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# On the Construction of the Resolvent of a Differential Pencil of the Fourth Order on the Whole Real Line

Konul H. Aliyeva

**Abstract.** A pencil of fourth-order differential equations on the entire axis with multiple characteristics is considered. Using special solutions of a fourth-order differential equation, the resolvent of a differential pencil is studied. By means of Green's function, the integral representation of the resolvent of a differential pencil is found.

**Key Words and Phrases:** pencil of fourth-order differential equations, Green's function, operator pencil, resolvent, spectrum, eigenvalues

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## 1. Introduction

We consider the equation

$$\ell\left(x, \frac{d}{dx}, \lambda\right) y = \left(\frac{d^2}{dx^2} + \lambda^2\right)^2 y + r(x)\, y' + (\lambda p(x) + q(x))\, y = 0, \tag{1}$$

where the complex-valued functions$r(x)$, $p(x)$ and $q(x)$ satisfy the conditions

$$\begin{array}{l} r(x) \in C^{(1)}(-\infty, +\infty),\ p(x) \in C^{(1)}(-\infty, +\infty),\ q(x) \in C(-\infty, +\infty) \\ \int_{-\infty}^{+\infty} \left(1 + x^4\right) \left\{\left|r^{(j)}(x)\right| + \left|p^{(j)}(x)\right| + |q(x)|\right\} dx < \infty, j = 0, 1. \end{array} \tag{2}$$

The left side of the equation (1) generates an operator pencil $L_\lambda$ in the space $L_2(-\infty, +\infty)$ . The present work is devoted to the construction of the Green's function for the operator pencil $L_\lambda$. Note that for a pencil $L_\lambda$ defined on a semi-axis, a similar problem was studied in [2], [5]. In the case of the entire axis, it is necessary to use special solutions of equation (1) with asymptotics at $-\infty$ . This raises additional difficulties. Some results for a differential beam on the entire axis were obtained in [3], [6].
The results of this work can be used in the study of direct and inverse scattering problems for equation (1). Note that inverse problems for differential equations of high orders have been studied in the works of many authors (see [1], [4], [7], [8] and references therein).

 

## 2. Preliminary information

In this section we formulate some auxiliary facts established in works [3], [6]. Consider the equation (1). Let's we put

$$\begin{aligned}\sigma_j^{\pm}(x) &= \pm\int_x^{\pm\infty} |s|^j \left\{|p(s)| + |r(s)| + |sr'(s)| + |sq(s)|\right\} ds,\\ \tau^{\pm}(x) &= \pm\int_x^{\pm\infty} s^2 \left\{|p(s)| + |r(s)| + |sr'(s)| + |sq(s)|\right\} ds.\end{aligned}$$

It is known [2] , [6] that under conditions (2) equation (1) has special solutions $f_j^{\pm}(x,\lambda)$, $g_j^{\pm}(x,\lambda)$ representable in the form

$$\begin{aligned} f_j^{\pm}(x,\lambda) &= x^{j-1}e^{\pm i\lambda x} + \int_x^{\infty} A_j^{\pm}(x,t)\, e^{\pm i\lambda t}dt,\ j=1,2,\\ g_j^{\pm}(x,\lambda) &= x^{j-1}e^{\mp i\lambda x} + \int_{-\infty}^{x} B_j^{\pm}(x,t)\, e^{\mp i\lambda t}dt,\ j=1,2,\end{aligned} \tag{3}$$

where $A_j^{\pm}(x,t)$ , $B_j^{\pm}(x,t)$ are four times continuously differentiable functions and the following relations hold:

$$\begin{aligned}\left|A_j^{\pm}(x,t)\right| &\le \frac{1}{4}\sigma_j^{+}\left(\frac{x+t}{2}\right)\exp\left\{\tau^{+}(x)\right\},\\ \left|B_j^{\pm}(x,t)\right| &\le \frac{1}{4}\sigma_j^{-}\left(\frac{x+t}{2}\right)\exp\left\{\tau^{-}(x)\right\}.\end{aligned} \tag{4}$$

According to relations (3), (4), for each fixed $x$, the functions$f_j^{\pm}(x,\lambda)$ , $g_j^{\pm}(x,\lambda)$ are analytic functions in the open half-plane $\pm Im\lambda > 0$ and are continuous up to the real axis. Further, from the general theory it is known that (see [3], [5], [6]) for all $\lambda \neq 0$, the equation (1) also has linearly independent solutions $\psi_j^{\pm}(x,\lambda)$ , $j = 1,2,3,4$ with asymptotics

$$\psi_j^{\pm}(x,\lambda) = e^{\mp i\omega_j\lambda x}\left(1 + o(1)\right),\ \omega_j = e^{\frac{i\pi(j-1)}{2}},\ x\to\pm\infty. \tag{5}$$

Now we consider the unperturbed equation

$$\left(\frac{d^2}{dx^2} + \lambda^2\right)^2 y = 0. \tag{6}$$

Obviously, for $\lambda \neq 0$, equation (6) has four linearly independent solutions of the form

$$y_1(x,\lambda) = e^{i\lambda x},\ y_2(x,\lambda) = xe^{i\lambda x},\ y_3(x,\lambda) = e^{-i\lambda x},\ y_4(x,\lambda) = xe^{-i\lambda x}.$$

It is easy to check that the Wronskian $W\{y_1, y_2, y_3, y_4\}$ of these solutions is equal to $16\lambda^4$.

## 3. Studying the Resolvent of the Pencil $L_\lambda$

Let us study the eigenvalues of the operator pencil $L_\lambda$. We denote by $W^{\pm}(\lambda)$ the Wronskian of the solutions $f_1^{\pm}(x,\lambda)$ , $f_2^{\pm}(x,\lambda)$ , $g_1^{\pm}(x,\lambda)$ , $g_2^{\pm}(x,\lambda)$ of equation (1):

$$W^{\pm}(\lambda) = W\left\{f_1^{\pm}(x,\lambda),\ f_2^{\pm}(x,\lambda),\ g_1^{\pm}(x,\lambda),\ g_2^{\pm}(x,\lambda)\right\}.$$

Since the Wronskian of solutions of the equation (1) does not depend on $x$, we can write:

$$W^{\pm}(\lambda)=\begin{vmatrix} f_1^{\pm}(0,\lambda) & f_2^{\pm}(0,\lambda) & g_1^{\pm}(0,\lambda) & g_2^{\pm}(0,\lambda) \\ f_1^{\pm(1)}(0,\lambda) & f_2^{\pm(1)}(0,\lambda) & g_1^{\pm(1)}(0,\lambda) & g_2^{\pm(1)}(0,\lambda) \\ f_1^{\pm(2)}(0,\lambda) & f_2^{\pm(2)}(0,\lambda) & g_1^{\pm(2)}(0,\lambda) & g_2^{\pm(2)}(0,\lambda) \\ f_1^{\pm(3)}(0,\lambda) & f_2^{\pm(3)}(0,\lambda) & g_1^{\pm(3)}(0,\lambda) & g_2^{\pm(3)}(0,\lambda) \end{vmatrix}. \tag{7}$$

As shown in work [6], in the half-plane $\pm Im\lambda>0$ the eigenvalues of the operator pencil $L_\lambda$ coincide with the zeros of the function $W^{\pm}(\lambda)$. In addition, on the real axis the operator pencil $L_\lambda$ has no eigenvalues.

Let us study the resolvent of the operator pencil $L_\lambda$. For this purpose, we construct the corresponding Green's function and find the integral representation of the resolvent.

**Theorem 1.** *If $\lambda$, $Im\lambda>0$ is not an eigenvalue of the operator pencil $L_\lambda$, then the resolvent $R_\lambda^+$ of this pencil $L_\lambda$ is an integral operator acting according to the formula*

$$R_\lambda^+ f(x)=\int_{-\infty}^{+\infty} G^+(x,\xi,\lambda)\, f(\xi)\, d\xi \tag{8}$$

*where the Green's function $G^+(x,\xi,\lambda)$ is defined by the formula*

$$G^+(x,\xi,\lambda)=\begin{cases} f_1^+(x,\lambda)\, g_2^-\left(\xi,\overline{\lambda}\right)+f_2^+(x,\lambda)\, g_1^-\left(\xi,\overline{\lambda}\right),\xi<x, \\ -\left[g_1^+(x,\lambda)\, f_2^-\left(\xi,\overline{\lambda}\right)+g_2^+(x,\lambda)\, f_1^-\left(\xi,\overline{\lambda}\right)\right],\xi\geq x. \end{cases} \tag{9}$$

Proof. Consider the equation

$$\ell\left(x,\frac{d}{dx},\lambda\right)Y=f,$$

where $f(x)\in L_2(0,\ \infty)$ is a function with bounded support. We are looking for a solution $Y=Y(x,\lambda)$ in the form

$$Y(x,\lambda)=C_1 f_1^+(x,\lambda)+C_2 f_2^+(x,\lambda)+D_1 g_1^+(x,\lambda)+D_2 g_2^+(x,\lambda), \tag{10}$$

where $C_1=C_1(x,\lambda)$, $C_2=C_2(x,\lambda)$, $D_1=D_1(x,\lambda)$, $D_2=D_2(x,\lambda)$ are the quantities to be determined. Let us denote by $W_i(x,\ \lambda)$ the algebraic complement of the $i$-th element of the last row of the determinant $W^+(\lambda)$. Applying the method of variation of constants, we obtain

$$C_1'(x,\lambda)=Z_4(x,\lambda)\, f(x),\ C_2'(x,\lambda)=Z_3(x,\lambda)\, f(x),$$
$$D_1'(x,\lambda)=Z_2(x,\lambda)\, f(x),\ D_2'(x,\lambda)=Z_1(x,\lambda)\, f(x)$$

where

$$Z_{5-k}(x,\lambda)=\frac{W_k(x,\lambda)}{W^+(\lambda)},\quad k=1,2,3,4.$$

From the general theory of differential equations it is known that $Z_k(x,\lambda)$, $k=1,2,3,4$ are solutions to the conjugate equation $\ell\left(x,\frac{d}{dx},\overline{\lambda}\right)Z=0$ . Moreover, the relations

$$Z_1(x,\lambda)=f_1^-\left(x,\overline{\lambda}\right),\ Z_2(x,\lambda)=f_2^-\left(x,\overline{\lambda}\right),$$
$$Z_3(x,\lambda)=g_1^-\left(x,\overline{\lambda}\right),\ Z_4(x,\lambda)=g_2^-\left(x,\overline{\lambda}\right).$$

are valid. From these considerations it follows that

$$C_1(x,\lambda)=c_1+\int_{-\infty}^{x} g_2^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\,,\;C_2(x,\lambda)=c_2+\int_{-\infty}^{x} g_1^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\;,$$

$$D_1(x,\lambda)=d_1-\int_{x}^{+\infty} f_2^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\,,\;D_2(x,\lambda)=d_2-\int_{x}^{+\infty} f_1^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\;.$$

Since $Y(x,\ \lambda)\in L_2(0,\ \infty)$ olması, we require that

$$C_1(x,\lambda)\to 0, C_2(x,\lambda)\to 0, x\to-\infty,$$

$$D_1(x,\lambda)\to 0, D_2(x,\lambda)\to 0, x\to+\infty.$$

Hence,

$$C_1(x,\lambda)=\int_{-\infty}^{x} g_2^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\,,\;C_2(x,\lambda)=\int_{-\infty}^{x} g_1^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\;, \tag{11}$$

$$D_1(x,\lambda)=-\int_{x}^{+\infty} f_2^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\,,\;D_2(x,\lambda)=-\int_{x}^{+\infty} f_1^-\left(\xi,\overline{\lambda}\right)f(\xi)\,d\xi\;. \tag{12}$$

Substituting instead of$C_1=C_1(x,\lambda)\,,\,C_2=C_2(x,\lambda)\,,\,D_1=D_1(x,\lambda)\,,\,D_2=D_2(x,\lambda)$ their expressions (11), (12) into equality (10), we obtain

$$\begin{array}{l} Y(x,\lambda)=\int_{-\infty}^{x}\left[f_1^+(x,\lambda)\,g_2^-\left(\xi,\overline{\lambda}\right)+f_2^+(x,\lambda)\,g_1^-\left(\xi,\overline{\lambda}\right)\right]f(\xi)\,d\xi- \\ -\int_{x}^{+\infty}\left[g_1^+(x,\lambda)\,f_2^-\left(\xi,\overline{\lambda}\right)+g_2^+(x,\lambda)\,f_1^-\left(\xi,\overline{\lambda}\right)\right]f(\xi)\,d\xi. \end{array} \tag{13}$$

From (13) it follows that function (9) is the Green's function for the operator pencil $L_\lambda$, and the resolvent of this pencil admits an integral representation (8).
The theorem has been proven.
The following theorem is proved in a similsr way.

**Theorem 2.** *If $\lambda$, $Im\lambda<0$ is not an eigenvalue of the operator pencil $L_\lambda$, then the resolvent $R_{\lambda}^-$ of this pencil $L_\lambda$is an integral operator acting according to the formula*

$$R_\lambda^- f(x)=\int_{-\infty}^{+\infty} G^-(x,\xi,\lambda)\,f(\xi)\,d\xi$$

*where the Green's function $G^+(x,\xi,\lambda)$ is defined by the formula*

$$G^+(x,\xi,\lambda)=\left\{\begin{array}{l} f_1^-(x,\lambda)\,g_2^+\left(\xi,\overline{\lambda}\right)+f_2^-(x,\lambda)\,g_1^+\left(\xi,\overline{\lambda}\right),\xi<x, \\ -\left[g_1^-(x,\lambda)\,f_2^+\left(\xi,\overline{\lambda}\right)+g_2^-(x,\lambda)\,f_1^+\left(\xi,\overline{\lambda}\right)\right],\xi\geq x. \end{array}\right.$$

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Konul H. Aliyeva
*Sumqait State University, Sumqait AZ5008, Azerbaijan*
*E-mail:* konul.aliyeva@sdu.edu.az

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