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A Short and Educational Proof of the Bolzano-Weierstrass Theorem

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Abstract. A new proof of the Bolzano-Weierstrass theorem is presented. The Heine-Borel theorem, Cantor's intersection method, the preliminary theorem of monotone subsequences, the recursive construction process and the axiom of choice are not used in this new proof. The presented proof may educationally preferable for instructors.

Key Words and Phrases: Bolzano- Weierstrass Theorem

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1. Introduction

It is appropriate to start with a theorem from Bernard Bolzano's article "Purely analytic proof of the theorem, that between any two values, which give results of opposite sign, there lies at least one real root of the equation" [2] in order to understand why this theorem is called by two names; Bolzano and Weierstrass.

Theorem 1.1. *Let M be a property that is true for all nonnegative variable x less than a given number u, but not true for all nonnegative variable x. In this case there is such a largest number U such that*

$$\{x : M(x)\} = \{x : x < U\}$$

Proof. Since the property M is not satisfied for all non-negative values of x but is satisfied for all values less than u, then for a positive number D, the property M is not satisfied for x less than $V = u + D$. For each $m = 0, 1, 2, 3$ consider the set

$$S_m = \{x : x < u + \frac{D}{2^m}\}$$

and consider the question that "Is there a smallest number m for the set S_m such that the property M is satisfied?"
If such a number m does not exist, we could take $U = u$ because if we assume that $U = u + d$ for a number d, then for sufficiently large m the inequality $u + \frac{D}{2^m} < u + d$ holds, which contradicts the non-existence of the smallest number m.

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Now suppose that there exists such a number m_0 such that the property M holds for every element of the set S_{m_0} and not for some elements of S_{m_0-1}.That is, it is satisfied for all x smaller than $u+\frac{D}{2^{m_0}}$, but not for all x smaller than $u+\frac{D}{2^{m_0-1}}$. There is also no reason why it should be $U=u+\frac{D}{2^{m_0}}$. Since the difference between $u+\frac{D}{2^{m_0}}$ and $u+\frac{D}{2^{m_0-1}}$ is $\frac{D}{2^{m_0}}$ For each $m^1=0,1,2,3$ consider the set

$$S_{m^1}=\{x : x<u+\frac{D}{2^{m_0}}+\frac{D}{2^{m_0+m^1}}\}$$

Now let's repeat our question that "Is there a smallest number m^1 for the set S_{m^1} such that the property M is satisfied?". If such a number m^1 does not exist, we could take $U=u+\frac{D}{2^{m_0}}$. If there is such a number m_0^1, then for each $m^2=0,1,2,\ldots$ consider the set

$$S_{m^2}=\{x : x<u+\frac{D}{2^{m_0}}+\frac{D}{2^{m_0+m_0^1}}+\frac{D}{2^{m_0+m_0^1+m^2}}\}$$

The same question is asked for these sets over m_2 ; "Is there a smallest number m^2 for the set S_{m^2} such that the property M is satisfied?" As a result, this process will end in two ways.

1. For a given number $i\in\mathbb{N}$ there does not exits a smallest number m_0^i for the set S_{m^i}, given below, such that the property M

   $$S_{m^i}=\{x : x<u+\frac{D}{2^{m_0}}+\frac{D}{2^{m_0+m_0^1}}+\frac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\frac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m^i}}\}$$

   then

   $$U=u+\frac{D}{2^{m_0}}+\frac{D}{2^{m_0+m_0^1}}+\frac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\frac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m_0^i}}$$

ii. For each number $j\in\mathbb{N}$ there exits a smallest number m_0^i for the set S_{m^i},given below, such that the property M

   $$S_{m^i}=\{x : x<u+\frac{D}{2^{m_0}}+\frac{D}{2^{m_0+m_0^1}}+\frac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\frac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m^j}}\}$$

   then U is the limit point of the series below,

   $$u+\frac{D}{2^{m_0}}+\frac{D}{2^{m_0+m_0^1}}+\frac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\frac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m^j}}\}$$

□

This theorem of Bolzano, known as the "greatest lower bound property", helped Weierstrass to prove the "every bounded infinite set of real numbers has a limit point" theorem [1].

Proof. Let (x_n) be a bounded real sequence and it has a monotone increasing subsequence. Consider the set

$$B = \{y : y \geq x, x = x_n \text{ for some } n \in \mathbb{N}\}$$

and the property $M :=$ "Not belongs to B ". Let us take any element b_0 from set B and any element a_0 from set A. For the values of x, it can be said that the property M is not true for all values less than b_0, but it is true for values less than a_0. Hence by the theorem of Bolzano there exists an element U such that the sequnce x_{n_k} converges to U. Similarly the proof will be given in case the sequence has a monotone decreasing subsequence. □

It will be convenient to list the outline of the proofs of the Bolzano-Weierstrass theorem

1. A proof is presented by using the lemma that "there exists a monotone increasing or monotone decreasing subsequence of a bounded sequence" and "the theorem that a bounded monotone decreasing or increasing sequence is convergences to its infumum or supremum, respectively". One can easliy find the proof in [3]

2. If (x_n) is a bounded sequence, then it is in a closed interval $[a, b]$. the interval with infinite elements of the sequence (x_n) is selected from the closed intervals $[a, c]$ and $[c, b]$ with the midpoint c of this closed $[a, b]$. When this process is repeated over the inclusion of infinite elements, then it is obtained nested closed intervals with infinite elements from the sequence (x_n). Then it is obtained that there is a unique element at the intersection of these nested closed intervals and a subsequence of (x_n) converges to this element by using the Cantor intersection theorem. One can easliy find the proof in [5].

3. An open cover is constructed for the bounded and infinite set $\{x_n\}_n$ set with the assumption that there is no limit point then by using Heine-Borel theorem a finite open cover is obtained for the set $\{x_n\}_n$ but infinite elements of $\{x_n\}_n$ does not belong to the finite cover. the theorem based on this contradiction is easily found in [4].

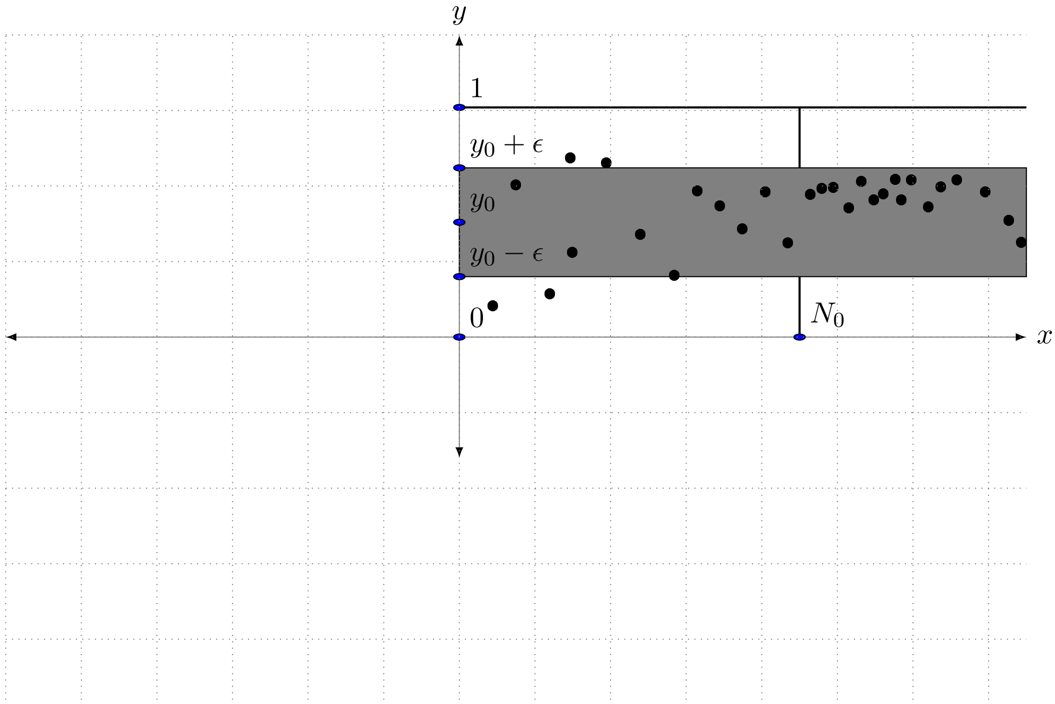
4. Firstly it is defined that the notions (x_n). $\liminf (x_n)$ and $\limsup (x_n)$ and prove that they exist and are unique. Then it is proved that (x_n). $\liminf (x_n)$ and $\limsup (x_n)$ are limit points for the set $\{(x_n)\}$.

5. It is proved that the supremum and infumum of $\{x_n\}$ exits and they are the limit point of the sequence By the using Stäckel-finite concept which is equivalent to the Dedekind and Tarski finiteness in ZFC [6].

2. New Proof of Bolzano-Weierstrass Theorem

Let (x_n) be a bounded real sequence. Without loss of generality it can be taken $[0, 1]$ as a domain of the sequence and also consider the function $f : \mathbb{N} \to [0, 1]$ instead of (x_n) since a sequence is a function whose domain is the set of natural numbers.

A real sequence (x_n) converges to a real number x if and only if for each $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that the implication "$n \geq N_0 \implies |x_n - x| < \epsilon$" holds. For a given $(0, y_0)$ where $0 \leq y_0 \leq 1$ and $\epsilon > 0$ the set $\{(x, y) : 0 \leq x, y_0 - \epsilon \leq y \leq y_0 + \epsilon\}$ is called ϵ-**band** of $(0, y_0)$ and denoted by $B(y_0, \epsilon)$. It is easily seen that if a sequence $f(n)$ converges to an element y if and only if for each $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that $\{(n, f(n)) : n \geq N_0\}$ belongs to ϵ-**band** of $(0, y)$.



If a sequence $f(n)$ has no convergent subsequence, for each $y \in [0, 1]$ there exist at least one $\epsilon > 0$ such that ϵ-**band** of $(0, y)$ contains finite elements of $f(n)$.

Theorem 2.1 (Bolzano-Weierstrass theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let's suppose that there is a sequence $f(n)$ without a convergent subsequence. Assume as well that our sequence doesn't have any terms that continuously repeat. If not, the convergent subsequence will arise. $f(n_k)$. Let we choose an arbitrary element $y_0 \in [0, 1]$. By the assumption there exists an $\epsilon_0 > 0$ such that the set $\{f(n) : n \in \mathbb{N}\} \cap B(y_0, \epsilon_0)$ is finite. Let consider the set $X := \{\epsilon : \{f(n) : n \in \mathbb{N}\} \cap B(y_0, \epsilon)$ is finite$\}$. The set X is non-empty since there exists at least $\epsilon_0 > 0$ and it is bounded above because of the inclusion of the $\{f(n) : n \in \mathbb{N}\} \subset B(y_0, 1)$. Therefore $\sup X$ exists, say $\overline{\epsilon}$. The set

$$B(y_0, \overline{\epsilon}) \cap \{f(n) : n \in \mathbb{N}\}$$

is finite because there exist $\epsilon_1, \epsilon_2 > 0$ such that the sets $\{f(n) : n \in \mathbb{N}\} \cap B(y_0 + \overline{\epsilon}, \epsilon_1)$ and $\{f(n) : n \in \mathbb{N}\} \cap B(y_0 - \overline{\epsilon}, \epsilon_2)$ are finite, moreover there are also finite for the real number $\epsilon^* := \min\{\epsilon_1, \epsilon_2\}$. The set $B(y_0, \epsilon^*) \cap \{f(n) : n \in \mathbb{N}\}$ is finite because $\overline{\epsilon}$ is supremum. Hence The set $B(y_0, \overline{\epsilon}) \cap \{f(n) : n \in \mathbb{N}\}$ is finite. On the other hand it is easily sen that the set $B(y_0, \overline{\epsilon} + \epsilon^*) \cap \{f(n) : n \in \mathbb{N}\}$ is also finite, it is a contradiction since it is assumed that $\sup X = \overline{\epsilon}$. □

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