Journal of Contemporary Applied Mathematics
V. 14, No 1, 2024, July
ISSN 2222-5498
DOI 10.5281/zenodo.10908206

Fundametal Solution of a Cauchy and Goursat Type Problem for Second Order Hyperbolic Equations and Related Optimal Control Problems

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Abstract. In the paper it is proved existence and uniqeness of the fundamental solution of Cauchy and Gourset problem with measurable and bounded coefficients for one type of hyperbolic equation of second order and integral representation of the considered problem was found by using it.

Key Words and Phrases: fundamental solution, S.L.Sobolev, Cauchy, Goursat spaces.
2010 Mathematics Subject Classifications: Primary 41A35, 47A58.

1. Introduction

In this paper, for a second order, hyperbolic equation we study a problem stated on lines consisting of finitely many characteristic and non-characteristic pieces. Thus problem includes both the Cauchy problem and the Goursat problem, as a special case for a second order hyperbolic equation.

Under very weak constraints as boundedness and summability on the coefficients for this problem, we introduce the notion of θ fundamental solution that in a natural way generalizes the notion of the Riemann function to the case of second-order hyperbolic equations with nonsmooth coefficients and allows to find integral representation of the solution of an inhomogeneous problem. The concept of a θ fundamental solution was first introduced in the general case in [1] by S.S.Akhiyev. Furthermore, be have found sufficient conditions under which this problem is everywhere well-defined solvable together with its coujugated system and there exists a unique θ fundamental solution. This problem includes as a particular case the Cauchu problem and the Goursat problem [2,3].

2. Problem statement

On a rectangular $G = G_1 \times G_2$, $G_1 = (x_0, x_1)$, $G_2 = (y_0, y_1)$ of the plane $X0Y$ we consider a continuous line Γ located on $\overline{G}$, connecting the points (x_0, y_1) and (x_1, y_0), and satisfying the following conditions: with all the straight lines of the form $x = \alpha = const$ and $y = \beta = const$ $\left(\alpha \in \overline{G}_1,\, \beta \in \overline{G}_2\right)$ the line Γ has a single point of intersection with the possible exception of finitely many lines $x = \alpha_k$, $k = 1, 2, ..., N_1$ and $y = \beta_k$, $k = 1, 2, ..., N_2$, $\left(\alpha_k \in \overline{G}_1, \beta_k \in \overline{G}_2\right)$, with which Γ may intersect along some segments. We call any such line a monotone line. We will consider monotone lines that can be determined as a union

$$\left\{(x, S_\Gamma(x)) \Big|' x \in \overline{G}_1\right\} \quad U \quad \left\{(\nu_\Gamma(y), y) \Big|' y \in \overline{G}_2\right\}$$

graphs of some pair of monotonically non-increasing function $y = S_\Gamma(x)$ and $x = \nu_\Gamma(y)$ being continuous on $\overline{G}_1$ and $\overline{G}_2$, with the possible exception of finitely many points $x = \alpha_k \in \overline{G}_2$, $k = 1, 2, ..., N_1$ and $y = \beta_k \in \overline{G}_2$, $k = 1, 2, ..., N_2$, in which they may have discontinuities of the first kind (in discontinuity points their values are considered equal to one of the limits at that point on the right or left). This time the function $x = \nu_\Gamma(y)$ can be considered as a generalized inverse function for the function $y = S_\Gamma(x)$ and vice versa.

Such monotone lines, in particular, can be composed as a union of finitely many pieces of straight lines parallel to coordinate axes.

Let $W_p^{(1,1)}(G)$, $1 \le p \le \infty$ be we a space of all $u \in L_p(G)$ with S.L.Sobolev generalized derivatives $D_x^i D_y^j u \in L_p(G)$, $i, j = 0, 1$, where $D_z = \frac{\partial}{\partial z}$. The space $W_p^{(1,1)}(G)$ is a Banach space in the norm

$$\|u\|_{W_p^{(1,1)}(G)} = \sum_{i,j=0}^{1} \left\|D_x^i D_y^j u\right\|_{L_p(G)}.$$

Let us consider the equation

$$\begin{gathered}\left(L^0 u\right)(x, y) = u_{xy}(x, y) + a_{1,0}(x, y)\, u_x(x, y) + \\ + a_{0,1}(x, y)\, u_y(x, y) + a_{0,0}(x, y)\, u(x, y) = \varphi^0(x, y), \ (x, y) \in G,\end{gathered} \tag{1}$$

where $\varphi^0 \in L_p(G)$

Let $a_{i,j}(x, y)$ be measurable on G and there exist such functions $a_{1,0}^0 \in L_p(G_{-2})$, $a_{0,1}^0 \in L_p(G_{-1})$ that $|a_{1,0}(x, y)| \le a_{1,0}^0(y)$, $|a_{0,1}(x, y)| \le a_{0,1}^0(x)$, $a_{0,0} \in L_p(G)$.

Under the imposed conditions, the operator L^0 of the equation (1) acts from $W_p^{(1,1)}(G)$ to $L_p(G)$ and is bounded. On some monotone line Γ for the equation (1) we give the conditions

$$\begin{array}{l}(L_1 u)(x) \equiv u_x(x, y)\big|_{y = S_\Gamma(x)} = \varphi_1(x), \ x \in G_1 \\ (L_2 u)(y) \equiv u_y(x, y)\big|_{y = \nu_\Gamma(y)} = \varphi_2(y), \ y \in G_2\end{array} \tag{2}$$

$$L_0 u \equiv u(x_0, y_1) = \varphi_0,$$

where $\varphi_1 \in L_p(G_1)$, $\varphi_2 \in L_p(G_2)$ and $\varphi_0 \in R$ are the given elements.

The operations of taking the traces are continuous from L_1, L_2, L_0 $W_p^{(1,1)}(G_1)$ to $L_p(G_1), L_p(G_2), R$, respectively. Therefore , the operator $L = \left(L_0, L_1, L_2, L^0\right)$ of problem (1), (2) acts from $W_p^{(1,1)}(G)$ to $E_p^{(1',1)} = R \times L_p(G_1) \times L_p(G_2) \times L_p(G)$ and is bounded.

We can write the problem (1), (2) also in the from of the equivalent operator equation

$$Lu = \varphi, \tag{3}$$

where $\varphi = \left(\varphi_0, \varphi_1, \varphi_2, \varphi^0\right) \in E_p^{(1,1)}$.

We will call the solution of problem (1), (2) the function $u \in W_p^{(1,1)}(G)$ for which the equality (1) is fulfilled almost everywhere on G, the first and second equality from (2) almost everywhere on G_1 and G_2, and also the third equality from (2) in the usual sense.

We call problem (1), (2) a Cauchy and Goursat type problem. If $\Gamma = \left\{(x, \gamma(x)) \left| x \in \overline{G}_1 \right.\right\}$, the problem (1), (2) is equivalent to the Cauchy problem of classic form, where $\gamma(x)$ is a continuously differentiable function on $\overline{G}_1$ $\gamma'(x) < 0,\ \gamma(x_0) = y_1,\ \gamma(x_1) = y_0$ (in this case the line Γ is non-characteristic).

If

$$\Gamma = \{(x,y)\,|y = S_\Gamma(x) = y_0,\ x \in [x_0, x_1]\}\, U\, \{(x,y)\,|x = \nu_\Gamma(y) = x_0,\ y \in [y_0, y_1]\}$$

or

$$\Gamma = \{(x,y)\,|y = S_\Gamma(x) = y_1,\ x \in [x_0, x_1]\}\, U\, \{(x,y)\,|x = \nu_\Gamma(y) = x_1, y \in [y_0, y_1]\},$$

the problem (1),(2) is equivalent to the Goursat problem of classic form (in this case Γ is a characteristic line) In this sense, the problem (1), (2) includes the Cauchy and Goursat problem as a special case.

If the problem (1), (2) for any $\varphi \in E_p^{(1,1)}$ has a unique solution $u \in W_p^{(1,1)}(G)$ and this time

$$\|u\|_{W_p^{(1,1)}}\ (G)\ \leq M\, \|\varphi\|_{E_p^{(1,1)}},$$

where $\|\varphi\|_{E_p^{(1,1)}} = |\varphi_0| + \|\varphi_1\|_{L_p(G_1)} + \|\varphi_2\|_{L_p(G_2)} + \left\|\varphi^0\right\|_{L_p(G)}$

and M is positive constant independent of φ, we say that problem (1), (2) is everywhere well-defined solvable.

Note that everywhere well-defined solvability of problem (1) (2) is equivalent to the fact that its operator L has a bounded inverse L^{-1} determined on $E_p^{(1,1)}$.

3. Constracting an adjoint operator

For the operator L of the problem (1), (2) we find an explicit form of the conjugated operator L^*.

Let us take some functional $f \in E_q^{(1,1)}$, $\frac{1}{p} + \frac{1}{q} = 1$ bounded on $E_p^{(1,1)}$. Then f is of the form $f = \left(f_0, f_1(x), f_2(x), f^0(x,y)\right)$, (1) where $f_1 \in L_q(G_1)$, $f_2 \in L_q(G_2)$, $f^0 \in L_p$ and by the definition we have:

$$\begin{aligned} f(Lu) = u(x_0, y_1) f_0 + \textstyle\int_{G_1} D_1 u(x, S_\Gamma(x)) f_1(x)\, dx + \\ + \textstyle\int_{G_2} D_2 u(\nu_\Gamma(y), y) f_2(y)\, dy + \iint_G \left(L^0 u\right)(x,y) f^0(x,y)\, dxdy \end{aligned} \tag{4}$$

Taking into account here the expression of the operator L^0 from (1), we reduce the right hand side of the equality (4) to the form

$$
\begin{gathered}
f(Lu) = (V_0 f) \cdot u(x_1, y_1) + \int_{G_1} (V_1 f)(x) D_1 u(x, y_1) + \\
+ \int_{G_2} (V_2 f)(y) D_2 u(x_1, y) dy + \iint_G \left(V^0 f\right)(x, y) D_1 D_2 u(x, y) dxdy,
\end{gathered} \tag{5}
$$

where

$$
\begin{gathered}
V_0 f = f_0 + \iint_G a_{0,0}(x, y) f^0(x, y) dxdy, \\
(V_1 f)(\tau) = f_1(\tau) - \iint_G \theta(\tau - x) f^0(x, y) a_{0,0}(x, y) dxdy + \\
+ \int_{G_2} f^0(\tau, y) a_{1,0}(\tau, y) dy - f_0, \ \tau \in G_1; \\
(V_2 f)(\xi) = f_2(\xi) - \iint_G \theta(\xi - y) a_{0,0}(x, y) f^0(x, y) dxdy + \\
+ \int_{G_1} a_{0,1}(x, \xi) f^0(x, \xi) dx, \ \xi \in G_2 \\
\left(V^0 f\right)(\tau, \xi) = f^0(\tau, \xi) - \theta(\xi - S_\Gamma(\tau)) f_1(\tau) - \\
-\theta(\tau - \nu_\Gamma(\xi) f_2(\xi)) + \iint_G \theta(\tau - x) - (\xi - y) a_{0,0}(x, y) f^0(x, y) dx
\end{gathered} \tag{6}
$$

where $\theta(t)$ is a Heaviside function on R.

Comparing equalities (4), (5) and taking into account general form of linear bounded functionals on $W_p^{(1,1)}(G)$ (see [1]), we obtain that L has a conjugated L^* that acts to $E_q^{(1,1)}$, is bounded and is of the form $L^* = \left(V_0, V_1, V_2, V^0\right)$, where the operators V_0, V_1, V_2, V^0 are determined by means of the equalities (5). Hence, it follows that the equation

$$
L^* f = \psi \tag{7}
$$

conjugated to the problem (1),(2), i.e. to the equation (3) can be written in the form of equivalent system of equations

$$
\begin{cases}
V_0 f = \psi_0; \\
(V_1 f)(x) = \psi_1(x), x \in G_1; \\
(V_2 f)(y) = \psi_2(y), y \in G_2; \\
\left(V^0 f\right)(x, y) = \psi^o(x, y), (x, y) \in G
\end{cases} \tag{8}
$$

where $\left(\psi_0, \psi_1, \psi_2, \psi^0\right) \in E_q^{(1,1)}$is the given, $f = \left(f_0, f_1, f_2, f^0\right) \in E_q^{(1,1)}$ are the desired elements.

4. Existence and uniqueness of the solution of problem (1), (2) and its adjoint system

Let us consider the fourth equation of the system (8) first on the domain $G^+ = \{(x, y) \,| y > S_\Gamma(x), x \in G_1\}_1$. Then we can reduce the fourth equation of the system (8) to the form

$$\begin{array}{l}\left(N_{(x_1,y_1)}f^0\right)(\tau,\xi)\equiv f^0(\tau,\xi)+\int_\tau^{x_1}\int_\xi^{y_1}a_{0,0}(x,y)\,f^0(x,y)\,dxdy+\\ +\int_{\xi_1}^{y_1}a_{1,0}(\delta,y)\,f^0(\tau,y)\,dy+\int_\tau^{x_1}a_{0,1}(x,\xi)\,f^0(x,\xi)\,dx\\ =\hat{\psi}(\tau,\xi)\equiv\psi^0(\tau,\xi)+\psi_1(\tau)+\psi_2(\xi)+\psi_0,\ (\tau,\xi)\in G^+\end{array}\qquad(9)$$

But if $(\tau,\xi)\in G^-=G/G^+$, then

$$\theta(\xi-S_\Gamma(\tau))=\theta(\tau-\nu_\Gamma(\xi))=0$$

Therefore, on G^- the fourth equation of the system (8) will take the form

$$\left(N_{(x_0,y_0)}f^0\right)(\tau,\xi)=\varphi^0(\tau,\xi)\,,\ (\tau,\xi)\in G^-\qquad(10)$$

The operators $N_{(x_1,y_1)}$ and $N_{(x_0,y_0)}$ are bounded operators acting in the spaces $L_q(G^+)$ and $L_q(G^-)$, respectively. Furthermore, $N_{(x_1,y_1)}u\,N_{(x_0,y_0)}$ are Volterra operators in negative and positive directions of variables $\tau,\ \xi$, respectively . Using this we can show that each of these equations (9) and (10) has a unique solution $f_+^0\in L_q(G^+)$ and $f_-^0\in L_q(G^-)$

This shows that the solution of the system (8) is equivalent to the solution of the pair of independent equations (9) and (10) having unique solutions f_+^0 and f_-^0 . The inverse is also valid in the sense that if $f_+^0\in L_q(G^+)$ and $f_-^0\in L_q(G^-)$ are the solution of the equations (9) and (10), the function $f^0(\tau,\xi)=f_\pm(\tau,\xi)\,,(\tau,\xi)\in G^\pm$ determines some solution $f=\left(f_0,f_1(\tau)\,,f_2(\xi)\,,f^0(\tau,\xi)\right)$ of the system (8) from $E_q^{(1,1)}$. Hence it follows that the system (8) for any $\psi\in E_q^{(1,1)}$ has a unique solution $f\in E_q^{(1,1)}$. Applying the Banach theorem, we obtain that L^* has a bounded inverse $(L^*)^{-1}$ acting to $E_q^{(1,1)}$. Since $(L^*)^{-1}$ is bounded, then L has a bounded inverse L^{-1} determined on $E_q^{(1,1)}$. Thus , we proved

5. Existence and uniqueness of θ fundamental solution and representation of the solution

Theorem 5.1. *The problem* (1), (2) *and its conjugated system* (8) *are unconditionally everywhere well-defined solvable.*

Now for each fixed point $(x,y)\in\overline{G}$ we consider the system of equations

$$\begin{array}{l}V_0f=1;\ \ (v,f)(\tau)=-\theta(\tau-x)\,,\ \tau\in G_1;\\ (V_2f)(\xi)=-\theta(\xi-y)\,,\xi\in G_2;\\ \left(V^0f\right)(\tau,\xi)=\theta(\tau-x)\,\theta(\xi-y)\,,\ (\tau,\xi)\in G,\end{array}\qquad(11)$$

where $f=\left(f_0,f_1(\tau)\,,f_2(\xi)\,,\,f^0(\tau,\xi)\right)$ is the desired four from $E_q^{(1,1)}$, while $(x,y)\in\overline{G}$ is a fixed point.

If the system for each fixed point $(x,y)\in\overline{G}$ has at least one solution $f(x,y)=\left(f_0(x,y)\,,f_1(.,x,y)\,,\,f_2(.,x,y)\,,\,f^0(.,x,y)\right)\in E_q$, we call this solution the θ- fundamental solution of problem (1), (2).

Theorem 5.2. *Problem* (1), (2) *has a unique θ fundamental solution*

$f(x,y)=\left(f_0(x,y), f_1(\tau,x,y), f_2(\xi,x,y), f^0(\tau,\xi,x,y)\right)$ *from* $E_q^{(1,1)}$ *and this time the solution* $u\in W_p^{(1,1)}(G)$ *of problem* (1), (2) *is represented in the form*

$$\begin{array}{l} u(x,y)=f_0(x,y)\varphi_0+\int_{G_1} f_1(\tau,x,y)\varphi_1(\tau)\,d\tau+ \\ +\int_{G_2} f_2(\xi,x,y)\varphi_2(\xi)\,d\xi+\iint_G f^0(\tau,\xi,x,y)\varphi^0(\tau,\xi)\,d\tau d\xi, \end{array} \tag{12}$$

Proof. According to Theorem 1, system (8) for any right-hand side of $\psi\in E_q$ has a unique solution $f\in E_q$. Therefore, the system (11) has a unique solution $f(x,y)\in E_q$. The existence and uniqueness of the θ-fundamental solution is proved. To prove equality (12), it suffices to compare the right-hand sides of equalities (11) on the solutions $u\in W_p^{(1,1)}(G)$ and $f(x,y)\in E_q^{(1,1)}$ of problem (1) (2) and system (11).

The theorem is proved.

6. Optimally can L.S. Pontraqinin maximum principia

Let as consider an optimal control problem: to minimize the functional

$$S(\hat{\upsilon})=\phi(u(x_0,y_0)) \tag{13}$$

on the solutions $u\in W_p^{(1,1)}(G)$ of the equation

$$\begin{array}{l} \left(L^0u\right)\equiv u_{xy}(x,y)+a_{1,0}(x,y)u_x(x,y)+a_{0,1}(x,y)uy(x,y)+ \\ +a_{0,0}(x,y)u(x,y)=\varphi^0(x,y,\upsilon(x,y)),(x,y)\in G \end{array} \tag{14}$$

satisfying the following conditions

$$\begin{array}{l} (L_1u)(x)\equiv u_k(x,y)\left|_{y=s_\Gamma(x)}\right.=\varphi_1(x,\upsilon_1(x)),\ x\in G_1 \\ (L_2u)(y)\equiv u_y(x,y)\left|_{x=\gamma_\Gamma(y)}\right.=\varphi_2(y,\upsilon_2(y)),\ y\in G_2 \end{array}$$

$$\begin{array}{l} L_0u=u(x_0,y_1)=\varphi_0(\upsilon_0); \\ G=G_1\times G_2=(x_0,h_1)\times(y_0,h_2), \\ \varphi(x,y,\upsilon),\ \varphi_1(x_1,\upsilon_1),\ \varphi_2(x_2,\upsilon_2)\text{ and } u\varphi_0(\upsilon_0) \end{array} \tag{15}$$

The given functions determined on

$$G\times R^r,\ G_1\times R^{r_1},\ G_2\times R^{r_2}8\ R^{r_0};\ \upsilon(x,y)=$$

$$=(\upsilon_1(x,y),\upsilon_2(x,y),...,\upsilon_r(x,y)),\upsilon_1(x)=$$

$$(\upsilon_{11}(x),\upsilon_{12}(x),...,\upsilon_{1r_1}(x)),\ \upsilon_2(y)=(\upsilon_{21}(y),\upsilon_{22}(y),...,\upsilon_{2r_2}(y))$$

are control functions , while $\upsilon_0=(\upsilon_{01},\upsilon_{02},...,\upsilon_{0r_0})$ is a control vector.

We call the four of vectors $(\upsilon(x,y),\upsilon_1(x),.\upsilon_2(y),\ \upsilon_0)$ admissible control if $\upsilon(x,y),\upsilon_1(x),.\upsilon_2(y)$ are measurable bounded functions with the values from some given sets $V\subset R^r, V_1\subset R^{r_1}, V_2\subset R^{r_2}$.

$(\upsilon(x,y) \in V, \upsilon_1(x) \in V_1,\ \upsilon_2(y) \in V_2)\ U\upsilon_0 \in V_0 \subset R^{r_0}\ V_1 \subset R^{r_1}, V_2 \subset R^{r_2}$ $(\upsilon(x,y) \in V, \upsilon_1(x) \in V_1,\ \upsilon_2(y) \in V_2)$ and $\upsilon_0 \in V_0 \subset R^{r_0}$

Assume that $\varphi(x,y,\upsilon)$, $\varphi_1(x_1,\upsilon_1)$, $\varphi_2(x_2,\upsilon_2)$ on $G \times R^r$, $G_1 \times R^{r_1} R^{r_2}$ and $G_2 \times R^{r_2}$ satisfy the Caratheodory conditions and for any $\ell > 0$ there exist such functions $\varphi_e^0 \in L_p(G)$, $\varphi_{1,\ell}^0 \in L_p(G_1)$ and $\varphi_{2,\ell}^0 \in L_p(G_2)$ that $|\varphi(x,y,\upsilon)| \le \varphi_\ell^0(x,y)$, $|\varphi_1(x,\upsilon_1)| \le \varphi_{1,\ell}^0(x)$ and $|\varphi_2(y,\upsilon_2)| \le \varphi_{2,\ell}^0(y)$ almost for all $x \in G$ and for all $\|\upsilon\| \le \ell, \|\upsilon_1\| \le \ell_1 u \|\upsilon_2\| \le \ell_2$. Furthermore, $\varphi_0(\upsilon)$ is continuous on R_0^r.

Under the conditions of the theorem, problem (14), (15) for any admissible control $(\upsilon(x,y), \upsilon_1(x), \upsilon_2(y), \upsilon_0)$ has a unique solution and $u \in W_p^{(1,1)}(G)$. Therefore, we can consider the problem of minimizing the functional (13) on the set of solutions to equation (14) that satisfy the given conditions (15).

Here it is assumed that $\phi(u)$ is continuous together with the derivative $\phi_u(u)$.

Assume

$$H(\tau,\zeta,\lambda,\upsilon) = \lambda\varphi(\tau,\zeta,\upsilon),\ H_1(\tau,\lambda_1,\upsilon_1) = \lambda_1 H_1(\tau,\upsilon),$$

$$H_2(\zeta,\lambda_2,\upsilon_2) = \lambda_2 H_2(\zeta,\upsilon_2)$$

and $H_0(\lambda_0,\upsilon_0) = \lambda_0\varphi_0(\upsilon_0)$; where λ, λ_1, $\lambda_2 u\,\lambda_0$ are determined as the solution of the system of equations.

Theorem 6.1. *For the optimality of the admissible control* $\hat{\upsilon} = (\upsilon(x,y), \upsilon_1(x), \upsilon_2(y), \upsilon_0)$*, it is necessary, and in the case when* $\phi^0(u)$ *is a convex function, it is sufficient that the following conditions be fulfilled.*

$$\begin{aligned}
&\max_{\upsilon\in V} H(\tau,\zeta,\lambda(\tau,\zeta)\upsilon) = H(\tau,\zeta,\lambda(\tau,\zeta), \upsilon(\tau,\zeta)),\\
&\max_{\upsilon_1\in V_1} H_1(\tau,\lambda_1(\tau),\upsilon_1) = H(\tau,\lambda(\tau), \upsilon_1(\tau)),\\
&\max_{\upsilon_2\in V_2} H_2(\zeta,\lambda_2(\zeta),\upsilon_2) = H(\zeta,\lambda_2(\zeta), \upsilon_2(\zeta)),\\
&\max_{\mu\in V_0} H_0(\lambda_0,\mu) = H_0(\lambda_0,\upsilon_0).
\end{aligned}$$

We call the totality of these conditions the L.S. Pontryagin maximum principle. Various special cases were earlier considered in works [2,3,4]. The application of the θ fundamental solution to optimal controls have been studied in the papers [5,6,7,8] as well.

References

[1] S.S. Akhiev, Representation of solutions of some linear operator equations. *DAN SSSR,* **251** (1980), no.5, 1037-1040.

[2] S. S.Akhiev, O.M. Guseynov, Vestnik Bakinskogo Universitet. *Baku Univ.* (1992), no. 1, 152-155.

[3] S.S Akhiev, A pariori estimate for discontinuous solutions of a second order linear hyperbolic problem. *Electron. J. Qual. Theory Differ. Equ.* (2010), No. 23, 1-12. http www.math.u-szeger.hu / ejqte

[4] S.S. Akhiev, F.Sh. Akhmedov, A pariori estimate of the solution of a linear algebraic system. *Development of Eucation, Science and Business: Result 2020: abstract of the International Scientific and Practical Internet Conference. December 3-4. 2020-Ukraine, Dnepr,* P,1, 638 p.

[5] S.S.Akhiev, F.Sh.Akhmedov, Fundamental solutions of a linear nonlocal boundary value problem for a hyperbolic semi loaded equation. *Integration of Education, Science and Business: in Modern Environment: Winter Debates: Proceedings of the 3 rd. Integration Scientific and Practical Internet Conference. February* 3-4. 2020-Ukraine 463 p. ISBN 978-617-95218-3-6

[6] O.M. Guseynov, N. Ismayilov, R.Mamedov, I. Kubishev, A.Gasimova, THERMAN 2015 and 4^{nd} Rostocker *International symposium.* Baku, Azerbaijan, (2015) 77-78.

[7] O.M. Guseynov, International conference, *Khazar University, Baku, Azerbaijan,* (2019) 158-160.

[8] O.M. Guseynov, T.D. Mamedov, Fundamental Solution of Cauchy and Goursat Type Problems in Lines with Characteristics and Non-characteristics Pieces for Third Order Hyporbolik Equation, Journal of Contemporary Applid Mathematics V.12, no.2, 2022, December ISSN 2222-5498

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Received 15 February 2024
Accepted 28 March 2024