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On the statistical approximation of q-Bernstein-Kantorovich operators on the symmetric interval

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Abstract. This study defines the q-type of the Bernstein-Kantorovich operator on the symmetric interval. It also calculates the statistical approximation of this new generalized operator and its rate of approximation with the help of the modulus of continuity.

Key Words and Phrases: Bernstein-Kantorovich operators, q-calculus, modulus of continuity, statistical approximation.
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1. Introduction

Bernstein operators are linear operators used to approximate continuous functions on the interval $[0,1]$. The n-th Bernstein polynomial of a function f is defined as:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

These polynomials were first introduced by Bernstein in 1912 [1]. Kantorovich operators are an integral modification of Bernstein operators where the function f is integrated over small intervals (see [2]):

$$L_n(f;x) = (n+1)\sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{1}$$

for $x \in [0,1]$. An estimate for the Korovkin-type approximation properties and the convergence rate of these operators can be found in [3]. Additionally, numerous authors have constructed and studied Kantorovich-type generalizations of various other operators [4], [5], [6].

In recent years, interesting generalizations of Bernstein polynomials have been proposed by Lupas [7] and Phillips [8]. Generalizations of Bernstein polynomials based on q-integers have attracted much attention and have been widely studied by many authors [9], [10], [11].

First, let us give some basic definitions from q-calculus (see [12]).

For any fixed real number $q > 0$,the q-integer $[r]$ is defined as

$$[r]_q := [r] = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1 \\ r, & q = 1 \end{cases}$$

for all nonnegative integers. The q-factorial $[r]!$ and q-binominal $\begin{bmatrix} n \\ k \end{bmatrix}$, $(n \geqslant r \geqslant 0)$ are also defined by

$$[r]! = \begin{cases} [r][r-1]...[1], & r = 1, 2, ... \\ 1, & r = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}$$

respectively.

For an arbitrary function $f(x)$, the q-differential is given by

$$d_q f(x) = f(qx) - f(x).$$

In particular, $d_q x = (1-q)x$.

Suppose $0 < a < b$. The definite q-integral is defined as

$$\int_0^b f(t) d_q t = (1-q) b \sum_{j=0}^{\infty} f(q^j b) q^j \quad 0 < q < 1$$

and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

The q-Bernstein polynomials were first introduced by Lupas [7] and then Ostrovska [5] studied the smooth approximation properties of these operators in a recent paper. In 1997, Philipps [8] introduced another modification of q-Bernstein polynomials as

$$B_n(f; q; x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x)$$

for each positive integer n, and $f \in C[0,1]$.

In this study, the Korovkin type approximation and statistical approximation of q-Bernstein-Kantorovich operators are investigated.

Let us now recall the concept of statistical convergence, which has become an important area of research in approximation theory, especially in the study of linear positive operators.

The natural density, δ, of a set $K \subseteq N$ is defined by

$$\delta(K) = \lim_n \frac{1}{n} \{\text{the number} \quad k \leq n : k \in K\}$$

provided the limit exists (see [13]). A sequence $x = (x_k)$ is called statistically convergent to a number L if, for every $\varepsilon > 0$

$$\delta\{k : |x_k - L| \geq \epsilon\} = 0$$

and it is denoted as $st - \lim_{n\to\infty} \frac{1}{[n]} = 0$.

Theorem 1.1. *(see [14]) If the sequence of linear positive operators $A_n : C[a,b] \to C[a,b]$ satisfies the conditions*

$$st - \lim_{n} \|A_n(e_i;.) - e_i\|_{C[a,b]} = 0; \quad e_i(t) = t^i$$

for $i = 0, 1, 2,$ then for any function $f \in C[a,b]$,

$$st - \lim_{n} \|A_n(f,.) - f\|_{C[a,b]} = 0$$

2. Construction of the Operators

Let us define the following generalization of the Bernstein-Kantorovich operator (1):

$$K_n(f;q;x) = \frac{[n+1]}{2} \sum_{k=0}^{n} \left(\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1} f(t) d_q t \right) \begin{bmatrix} n \\ k \end{bmatrix} q^{-k} (1-x)_q^k (1+x)_q^{(n-k-1)}, \qquad (2)$$

where $K_n : C[-1,1] \to C[-1,1]$.

Lemma 2.1. *Let $x \in [-1,1]$, $\quad e_i = x^i \quad i = 0,1,2$. Then the following statements are true for the q-Bernstein-Kantorovich operators (2):*

$$\begin{aligned}
K_n(e_0;q;x) &= 1 \\
K_n(e_1;q;x) &= \frac{2}{[n+1][2]} + \frac{[n]}{[n+1]}(1+x) - \frac{2}{[2]} \\
K_n(e_2;q;x) &= \frac{[n]^2}{[n+1^2]}x^2 + \left(\frac{2[n]^2}{[n+1]^2} + \frac{(2+4q)[n]}{[3][n+1]^2} - \frac{3[2][n]}{[n+1][3]} \right) x \\
&\quad + \frac{[n]^2}{[n+1]^2} + \frac{(2+4q)[n]}{[3][n+1]^2} - \frac{3[2][n]}{[n+1][3]} \\
&\quad + \frac{4}{3[n+1]^2} - \frac{6}{[n+1][3]} + \frac{3}{[3]}.
\end{aligned} \qquad (3)$$

To prove a Korovkin-type theorem, the operator must be linear and positive. It is clear that the operator $K_n(f;q;x)$ is linear. Let us give the following lemma for the positivity of the operator $K_n(f;q;x)$.

Lemma 2.2. *Let $0 \leq a < b$, $0 < q < 1$ and let f be a pozitive function defined on the interval $[0,b]$. If f is monotone increasing on $[0,b]$, then $\int_a^b f(t)d_qt \geq 0$ in this interval.*

Let $q := (q_n)$ be a sequence satisfying the following properties:

$$st - \lim_n q_n = 1 \quad and \quad st - \lim_n q^n = a : \quad a < 1 \tag{4}$$

Theorem 2.3. *Let be a sequence satisfying for $0 < q_n < 1$ and $K_n(f;q;x)$ be a sequence of operators defined by (2). Then for any monotone increasing positive function $f \in C[-1,1]$,*

$$st - \lim_n \|K_n(f;q;.) - f\|_{C[-1,1]} = 0 \tag{5}$$

is satisfied.

Due to the general definition of the q-integral, it is not easy to calculate the convergence rate of the operator. Because the q-integral is the difference of two infinite sums, it does not satisfy some integral inequalities. Therefore, the Riemann type q-integral will be used here instead of the q-integral .

Definition 2.4. *(See [15]). Let $0 < q < 1, 0 < a < b$. The Riemann type q-integral is defined as*

$$\int_a^b f(x)d_q^R x = (1-q)(b-a)\sum_{j=0}^{\infty} f(a+(b-a)q^j)q^j$$

Now let us redefine the operator (2) using the Riemann type q-integral:

$$K_n^*(f;q;x) = \frac{[n+1]}{2}\sum_{k=0}^{n}\left(\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1} f(t)d_q^R t\right)\begin{bmatrix} n \\ k \end{bmatrix} q^{-k}(1-x)_q^k(1+x)_q^{(n-k-1)}. \tag{6}$$

It is clear that the Riemann type q-integral is linear and positive.

Lemma 2.5. *For the operator (6), the following equations are true:*

$$K_n^*(1;q;x) = 1 \tag{7}$$

$$K_n^*(t;q;x) = \frac{[n]}{[n+1]}\left(1+\frac{q-1}{[2]}\right)x + \frac{[n]}{[n+1]}\left(1+\frac{q-1}{[2]}\right) + \frac{2}{[n+1][2]} - 1 \tag{8}$$

$$\begin{aligned} K_n^*(t^2;q;x) = {} & \left(\frac{q^2[n][n-1]}{[n+1]^2} + \frac{2q^2(q-1)[n][n-1]}{[n+1]^2[2]} + \frac{q^2(q-1)^2[n][n-1]}{[n+1]^2[3]}\right)x^2 \\ & + \left(\frac{[2]q[n][n-1]}{[n+1]^2} + \frac{2[n]}{[n+1]^2} - \frac{2[n]}{[n+1]} + \frac{4[n]}{[n+1]^2[2]} + \frac{2q(q-1)[n][n+1]}{[n+1]^2}\right. \\ & \left. - \frac{2(q-1)[n]}{[n+1][2]} + \frac{4(q-1)[n]}{[n+1]^2[3]} + \frac{q(q-1)^2[n][n-1][2]}{[n+1]^2[3]} + \frac{2(q-1)^2[n]}{[n+1]^2[3]}\right)x \\ & + \frac{q[n][n-1]}{[n+1]^2} + \frac{2[n]}{[n+1]^2} - \frac{2[n]}{[n+1]} + \frac{4[n]+2q(q-1)[n][n-1]}{[n+1]^2[2]} \\ & - \frac{4+2(q-1)[n]}{[n+1][2]} + \frac{4+4(q-1)[n]+q(q-1)^2[n][n-1]+2(q-1)^2[n]}{[n+1]^2[3]} + 1 \end{aligned} \tag{9}$$

Lemma 2.6. *The second moment of the operator $K_n^*(f;q;x)$ is given below:*

$$\begin{aligned}
&K_n^*((t-x)^2;q;x) \\
&= \left(\frac{q^2[n][n-1]}{[n+1]^2}\left(1+\frac{2(q-1)}{[2]}+\frac{(q-1)^2}{[3]}\right)-\frac{4[n]q}{[n+1][2]}+1\right)x^2 \\
&+\left(\frac{[n]}{[n+1]^2}\left([2]q[n-1]+2+\frac{4}{[2]}+2q(q-1)[n-1]\right.\right. \\
&\left.\left.+\frac{4(q-1)+q(q-1)^2[n-1][2]+2(q-1)^2}{[3]}\right)-\frac{8[n]q}{[n+1][2]}-\frac{4}{[n+1][2]}+2\right)x \qquad (10)\\
&+\frac{q[n][n-1]}{[n+1]^2}\left(1+\frac{2(q-1)}{[2]}+\frac{(q-1)^2}{[3]}\right)+\frac{2[n]}{[n+1]^2}\left(1+\frac{2}{[2]}+\frac{[2](q-1)}{[3]}\right) \\
&-\frac{4q[n]}{[n+1][2]}+\frac{4}{[n+1]}\left(\frac{1}{[3][n+1]}-\frac{1}{[2]}\right)+1.
\end{aligned}$$

3. Rates of statistical convergence

This section will calculate the order of approximation of the operator $K_n^*(f;q;x)$ with the help of the modulus of continuity. First, we will recall the definition of the modulus of continuity.

The modulus of continuity of a continuous function f is given by

$$\omega(f;\delta) := \sup_{|x-y|\le\delta_{x,y\in[0,a]}} |f(x)-f(y)| \qquad (11)$$

where we are implicitly assuming that

$$\lim_{\delta\to 0^+}\omega(f;\delta)=0, \qquad (12)$$

and for any $\delta>0$,

$$|f(x)-f(y)|\le\omega(f;\delta)\left(\frac{|x-y|}{\delta}+1\right). \qquad (13)$$

Theorem 3.1. *If the sequence $q:=(q_n)$ satisfies the condition given in (4), then*

$$|K_n^*(f;q_n;x)-f(x)|\le 2\omega(f;\sqrt{\delta_{n,x}})$$

for all $f\in C[-1,1]$, where

$$\delta_{n,x}=K_n^*((t-x)^2;q_n;x).$$

Proof. Since the operator $K_n^*(f;q;x)$ is the linear and positiv, using inequality (13) gives

$$\begin{aligned}|K_n^*(f;q_n;x)-f(x)| &\le K_n^*(|f(t)-f(x)|;q;x)\\ &= \frac{[n+1]}{2}\sum_{k=0}^{n}\frac{1}{2^n}q^{-k}\begin{bmatrix}n\\k\end{bmatrix}(1-x)_q^k(1+x)_q^{(n-k-1)}\\ &\times\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}\left(1+\frac{|t-x|}{\delta}\right)\omega(f,\delta)d_q^Rt.\end{aligned}$$

Let us denote

$$\Phi_{n,k,q}(x)=\frac{[n+1]}{2}\sum_{k=0}^{n}\frac{1}{2^n}q^{-k}\begin{bmatrix}n\\k\end{bmatrix}(1-x)_q^k(1+x)_q^{(n-k-1)}$$

and rewrite the above inequality:

$$\begin{aligned}|K_n^*(f;q_n;x)-f(x)| &\le \omega(f,\delta)\sum_{k=0}^{n}\Phi_{n,k,q}(x)\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}d_q^Rt\\ &+\frac{1}{\delta}\omega(f,\delta)\sum_{k=0}^{n}\Phi_{n,k,q}(x)\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}|t-x|d_q^Rt\end{aligned}$$

Using equality $K_n^*(1;q_n;x)=1$ and the H/”older's inequality, we get

$$\begin{aligned}&|K_n^*(f;q_n;x)-f(x)|\\ &\le\omega(f,\delta)\left\{1+\frac{1}{\delta}\sum_{k=0}^{n}\Phi_{n,k,q}(x)\left(\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}(t-x)^2d_q^Rt\right)^{\frac{1}{2}}\left(\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}d_q^Rt\right)^{\frac{1}{2}}\right\}\\ &=\omega(f,\delta)\left\{1+\frac{1}{\delta}\sum_{k=0}^{n}\left(\Phi_{n,k,q}(x)\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}(t-x)^2d_q^Rt\right)^{\frac{1}{2}}\left(\Phi_{n,k,q}(x)\sum_{k=0}^{n}\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}d_q^Rt\right)^{\frac{1}{2}}\right\}\end{aligned}$$

Now, using Hölder's inequality for sums again, we get

$$\begin{aligned}&|K_n^*(f;q_n;x)-f(x)|\\ &\le\omega(f,\delta)\left\{1+\frac{1}{\delta}\left(\sum_{k=0}^{n}\Phi_{n,k,q}(x)\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}(t-x)^2d_q^Rt\right)^{\frac{1}{2}}\right.\\ &\left.\times\left(\sum_{k=0}^{n}\Phi_{n,k,q}(x)\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}d_q^Rt\right)^{\frac{1}{2}}\right\}\\ &=\omega(f,\delta)\left\{1+\frac{1}{\delta}\left(\sum_{k=0}^{n}\Phi_{n,k,q}(x)\int_{\frac{2[k]}{[n+1]}-1}^{\frac{2[k+1]}{[n+1]}-1}(t-x)^2d_q^Rt\right)^{\frac{1}{2}}\right\},\end{aligned}$$

so,

$$|K_n^*(f;q_n;x)-f(x)| \le \omega(f,\delta)\left\{1+\frac{1}{\delta}(K_n^*(t-x)^2;q;x)^{\frac{1}{2}}\right\}. \tag{14}$$

If $q := (q_n)$ is a sequence satisfying condition (4), and
$\delta := \sqrt{\delta_{n,x}} = (K_n^*((t-x)^2;q;x)^{\frac{1}{2}}$, the proof is complete. □

From the conditions (4) we obtain $st - \lim_n K_n^*((t-x)^2;q_n;x) = 0$, so that $st - \lim_n \delta(f,\delta_n) = 0$ from (12). This shows the pointwise statistical convergence rate of the operator $K_n^*(f;q;x)$ to the function f.

Now we examine the second moment of the operator $K_n^*(f;q;x)$ given by Eq. (10). Let's denote the coefficient of x^2 as

$$A := \frac{q^2[n][n-1]}{[n+1]^2}\left(1+\frac{2(q-1)}{[2]}+\frac{(q-1)^2}{[3]}\right)-\frac{4[n]q}{[n+1][2]}+1. \tag{15}$$

In the expression (15),

$$1+\frac{2(q-1)}{[2]}+\frac{(q-1)^2}{[3]} \le \frac{3q^2}{[3]}$$

and since $[n-1] < [n]$,

$$A = \frac{q^2[n][n-1]}{[n+1]^2}\frac{3q^2}{[3]}-\frac{4[n]q}{[n+1][2]}+1 \le \left(\frac{2[n]q}{[2][n+1]}-1\right)^2 \tag{16}$$

is obtained. Again, let's denote the coefficient of x as

$$\begin{aligned} B := \frac{[n]}{[n+1]^2}&\Bigg([2]q[n-1]+2+\frac{4}{[2]}+2q(q-1)[n-1] \\ &+\frac{4(q-1)+q(q-1)^2[n-1][2]+2(q-1)^2}{[3]}\Bigg) \\ &-\frac{8[n]q}{[n+1][2]}-\frac{4}{[n+1][2]}+2. \end{aligned} \tag{17}$$

In the expression (17),

$$\left([2]q+2q(q-1)+\frac{q(q-1)^2[2]}{[3]}\right)[n-1] \le q^2(q+1)[n-1],$$

and since $[n] < [n+1]$,

$$B < \frac{12}{[2][n+1]} \tag{18}$$

is obtained.

Finally, let's denote the constant term in equation (10) as

$$
\begin{aligned}
C := &\frac{q[n][n-1]}{[n+1]^2}\left(1+\frac{2(q-1)}{[2]}+\frac{(q-1)^2}{[3]}\right)\\
&+\frac{2[n]}{[n+1]^2}\left(1+\frac{2}{[2]}+\frac{[2](q-1)}{[3]}\right)\\
&-\frac{4q[n]}{[n+1][2]}+\frac{4}{[n+1]}\left(\frac{1}{[3][n+1]}-\frac{1}{[2]}\right)+1.
\end{aligned}
\tag{19}
$$

When the necessary evaluations are made on the right side of the last equation, we get

$$
C < \frac{28}{[2][3][n+1]}. \tag{20}
$$

Substituting inequalities (16), (18), and (20) into (10) gives

$$
\|K_n^*(f;q;x) - f(x)\| \le \omega(f,\delta_n)\left[1+\frac{1}{\delta}(A+B+C)^{\frac{1}{2}}\right]
$$

where

$$
\delta := \delta_n = \sqrt{\left(\frac{2[n]_{q_n}q_n}{[2]_{q_n}[n+1]_{q_n}}-1\right)^2+\frac{12}{[2]_{q_n}[n+1]_{q_n}}+\frac{28}{[2]_{q_n}[3]_{q_n}[n+1]_{q_n}}}.
$$

In this way, the following theorem is proved.

Theorem 3.2. *Let $q := (q_n)$ be a sequence satisfying (4) and $0 < q_n < 1$. If $f \in C[-1,1]$, then*

$$
\|K_n^*(f;q;x) - f(x)\| \le 2\omega(f,\delta_n)
$$

where

$$
\delta := \delta_n = \sqrt{\left(\frac{2[n]_{q_n}q_n}{[2]_{q_n}[n+1]_{q_n}}-1\right)^2+\frac{12}{[2]_{q_n}[n+1]_{q_n}}+\frac{28}{[2]_{q_n}[3]_{q_n}[n+1]_{q_n}}}.
$$

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