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On Operator Pencil with a Parameter and Applications

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Abstract. Spectral properties of a class of operator pencils with a parameter are studied in this work. The obtained results are used to derive the exact estimates for the norms of intermediate derivative operators in Sobolev type spaces.

Key Words and Phrases: Hilbert space, operator norms, intermediate derivatives, Sobolev type space.

Let be a separable Hilbert space and be a self-adjoint positive definite operator with the domain of definition $D(A)$. The linear set $D(A^\gamma)$ becomes a Hilbert space with respect to the norm $(x, y)_\gamma = (D^\gamma x, A^\gamma y)$, $\gamma \geq 0$. For $\gamma = 0$ we assume $(x, y)_0 = (x, y)$, $H_0 = H$.

Let

$$P_0(\lambda, A) = (\lambda E - A)^2(\lambda E + A). \tag{1}$$

For $\beta \in R = (-\infty, \infty)$, consider a polynomial operator pencil

$$P_j(\lambda, \beta, A) = P_0(\lambda, A)P_0(-\lambda, A) - \beta(i\lambda)^{2j} A^{6-2j}, j = 1, 2, . \tag{2}$$

The following theorem is true:

Theorem 1. For $\beta \in \left(0, \left(\frac{27}{4}\right)^{1/2}\right)$, the operator pencil $P_j(\lambda, \beta, A)$ $(j = 1, 2)$ has no spectrum on the imaginary axis, and it can be represented as follows:

$$P_j(\lambda, \beta, A) = F_j(\lambda, \beta, A) \cdot F_j(-\lambda, \beta, A) \quad (j = 1, 2), \tag{3}$$

where

$$F_j(\lambda, \beta, A) = \prod_{k=1}^{3} (\lambda E - \omega_{k,j}(\beta)A) = \lambda^3 E + a_2(\beta)\lambda^2 A + a_1(\beta)\lambda A^2 + A^3, \tag{4}$$

with $Re\omega_{k,j}(\beta) < 0$, $a_{1,j}(\beta) > 0$, $a_{2,j}(\beta) > 0$, $j = 1, 2$, for every $\beta \in \left(0, \left(\frac{27}{4}\right)^{1/2}\right)$, and

$$a_{1,1}^2(\beta) - 2\,a_{2,1}(\beta) = 3 - \beta, \quad a_{2,1}^2(\beta) - 2\,a_{1,1}(\beta) = 3, \quad \text{for} \quad j = 1, \tag{5}$$

$$a_{1,2}^2(\beta) - 2\,a_{2,2}(\beta) = 3, \quad a_{2,2}^2(\beta) - 2\,a_{1,2}(\beta) = 3 - \beta, \quad \text{for} \quad j = 2. \tag{6}$$

Proof. Let $\beta \in \sigma()$. Then

$$P_j(\lambda, \beta, \mu) = P_0(\lambda, \sigma) P_0(-\lambda, \sigma) - \beta (i\lambda)^{2j} \sigma^{6-2j}, \ \ j = 1, 2.$$

For $\lambda = i\xi, \ \xi \in R = (-\infty, \infty)$ we have

$$\begin{aligned} &P_j(\lambda, \beta, \mu) = (i\xi - \sigma)^2 (i\xi + \sigma)(-i\xi - \sigma)^3 (-i\xi + \sigma) - \beta \xi^{2j} \sigma^{6-2j} = \\ &= (i\xi - \sigma)^2 (i\xi + \sigma)^3 - \beta \xi^{2j} \mu^{6-2j} = (\xi^2 + \sigma^2)^3 - \beta \xi^{2j} \sigma^{6-2j} = \\ &= (\xi^2 + \sigma^2)^3 \left(1 - \beta \tfrac{\xi^{2j} \sigma^{6-2j}}{(\xi^2 + \sigma^{6-2j}}\right) > (\xi^2 + \sigma^2)^3 \left(1 - \beta \sup_{s \geq 0} \tfrac{s^j}{(1+s)^3}\right) \geq \\ &\geq (\xi^2 + \sigma^2)^3 \left(1 - \beta \tfrac{27}{2}\right) > 0 \ . \end{aligned}$$

So, the polynomials $P_j(\lambda, \beta, \sigma)$, $j = 1, 2$ have no roots on the imaginary axis. On the other hand, if $P_j(\lambda_0, \beta, \sigma) = 0$, then $P_j(-\lambda_0, \beta, \sigma) = 0$. Since $P_j(\lambda, \beta, \sigma)$ is a polynomial with real coefficients, $\bar{\lambda}_0$ is also a root of the polynomial $P_j(\lambda, \beta, \sigma)$. Therefore, for $\beta \in \left(0, \left(\frac{27}{4}\right)^{1/2}\right)$ the polynomial $P_j(\lambda, \beta, \sigma)$ can be represented as follows:

$$P_j(\lambda, \beta, \sigma) = F_j(\lambda, \beta, \sigma) \cdot F_j(-\lambda, \beta, \sigma), \quad j = 1, 2; \ , \tag{7}$$

where

$$F_j(\lambda, \beta, \sigma) = \prod_{k=1}^{3} (\lambda - \omega_{k,j}(\beta)\sigma) = a_{3,j}(\beta)\lambda^3 E + a_{2,j}(\beta)\lambda^2 \sigma + \\ + a_{1,j}(\beta)\lambda^2 \sigma + a_{0,j}(\beta)\sigma^3. \tag{8}$$

On the other hand, it is clear that $a_{3,j}(\beta) > 0$, and, by Vieta theorem, $a_{1,j}(\beta) > 0, \ a_{2,j}(\beta) > 0, \ a_{0,j}(\beta) > 1$. But $a_{0,j}^2(\beta) = 1$, so $a_{0,j}(\beta) = 1$.

From (7),(8), using the spectral decomposition of the operator , we get the validity of the equalities (4) and (8). Comparing the coefficients of λ in (7), we get the validity of the equalities (5) and (7).

The theorem is proved.

Let $L_2(R_+; H)$ be a Hilbert space of vector functions $f(t)$ defined for almost every $R_+ = (0, \infty)$ with the norm

$$\|f\|_{L_2(R_+;H)} = \int_0^\infty \|f(t)\|^2 \, dt < \infty.$$

Following [3], define the Hilbert space

$$W_2^3(R_+; H) = \{u; u^{(3)} \in L_2(R_+; H), \ A^3 u \in L_2(R_+; H)\}$$

equipped with the norm

$$\|u\|_{W_2^3(R_+;H)} = \left(\left\| u^{(3)} \right\|^2_{L_2(R_+;H)} + \left\| A^3 u \right\|^2_{L_2(R_+;H)} \right)^{\frac{1}{2}},$$

and denote it by

$$\overset{0}{W}_2(R_+;H)=\{u: u\in W_2^3(R_+;H),\ u^{(v)}(0)=0,\ \ v=\overline{0,2}\}.$$

Obviously, by the trace theorem stated in [3], $\overset{0}{W}_2(R_+;H)$ is a complete subspace of the space $W_2^3(R_+;H)$.

The spaces $L_2(R;H)$ and $W_2^3(R;H)$ for $R=(-\infty,\infty)$ are introduced similarly (see [3]).

It was proved in [ ] that

$$N_j(R)=\sup_{0\neq u\in W_2^3(R;H)}\frac{\left\|A^{3-j}u\right\|_{L_2(R;H)}}{\|P_0(d/dt)u\|_{L_2(R;H)}}=\left(\frac{4}{27}\right)^{\frac{1}{2}},\ j=1,2. \tag{9}$$

Now let's find the following norms in the space $\overset{0}{W}_2(R_+;H)$:

$$\overset{0}{N}_j(R_+)=\sup_{0\neq 8\in\overset{0}{W}_2(R_+;H)}\frac{\left\|Au^{(j)}\right\|_{L_2(R_+;H)}}{\|P_0(d/dt)u\|_{L_2(R_+;H)}},\ \ j=1,2. \tag{10}$$

First, let's prove the theorem below:

Theorem 2. Let $8\in\overset{0}{W}_2(R_+;H)$. Then, for every $\beta\in\left(0,\left(\frac{27}{4}\right)^{1/2}\right)$ we have

$$\|P_0(d/dt)u\|^2_{L_2(R_+;H)}-\beta\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}=\|F_j(d/dt;\beta;A)u\|^2_{L_2(R_+;H)}. \tag{11}$$

Proof. In fact, the simple calculations for $8\in\overset{0}{W}_2(R_+;H)$ $(8^{(\nu)}=0,\ \nu=1,2,3)$ show that

$$\begin{aligned}&\|F_j(d/dt;\beta;A)\|^2_{L_2(R_+;H)}=\left\|u^{(3)}+a^2_{2,j}(\beta)Au''+a_{1,j}(\beta)A^2u'+A^3u\right\|^2_{L_2(R_+;H)}=\\&=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+a^2_{2,j}(\beta)\left\|Au''\right\|^2_{L_2(R_+;H)}+a_{1,j}(\beta)\left\|A^2u'\right\|^2_{L_2(R_+;H)}+\left\|A^3u\right\|^2_{L_2(R_+;H)}=\\&=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+\left\|A^3u\right\|^2_{L_2(R_+;H)}+(a^2_{2,j}(\beta)-2a_{1,j}(\beta))\left\|Au''\right\|^2_{L_2(R_+;H)}+\\&\qquad+(a^2_{1,j}(\beta)-2a_{1,j}(\beta))\left\|A^2u'\right\|^2_{L_2(R_+;H)}.\end{aligned} \tag{12}$$

On the other hand,

$$\|P_0(d/dt)u\|^2_{L_2(R_+;H)}=\left\|u^{(3)}\right\|^2+3\left\|Au''\right\|^2_{L_2(R_+;H)}+3\left\|A^2u'\right\|^2_{L_2(R_+;H)}. \tag{13}$$

Thus, considering (13) in (12), we obtain

$$\begin{aligned}&\|F_j(d/dt;\beta;A)8\|^2=\|P_0(d/dt)u\|^2_{L_2(R_+;H)}+(a^2_{1,j}(\beta)-2a_{1,j}(\beta)-3)\left\|Au''\right\|^2_{L_2(R_+;H)}+\\&+(a^2_{1,j}(\beta)-2a_{2,j}(\beta)-3)\left\|Au'\right\|^2_{L_2(R_+;H)}.\end{aligned}$$

By (5) and (7), we get the validity of the theorem.

Theorem 3. The following relations are true:

$$\overset{0}{N}_j(R_+) = \left(\frac{4}{27}\right)^{\frac{1}{2}}, \quad j = 1, 2.$$

Proof. From (11) it follows that for every $8 \in \overset{0}{W}{}_2^3(R_+; H)$ the inequality

$$\|P_0(d/dt)u\|^2_{L_2(R_+;H)} > \beta \left\|A^{3-j}u^{(j)}\right\|^2_{L_2(R_+;H)}, \ j = 1, 2,$$

holds. Passing to the limit as $\beta \to \left(\frac{27}{4}\right)^{\frac{1}{2}}$, we obtain

$$\overset{0}{N}_j(R) \leq \left(\frac{4}{27}\right)^{\frac{1}{2}}. \tag{14}$$

Let's show that this inequality is in fact an equality.

By the definition of $N_j(R)$ ($j = 1, 2$), for every $\varepsilon > 0$ there exists a function $v_0(t) \in W_2^2(R, H)$ such that

$$\left\|A^{3-j}v_0(t)\right\|_{L_2(R;H)} > (N_j(R) - \varepsilon) \left\|P_0(d/dt)v_0(t)\right\|^2_{L_2(R;H)}.$$

Now let's find the function $v_k(t) \in W_2^3(R, H)$ with support in $[-k, k]$ such that $v_k(t) \to v_0(t)$ with respect to the norm of $W_2^3(R, H)$. Then, for large ?,

$$\left\|A^{n-j}v_k(t)\right\|_{L_2(R;H)} > (N_j(R) - \varepsilon) \left\|P_0(d/dt)v_k(t)\right\|_{L_2(R;H)}, \ k \geq n_0.$$

Now let's consider the function

$$8_{n_0}(t) = v_{n_0}(t - k) \in \overset{0}{W}{}_2^3(R_+; H).$$

Obviously, $\|8_{n_0}(t)\|_{L_2(R_+;H)} = \|v_{n_0}(t)\|_{L_2(R_+;H)}$, $\|P(d/dt)u_{n_0}\|_{L_2(R_+;H)} = = \|P(d/dt)v_{n_0}(t)\|_{L_2(R;H)}$. Therefore we have

$$\left\|A^{n-j}u_{n_0}\right\|_{L_2(R_+;H)} \geq \left(\left(\frac{4}{27}\right)^{\frac{1}{2}} - \varepsilon\right) \left\|P(d/dt)u_{n_0}(t)\right\|_{L_2(R_+;H)}.$$

Taking into account the inequality (14), we obtain

$$\overset{0}{N}_j(R_+) = \left(\frac{4}{27}\right)^{\frac{1}{2}}, \ j = 1, 2.$$

The theorem is proved.

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