ are of the form $(3Z, 0, Z) = Z(3, 0, 1)$ or $(0, Y, 0) = Y(0, 1, 0)$.
We thus find the points $P = (3, 0, 1)$ with multiplicity 2 for $Z = 1$ and $\infty = (0, 1, 0)$ with multiplicity 3 for $Y = 1$. Thus $(X - 3Z = 0) \cdot \mathcal{C} = 2P + 3\infty$.
In the same way as for $(X - 3Z = 0) \cdot \mathcal{C}$ we have $(Z = 0) \cdot \mathcal{C} = 5\infty$.
We conclude that

$$div(x - 3) = 2P - 2\infty.$$

Similarly as $(i)$ we determine the following divisors :

$$(ii) \quad div(y) = B_0 + B_1 + B_2 + B_3 + B_4 - 5\infty,$$
$$(iii) \quad div(x) = A_0 + A_1 - 2\infty.$$

□

**Consequence of Lemma 2.1.** : $2j\,(P) = 0$.

**Lemma 2.2.**

- $\mathcal{L}(\infty) = \langle\ 1\ \rangle$,
- $\mathcal{L}(2\infty) = \langle\ 1,\ x\ \rangle = \mathcal{L}(3\infty)$,
- $\mathcal{L}(4\infty) = \Big\langle\ 1,\ x,\ x^2\ \Big\rangle$,
- $\mathcal{L}(5\infty) = \Big\langle\ 1,\ x,\ x^2,\ y\Big\rangle$,

- $\mathcal{L}(6\infty) = \langle\ 1,\ x,\ x^2,\ y,\ x^3 \rangle$.

**Proof**. This is a consequence of the Lemma 2.1 and of the fact that according to the Riemann-Roch theorem we have $l(m\infty) = m - 1$ as soon as $m \geq 3$.

□

**Lemma 2.3.**
*A $\mathbb{Q}$-base of $\mathcal{L}(m\infty)$ is given by :*

$$\mathcal{B}_m = \left\{ x^r \ :\ r \in \mathbb{N}\ et\ r \leq \frac{m}{2} \right\} \cup \left\{ x^s y \ :\ s \in \mathbb{N}\ et\ s \leq \frac{m-5}{2} \right\}.$$

**Proof**. Refer to [1].

□

**Lemma 2.4.**

$$J_{\mathcal{C}}(\mathbb{Q}) \cong (\mathbb{Z}\ /\ 2\mathbb{Z}) = \langle\ j(P)\ \rangle\ .$$

**Proof**. Refer to [3].

□

## 3. Proof of the Theorem

Let $R$ be a point of $\mathcal{C}\left(\overline{\mathbb{Q}}\right)$ with $[\mathbb{Q}(R) : \mathbb{Q}] = k$.
The works of Mulholland in [3] allows us to assume $k \geq 2$. Let's note $R_1,\ R_2, ...., R_k$ the Galois conjugate points of $R$ and let's work with

$$t = [R_1 + R_2 + \cdots + R_k - k\infty].$$

We have $t \in J_{\mathcal{C}}(\mathbb{Q})$ and the Lemma 2.4 gives $t = mj(P)$ with $0 \leq m \leq 1$.
Thus we obtain:

$$[R_1 + R_2 + \cdots + R_k - k\infty] = mj(P) \text{ with } 0 \leq m \leq 1. \tag{1}$$

Our proof is divided into the following two cases :

<u>**Case**</u> : $m = 0$.

The formula (1) becomes $[R_1 + R_2 + \cdots + R_k - k\infty] = 0$.
The Abel-Jacobi theorem implies the existence of a rational function f defined over $\mathbb{Q}$ such that $div(\text{f}) = R_1 + R_2 + \cdots + R_k - k\infty$.
Therefore $\text{f} \in \mathcal{L}(k\infty)$ and according to the Lemma 2.3 we have

$$\text{f}(x, y) = \left( \sum_{r \leq \frac{k}{2}} a_r x^r \right) + y \left( \sum_{s \leq \frac{k-5}{2}} b_s x^s \right).$$

At the points $R_i$ we have

$$\left(\sum_{r\leq \frac{k}{2}} a_r x^r\right) + y\left(\sum_{s\leq \frac{k-5}{2}} b_s x^s\right) = 0 \text{ so}$$

$$y = -\frac{\left(\sum_{r\leq \frac{k}{2}} a_r x^r\right)}{\left(\sum_{s\leq \frac{k-5}{2}} b_s x^s\right)} \text{ and then}$$

the relation $y^2 = x^5 - 243$ gives the following equation

$$(\mathcal{E}_0) : \left(\sum_{r\leq \frac{k}{2}} a_r x^r\right)^2 = \left(x^5 - 243\right)\left(\sum_{s\leq \frac{k-5}{2}} b_s x^s\right)^2.$$

We thus find a family of points given by :

$$\mathcal{H}_0 = \left\{ \left(x, -\frac{\sum_{r\leq \frac{k}{2}} a_r x^r}{\sum_{s\leq \frac{k-5}{2}} b_s x^s}\right) \mid a_r, b_s \in \mathbb{Q} \text{ and } x \text{ the root of the equation } (\mathcal{E}_0) \right\}.$$

**Cas**: $m = 1$.

The formula (1) becomes $[R_1 + R_2 + \cdots + R_k - k\infty] = j(P) = -j(P)$; hence $[R_1 + R_2 + \cdots + R_k + P - (k+1)\infty] = 0$.
The Abel-Jacobi theorem implies the existence of rational function f defined over $\mathbb{Q}$ such that $div(\mathrm{f}) = R_1 + R_2 + \cdots + R_k + P - (k+1)\infty$.
Therefore $\mathrm{f} \in \mathcal{L}((k+1)\infty)$ and according to the Lemma 2.3 we have

$$\mathrm{f}(x, y) = \left(\sum_{r\leq \frac{k+1}{2}} a_r x^r\right) + y\left(\sum_{s\leq \frac{k-4}{2}} b_s x^s\right).$$

The function f is of order 1 at the point $P$ so we must have $\sum_{r\leq \frac{k+1}{2}} a_r (3)^r = 0$.
At the points $R_i$ we have

$$\left(\sum_{r\leq \frac{k+1}{2}} a_r x^r\right) + y\left(\sum_{s\leq \frac{k-4}{2}} b_s x^s\right) = 0 \text{ hence}$$

$$y = -\frac{\left(\displaystyle\sum_{r \leq \frac{k+1}{2}} a_r x^r\right)}{\left(\displaystyle\sum_{s \leq \frac{k-4}{2}} b_s x^s\right)} \text{ and then}$$

the relation $y^2 = x^5 - 243$ gives the following equation

$$(\mathcal{E}_1) : \left(\sum_{r \leq \frac{k+1}{2}} a_r x^r\right)^2 = \left(x^5 - 243\right) \left(\sum_{s \leq \frac{k-4}{2}} b_s x^s\right)^2.$$

We thus find a family of points given by :

$$\mathcal{H}_1 = \left\{ \begin{array}{l} \left( x, -\frac{\displaystyle\sum_{r \leq \frac{k+1}{2}} a_r x^r}{\displaystyle\sum_{s \leq \frac{k-4}{2}} b_s x^s} \right) \mid a_r, b_s \in \mathbb{Q} \text{ satisfying } \sum_{r \leq \frac{k+1}{2}} a_r (3)^r = 0 \\ \\ \text{and } x \text{ the root of the equation } (\mathcal{E}_1) \end{array} \right\}.$$

In conclusion the set of algebraic points of any degree at most $d$ over $\mathbb{Q}$ on the curve $\mathcal{C}$ is given by :

$$\bigcup_{[K:\mathbb{Q}] \leq d} \mathcal{C}(K) = \mathcal{H}_0 \cup \mathcal{H}_1. \qquad \square$$

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