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Basis Property of the System of Eigenfunctions Corresponding to a Problem with a Spectral Parameter in the Boundary Condition

Sevinj S.Hadiyeva

Abstract. We consider the Sturm-Liouville operator with the same spectral parameter in the equation and in one of the boundary conditions. We study the basis property of the system of eigenfunctions of this operator in the space$L_p(0,1)$ $(p>1)$.

Key Words and Phrases: Sturm-Liouville operator, transformation operator, eigenfunction, orthogonal system, basis.

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1. Introduction

Consider the spectral problem

$$-y'' + q(x)y = \lambda y, \quad 0 < \mathrm{x} < 1 \tag{1}$$

$$y(0) = 0 \tag{2}$$

$$(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1) \tag{3}$$

where λ is a spectral parameter, $q(x) \in L_2(0,1)$ is a real-valued function and a_1, b_1, c_1, d_1 are real constants and $a_1d_1 - b_1c_1 > 0$. This problem arises when modeling the transrelaxation heat process and in the mathematical description of vibrations of a loaded string(see [1]- [4]). In [5], the spectral properties of problem (1)-(3) were studied. It is known [5] that there is an infinitely increasing sequence of eigenvalues s of problem (1)-(3):$\lambda_0 < \lambda_1 < .... < \lambda_n < ....$ Let us denote by $\varphi(x,\lambda)$the solution to equation (1) with initial conditions $\varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = \sqrt{\lambda}$. Obviously, the eigenvalues of problem (1)-(3) serve as zeros of the entire function $\Delta(\lambda) = \quad (a_1\lambda + b_1)\varphi(1,\lambda) - (c_1\lambda + d_1)\varphi'(1,\lambda)$. In addition, $\varphi(x,\lambda_n)$ is an eigenfunction of problem (1)-(3), corresponding to the eigenvalue λ_n. It was proven in [5], for a sufficiently large n the formula

$$\sqrt{\lambda_n} = \pi(n+\nu) + \mathrm{O}(\frac{1}{n}), \tag{4}$$

holds, where

$$\nu = \begin{cases} 0 & \text{if} \quad c_1 = 0 \\ -\frac{1}{2} & \text{if} \quad c_1 \neq 0 \end{cases}$$

In the present paper, we prove the basis property of the system of eigenfunctions of the boundary value problem (1)-(3) in the space $L_p(0,1)$ $(p > 1)$. We note that in [6] the basis property in $L_2(0,1)$ is studied for the system of eigenfunctions of the boundary value problem (1)-(3), where $q(x)$is a real-valued continuous function on the interval $[0,1]$. At the same time, the approach we propose differs significantly from the approach used in [6] and is based on the properties of transformation operators (see[7], [8]) for the Sturm-Liouville equations.

2. The Basis Property of the System of Eigenfunctions of the Boundary Value Problem (1)–(3)

Theorem 1. *The system* $\{\varphi(x, \lambda_n)\}_{n=0}^{\infty}$ *, that is, the system of eigenfunctions of boundary value problem* (1)-(3) *with one function deleted, is a basis in the space* $L_p(0,1)$ $(p > 1)$ *and when p=2 the basis is unconditional.*

Proof. We will need the basic properties of the system

$$\left\{\sin\sqrt{\lambda_n}x\right\} \quad (n = 0, 1, ...n \neq k_0) \tag{5}$$

where k_0 is an arbitrarily fixed non-negative integer. Let us show that system (5) is a basis in the space $L_p(0,1)$ $(p > 1)$. We consider formula (4). Let, for example, $\nu = -\frac{1}{2}$ ($c_1 = 0$). It is known that the system of functions

$$\left\{\sin(n - \frac{1}{2})\pi x\right\}_{n=1}^{\infty} \tag{6}$$

is a basis[3] in the space $L_p(0,1)$ $(p > 1)$and for $p = 2$ the basis is orthogonal. By virtue of (4), the system (5) is quadratically close to the system (6). Then from the completeness of system (5) in $L_2(0,1)$ (see, for example, [9]) it follows that this system forms a Riesz basis. The latter entails the unconditionality of the basis.

Suppose $1 < p < 2$ and $f(x) \in L_p(0,1)$. Let $c_n(f)$, $n = 1, 2, ...$ denote the Fourier coefficients of the function $f(x)$ according to system (6). Since the system $\left\{\sqrt{2}\sin(n - \frac{1}{2})\pi x\right\}_{n=1}^{\infty}$ is uniformly bounded and orthonormal in $L_p(0,1)$, then by the Riesz theorem (see [10]) we have

$$\left(\sum_{n=1}^{\infty} |c_n(f)|^q\right)^{\frac{1}{q}} \leq M \|f\|_{L_p(0,1)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Whence it follows that system (6) is a q-basis in $L_p(0,1)$ (see [11]). In addition, using (4) we obtain

$$\left\|\sin\sqrt{\lambda_n}x - \sin\left(n - \frac{1}{2}\right)\pi x\right\|_{L_p(0,1)}^{p} = O\left(\frac{1}{n^p}\right), \quad n \to \infty,$$

according to which system (5) is p-close in $L_p(0,1)$ to system (6). Since system (5) is complete in $L_p(0,1)$ for $1 < p < 2$, it forms [11] a basis in $L_p(0,1)$ that is isomorphic to (6). Similarly, if $p > 2$, then system (6) is a p-basis in $L_p(0,1)$. Obviously, system (5) is q-close in $L_p(0,1)$ to system (6). Moreover, system (5) is ω-linearly independent in $L_p(0,1)$, since it forms a basis in $L_2(0,1)$. Whence it follows that system (5) forms[11] a basis in $L_p(0,1)$ that is isomorphic to system (6). Thus, we have proven that the system of functions $\left\{\sin\sqrt{\lambda_n}x\right\}_{n=0}^{\infty}$ with one function deleted, is a basis in the space $L_p(0,1)$ $(p > 1)$ and when $p = 2$ the basis is unconditional.

Let us now consider the solution $\varphi(x,\lambda)$ of the equation (1). As is known [7], [8], for this solution is true representation using the transformation operator

$$\varphi(x,\lambda) = \sin\sqrt{\lambda}x + \int_0^x K(x,t)\sin\sqrt{\lambda}t dt \tag{7}$$

where *K(x,t)* is a real continuous function and $K(x,x) = h + \frac{1}{2}\int_0^x q(t)dt$.

Consider the transformation operator defined by the formula

$$(I + \Omega)f = f(x) + \int_0^x K(x,t)f(t)dt.$$

Since Ω is a Voltaire integral operator, the operator $I + \Omega$ has an inverse operator of the same form. This means that the operator $I + \Omega$ carries out a one-to-one mapping of the space $L_p(0,1)$ $(p > 1)$ onto itself. Using then the basic property of the system (5) and formula (7), we complete the proof of the theorem .

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Sevinj S.Hadiyeva
Sumqait State University, Sumqait AZ5008, Azerbaijan
E-mail: sevinc.hadiyeva@sdu.edu.az

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