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Fundamental Solution of One Boundary Value Problem on the Geometric Middle of the Domain for the 3d Bianchi Integrodifferential Equation

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Abstract. In this paper, we construct the fundamental solution of one boundary value problem on the geometric middle of the domain for an integro-differential equation of 3D Bianchi type with a dominant mixed derivative of third order with nonsmooth coefficients.

Key Words and Phrases: integro-differential equations, boundary value problem on the geometric middle of the domain, fundamental solutions.

2010 Mathematics Subject Classifications: 35L25, 35L35

1. Introduction

In the mathematical literature, there are only separate scientific papers that investigate the construction of fundamental solutions for hyperbolic equations with dominant mixed derivatives (or pseudo-parabolic equations) with variable coefficients. The works of D. Colton [1], M. Kh. Shkhanukov and A. P. Soldatov [12], performed in this direction, show that for some classes of such equations with sufficiently smooth coefficients the fundamental solution can be defined as an analogue of the classical Riemann function. However, the method of Riemann characteristics applied in these works is very limited, and in general, does not admit generalisation even to the case of simple nonlocal problems, even in the case of equations with constant coefficients. It should be especially noted that in the literature so far the Riemann function for various classes of equations has been constructed only for the case of sufficiently smooth coefficients. Therefore, there arises a very relevant question of investigating the correct solvability and constructing fundamental solutions for boundary value problems associated with hyperbolic equations (or pseudo-parabolic equations) with dominant mixed derivatives, generally speaking, with non-smooth variable coefficients. In this connection, this paper is devoted to the study of boundary value problems of a new type for integro-differential equations of 3D Bianchi type with dominant mixed derivatives having, in general, nonsmooth Lp-coefficients (i.e., coefficients from Lp-type spaces) and to the development of a method for constructing their fundamental solutions. It is stated mainly for third-order integro-differential equations of the 3D Bianchi type with threefold characteristics. An important fundamental point is that the considered equation has nonsmooth coefficients that satisfy only some P-integrability and boundedness conditions, i.e., the considered integrodifferential operator of the 3D Bianchi type does not have a traditional conjugate operator. Therefore, the Riemann function for such an equation cannot be investigated by the classical characteristics method. In this paper, a methodology is developed to investigate such problems, which makes substantial use of modern methods of function theory and functional analysis. With the help of this methodology, a new concept of conjugate problems is introduced for boundary value problems. Such conjugate problems, unlike conjugate problems of the traditional kind defined using formally conjugate differential operators, by definition have the form of an integral equation, and therefore make sense under sufficiently weak conditions on the coefficients. With the help of such a conjugate problem, the notion of a fundamental solution is introduced and an integral representation for the solutions of the corresponding boundary value problems is found.

2. Problem Statement.

Consider an integrodifferential equation of 3D Bianchi type (3D means three-dimensional)

$$\begin{aligned}(V_{1,1,1}u)(x,y,z) \equiv u_{xyz}(x,y,z) + A_{0,0,0}u(x,y,z) + A_{1,0,0}u_x(x,y,z)+ \\ +A_{0,1,0}u_y(x,y,z) + A_{0,0,1}u_z(x,y,z) + A_{1,1,0}u_{xy}(x,y,z) + A_{0,1,1}u_{yz}(x,y,z)+ \\ +A_{1,0,1}u_{xz}(x,y,z) + \int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z}[K_{0,0,0}(\tau,\xi,\eta;x,y,z)\,u(\tau,\xi,\eta)+ \\ +K_{1,0,0}(\tau,\xi,\eta;x,y,z)\times \\ \times u_x(\tau,\xi,\eta) + K_{0,1,0}(\tau,\xi,\eta;x,y,z)u_y(\tau,\xi,\eta) + K_{0,0,1}(\tau,\xi,\eta;x,y,z)\times \\ \times u_z(\tau,\xi,\eta) + K_{1,1,0}(\tau,\xi,\eta;x,y,z)u_{xy}(\tau,\xi,\eta + K_{0,1,1}(\tau,\xi,\eta;x,y,z)\times \\ \times u_{yz}(\tau,\xi,\eta) + K_{1,0,1}(\tau,\xi,\eta;x,y,z)u_{xz}(\tau,\xi,\eta)]\,d\tau d\xi d\eta = \varphi_{1,1,1}(x,y,z), \quad (x,y,z)\in G,\end{aligned} \tag{1}$$

Here$u = u(x,y,z)$ is the desired function defined on G; $A_{i,j,k} = A_{i,j,k}(x,y,z)$ are given measurable functions on$G = G_1 \times G_2 \times G_3$, where $G_1 = (x_0,x_1),\ x_0 \geq 0, G_2 = (y_0,y_1)$, $y_0 \geq 0, G_3 = (z_0,z_1),\ z_0 \geq 0;\quad \varphi_{1,1,1}(x,y,z)$is a given measurable function on$G$.

Equation (1) is a hyperbolic integrodifferential equation that has three real simple characteristics $x = const,\ y = const,\ z = const$.

Moreover, in the literature so far, the Riemann function of equation (1) has been able to be constructed only for the case when$K_{i,j,k}(\tau,\xi,\eta;x,y,z) \equiv 0$, the functions$A_{i,j,k}(x,y,z)$ are sufficiently smooth /i.e., when the functions$A_{i,j,k}(x,y,z)$ are continuous together with the derivatives$D_x^i\,D_y^j\,D_z^k\,A_{i,j,k}(x,y,z)$ in the domain $\overline{G}$ /. In [5], an equation with a dominant derivativeu_{xyz} is studied. This type of equation finds application in models of vibration processes and is also of considerable importance in approximation theory. Therefore, there are problems in practice that are directly related to the three-dimensional boundary value problem or the hyperbolic equation. The three-dimensional contact boundary value problem is considered in [7]. Boundary value problems set at the middle and geometric middle of the domain for integrodifferential equations with dominant mixed derivativesu_{xyz} are studied in the works of the authors [8]- [9].

In this paper, equation (1) is first investigated in the general case when the coefficients$A_{i,j,k}(x,y,z)$ and$K_{i,j,k}(\tau,\xi,\eta;x,y,z)$ are nonsmooth functions satisfying only the following conditions:

$$\begin{aligned}
&A_{0,0,0}(x,y,z)\in L_p(G),\ A_{1,0,0}(x,y,z)\in L^{x,y,z}_{\infty,p,p}(G),\ A_{0,1,0}(x,y,z)\in L^{x,y,z}_{p,\infty,p}(G),\\
&A_{0,0,1}(x,y,z)\in L^{x,y,z}_{p,p,\infty}(G),\ A_{1,1,0}(x,y,z)\in L^{x,y,z}_{\infty,\infty,p}(G),\ A_{0,1,1}(x,y,z)\in L^{x,y,z}_{p,\infty,\infty}(G),\\
&A_{1,0,1}(x,y,z)\in L^{x,y,z}_{\infty,p,\infty}(G),\ \ K_{i,j,k}(\tau,\xi,\eta;x,y,z)\in L_\infty(G\times G)\,.
\end{aligned}$$

Under these conditions, we will look for the solution$u(x,y,z)$ of equation (1) in the S.L. Sobolev space $W_p^{(1,1,1)}(G)=\{\,u\in L_p(G)\ /D_x^i\,D_y^j\,D_z^k u\in L_p(G);\ i,\ j,\ k=0,\,1\}$, where $1\le p\le\infty$. The norm in the space$W_p^{(1,1,1)}(G)$ will be defined by equality$\|u\|_{W_p^{(1,1,1)}(G)}=\sum_{i,j,k=0}^{1}\left\|D_x^iD_y^jD_z^ku\right\|_{L_p(G)}$.

For equation (1), the conditions on the geometric middle of the non-classical domain can be given in the form

$$\begin{cases}
V_{0,0,0}u\equiv u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})=\varphi_{0,0,0}\\
(V_{1,0,0}u)(x)\equiv u_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1})=\varphi_{1,0,0}(x),\\
(V_{0,1,0}u)(y)\equiv u_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1})=\varphi_{0,1,0}(y),\\
(V_{0,0,1}u)(z)\equiv u_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z)=\varphi_{0,0,1}(z),\\
(V_{1,1,0}u)(x,y)\equiv u_{xy}(x,y,\sqrt{z_0z_1})=\varphi_{1,1,0}(x,y),\\
(V_{0,1,1}u)(y,z)\equiv u_{yz}(\sqrt{x_0x_1},y,z)=\varphi_{0,1,1}(y,z),\\
(V_{1,0,1}u)(x,z)\equiv u_{xz}(x,\sqrt{y_0y_1},z)=\varphi_{1,0,1}(x,z),
\end{cases} \tag{2}$$

where$\varphi_{0,0,0}\in R$ is a given number and the remaining$\varphi_{i,j,k}$ are the given functions satisfying the conditions:

$$\varphi_{1,0,0}(x)\in L_p(G_1),\varphi_{0,1,0}(y)\in L_p(G_2),\varphi_{0,0,1}(z)\in L_p(G_3),\varphi_{1,1,0}(x,y)\in L_p(G_1\times G_2),$$

$$\varphi_{0,1,1}(y,z)\in L_p(G_2\times G_3),\ \varphi_{1,0,1}(x,z)\in L_p(G_1\times G_3).$$

3. Operator form of the boundary value problem defined on the geometric middle of the domain

We will investigate the problem (1), (2) by the method of operator equations and follow the scheme of [10]. Preliminary problems (1), and (2) will be written in the form of an operator equation

$$Vu=\varphi, \tag{3}$$

where V is a vector operator defined by the equality

$V=(V_{0,0,0},V_{1,0,0},V_{0,1,0},V_{0,0,1},V_{1,1,0},V_{0,1,1},V_{1,0,1},V_{1,1,1}):W_p^{(1,1,1)}(G)\to \mathrm{E}_p^{(1,1,1)}$, and φ is a given vector element of the form

$$\varphi=(\varphi_{0,0,0},\varphi_{1,0,0},\varphi_{0,1,0},\varphi_{0,0,1},\varphi_{1,1,0},\varphi_{0,1,1},\varphi_{1,0,1},\varphi_{1,1,1})$$

from the space

$$\mathrm{E}_p^{(1,1,1)}\equiv R\times L_p(x_0,x_1)\times L_p(y_0,y_1)\times L_p(z_0,z_1)\times L_p(G_1\times G_2)\times L_p(G_2\times G_3)\times \\ \times L_p(G_1\times G_3)\times L_p(G).$$

Note that in the space$\mathrm{E}_p^{(1,1,1)}$ we will naturally define the norm, using the equality

$$\|\varphi\|_{\mathrm{E}_p^{(1,1,1)}}=\|\varphi_{0,0,0}\|_R+\|\varphi_{1,0,0}\|_{L_p(x_0,x_1)}+\|\varphi_{0,1,0}\|_{L_p(y_0,y_1)}+\|\varphi_{0,0,1}\|_{L_p(z_0,z_1)}+ \\ +\|\varphi_{1,1,0}\|_{L_p(G_1\times G_2)}+\|\varphi_{0,1,1}\|_{L_p(G_2\times G_3)}+\|\varphi_{1,0,1}\|_{L_p(G_1\times G_3)}+\|\varphi_{1,1,1}\|_{L_p(G)}.$$

First of all, we note that integral representations of functions from spaces of$W_p^{(1,1,1)}(G)$ type (from Sobolev spaces with dominating mixed derivatives of general form) have been studied in the works of T.I. Amanov [2], S.M. Nikolsky [11], P.I. Lizorkin and S.M. Nikolsky [6], O.V. Besov, V.P. Ilyin and S.M. Nikolsky [4], S.S. Akhiev [3] and others. But from them we will use the integral representation from [3], by which any function$u(x,y,z)\in W_p^{(1,1,1)}(G)$ is uniquely representable as

$$u(x,y,z)=(Qb)(x,y,z)\equiv b_{0,0,0}+\int_{\sqrt{x_0x_1}}^{x}b_{1,0,0}(\alpha)d\alpha+ \\ +\int_{\sqrt{y_0y_1}}^{y}b_{0,1,0}(\beta)d\beta+\int_{\sqrt{z_0z_1}}^{z}b_{0,0,1}(\gamma)d\gamma+ \\ +\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}b_{1,1,0}(\alpha,\beta)d\alpha d\beta+\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z}b_{0,1,1}(\beta,\gamma)d\beta d\gamma+ \\ +\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{z_0z_1}}^{z}b_{1,0,1}(\alpha,\gamma)d\alpha d\gamma+ \\ +\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z}b_{1,1,1}(\alpha,\beta,\gamma)d\alpha d\beta d\gamma \tag{4}$$

using the single-element

$$b=(b_{0,0,0},b_{1,0,0},b_{0,1,0},b_{0,0,1},b_{1,1,0},b_{0,1,1},b_{1,0,1},b_{1,1,1})\in \mathrm{E}_p^{(1,1,1)}.$$

There exist such positive constantsM_1^0 andM_2^0 that

$$M_1^0\|b\|_{\mathrm{E}_p^{(1,1,1)}}\le\|Qb)(x,y,z)\|_{W_p^{(1,1,1)}(G)}\le M_2^0\|b\|_{\mathrm{E}_p^{(1,1,1)}},\ \text{ for any } b\in\mathrm{E}_p^{(1,1,1)} \tag{5}$$

Obviously, the operator$Q : \mathrm{E}_p^{(1,1,1)} \to W_p^{(1,1,1)}(G)$ is a linear bounded operator. Inequality (5) shows that the operator$Q$ also has a bounded inverse operator defined on the space$W_p^{(1,1,1)}(G)$. Hence, the operator$Q$ is a homeomorphism between the Banach spaces$\mathrm{E}_p^{(1,1,1)}$ and$W_p^{(1,1,1)}(G)$. Therefore, the solution of equation (3) is equivalent to the solution of the equation

$$VQb = \varphi \tag{6}$$

Equation (6) will be called the canonical form of equation (3).

Furthermore, formula (4) shows that any function$u \in W_p^{(1,1,1)}(G)$ has the traces:

$$u(\sqrt{x_0x_1}, \sqrt{y_0y_1}, \sqrt{z_0z_1}), u_x(x, \sqrt{y_0y_1}, \sqrt{z_0z_1}), u_y(\sqrt{x_0x_1}, y, \sqrt{z_0z_1}),$$
$$u_z(\sqrt{x_0x_1}, \sqrt{y_0y_1}, z), u_{xy}(x, y, \sqrt{z_0z_1}), u_{yz}(\sqrt{x_0x_1}, y, z), u_{xz}(x, \sqrt{y_0y_1}, z)$$

and the operations of taking these traces are continuous from$W_p^{(1,1,1)}(G)$ to$R$,

$$L_p(x_0, x_1), \quad L_p(y_0, y_1),\ L_p(z_0, z_1),\ L_p(G_1 \times G_2),\ L_p(G_2 \times G_3),\ L_p(G_1 \times G_3)$$

respectively. Further, for these traces the following equalities are also valid:

$$u(\sqrt{x_0x_1}, \sqrt{y_0y_1}, \sqrt{z_0z_1}) = b_{0,0,0},$$
$$u_x(x, \sqrt{y_0y_1}, \sqrt{z_0z_1}) = b_{1,0,0}(x),$$

$$u_y(\sqrt{x_0x_1}, y, \sqrt{z_0z_1}) = b_{0,1,0}(y),$$
$$u_z(\sqrt{x_0x_1}, \sqrt{y_0y_1}, z) = b_{0,0,1}(z),$$
$$u_{xy}(x, y, \sqrt{z_0z_1}) = b_{1,1,0}(x, y),$$

$$u_{yz}(\sqrt{x_0x_1}, y, z) = b_{0,1,1}(y, z),$$
$$u_{xz}(x, \sqrt{y_0y_1}, z) = b_{1,0,1}(x, z).$$

4. Construction of the conjugate operator

Let

$$f = (f_{0,0,0}, f_{1,0,0}(x),\ f_{0,1,0}(y),\ f_{0,0,1}(z),\ f_{1,1,0}(x, y),$$
$$f_{0,1,1}(y, z),\ f_{1,0,1}(x, z), f_{1,1,1}(x, y, z)) \in \mathrm{E}_q^{(1,1,1)} \equiv$$
$$\equiv R \times L_q(x_0, x_1) \times L_q(y_0, y_1) \times L_q(z_0, z_1) \times L_q(G_1 \times G_2) \times L_q(G_2 \times G_3) \times L_q(G_1 \times G_3) \times L_q(G)$$

where,$1/p + 1/q = 1$ is some linear bounded functional on $\mathrm{E}_p^{(1,1,1)}$. Then by definition, we have

$$f(Vu) = \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{y_0y_1}}^{y_1} \int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(x, y, z)(V_{1,1,1}u)(x, y, z)dxdydz + f_{0,0,0}(V_{0,0,0}u)+$$

$$+ \int_{\sqrt{x_0x_1}}^{x_1} f_{1,0,0}(x)(V_{1,0,0}u)(x)dx+$$

$$+\int_{\sqrt{y_0y_1}}^{y_1} f_{0,1,0}(y)(V_{0,1,0}u)(y)dy+\int_{\sqrt{z_0z_1}}^{z_1} f_{0,0,1}(z)(V_{0,0,1}u)(z)dz+$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1} f_{1,1,0}(x,y)(V_{1,1,0}u)(x,y)dxdy+$$

$$+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{0,1,1}(y,z)(V_{0,1,1}u)(y,z)dydz+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,0,1}(x,z)(V_{1,0,1}u)(x,z)dxdz \tag{7}$$

Considering here the expressions of the operators$V_{i,j,k}$ and using the integral representation (4) of the functions$u \in W_p^{(1,1,1)}(G)$, we obtain from (7):

$$f(Vu)=u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1})(\omega_{0,0,0}f)+\int_{\sqrt{x_0x_1}}^{x_1} u_x(x,\sqrt{y_0y_1},\sqrt{z_0z_1})(\omega_{1,0,0}f)(x)dx+$$
$$+\int_{\sqrt{y_0y_1}}^{y_1} u_y(\sqrt{x_0x_1},y,\sqrt{z_0z_1})(\omega_{0,1,0}f)(y)dy+\int_{\sqrt{z_0z_1}}^{z_1} u_z(\sqrt{x_0x_1},\sqrt{y_0y_1},z)(\omega_{0,0,1}f)(z)dz+ \tag{8}$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1} u_{xy}(x,y,\sqrt{z_0z_1})(\omega_{1,1,0}f)(x,y)dxdy+$$

$$+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} u_{yz}(\sqrt{x_0x_1},y,z)(\omega_{0,1,1}f)(y,z)dydz+$$

$$\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1} u_{xz}(x,\sqrt{y_0y_1},z)(\omega_{1,0,1}f)(x,z)dxdz+$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} u_{xyz}(x,y,z)(\omega_{1,1,1}f)(x,y,z)dxdydz \equiv (V^*f)(u),$$

where

$$\omega_{0,0,0}f \equiv \int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(x,y,z)[A_{0,0,0}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,0}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dG+f_{0,0,0},$$

$$(\omega_{1,0,0}f)(x) \equiv \int_{x}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(\alpha,y,z)[A_{0,0,0}(\alpha,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,0}(\tau,\xi,\eta;\alpha,y,z)d\tau d\xi d\eta]d\alpha dydz+$$

$$+\int_{\sqrt{y_0y_1}}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1.1.1}(x,y,z)[A_{1,0,0}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{1,0,0}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dydz+f_{1,0,0}(x),$$

$$(\omega_{0,1,0}f)(y) \equiv \int_{\sqrt{x_0x_1}}^{x_1}\int_{y}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(x,\beta,z)[A_{0,0,0}(x,\beta,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,0}(\tau,\xi,\eta;x,\beta,z)d\tau d\xi d\eta]\times$$

$$\times dxd\beta dz + \int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(x,y,z)[A_{0,1,0}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,1,0}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dxdz + f_{0,1,0}(y),$$

$$(\omega_{0,0,1}f)(z) \equiv \int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{z}^{z_1} f_{1,1,1}(x,y,\gamma)[A_{0,0,0}(x,y,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{\gamma} K_{0,0,0}(\tau,\xi,\eta;x,y,\gamma)d\tau d\xi d\eta]dxdyd\gamma+$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1} f_{1,1,1}(x,y,z)[A_{0,0,1}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,1}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dxdy + f_{0,0,1}(z),$$

$$(\omega_{1,1,0}f)(x,y) \equiv \int_{x}^{x_1}\int_{y}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(\alpha,\beta,z)[A_{0,0,0}(\alpha,\beta,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,0}(\tau,\xi,\eta;\alpha,\beta,z)d\tau d\xi d\eta]d\alpha d\beta dz+$$

$$+\int_{y}^{y_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(x,\beta,z)[A_{1,0,0}(x,\beta,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{z} K_{1,0,0}(\tau,\xi,\eta;x,\beta,z)d\tau d\xi d\eta]d\beta dz+$$

$$+\int_{x}^{x_1}\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(\alpha,y,z)[A_{0,1,0}(\alpha,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,1,0}(\tau,\xi,\eta;\alpha,y,z)d\tau d\xi d\eta]d\alpha dz+$$

$$+\int_{\sqrt{z_0z_1}}^{z_1} f_{1,1,1}(x,y,z)[A_{1,1,0}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{1,1,0}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dz + f_{1,1,0}(x,y),$$

$$(\omega_{0,1,1}f)(y,z) \equiv \int_{\sqrt{x_0x_1}}^{x_1}\int_{y}^{y_1}\int_{z}^{z_1} f_{1,1,1}(x,\beta,\gamma)[A_{0,0,0}(x,\beta,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{\gamma} K_{0,0,0}(\tau,\xi,\eta;x,\beta,\gamma)d\tau d\xi d\eta]dxd\beta d\gamma+$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{z}^{z_1} f_{1,1,1}(x,y,\gamma)[A_{0,1,0}(x,y,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{\gamma} K_{0,1,0}(\tau,\xi,\eta;x,y,\gamma)d\tau d\xi d\eta]dxd\gamma+$$

$$+\int_{\sqrt{x_0x_1}}^{x_1}\int_{y}^{y_1} f_{1,1,1}(x,\beta,z)[A_{0,0,1}(x,\beta,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,1}(\tau,\xi,\eta;x,\beta,z)d\tau d\xi d\eta]dxd\beta+$$

$$+\int_{\sqrt{x_0x_1}}^{x_1} f_{1,1,1}(x,y,z)[A_{0,1,1}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,1,1}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dx+f_{0,1,1}(y,z),$$

$$(\omega_{1,0,1}f)(x,z)\equiv\int_{x}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1}\int_{z}^{z_1} f_{1,1,1}(\alpha,y,\gamma)[A_{0,0,0}(\alpha,y,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{\gamma} K_{0,0,0}(\tau,\xi,\eta;\alpha,y,\gamma)d\tau d\xi d\eta]d\alpha dyd\gamma+$$

$$+\int_{y_0}^{y_1}\int_{z}^{z_1} f_{1,1,1}(x,y,\gamma)[A_{1,0,0}(x,y,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{\gamma} K_{1,0,0}(\tau,\xi,\eta;x,y,\gamma)d\tau d\xi d\eta]dyd\gamma+$$

$$+\int_{x}^{x_1}\int_{\sqrt{y_0y_1}}^{y_1} f_{1,1,1}(\alpha,y,z)[A_{0,0,1}(\alpha,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,1}(\tau,\xi,\eta;\alpha,y,z)d\tau d\xi d\eta]d\alpha dy+$$

$$+\int_{\sqrt{y_0y_1}}^{y_1} f_{1,1,1}(x,y,z)[A_{1,0,1}(x,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{1,0,1}(\tau,\xi,\eta;x,y,z)d\tau d\xi d\eta]dy+f_{1,0,1}(x,z),$$

$$(\omega_{1,1,1}f)(x,y,z)\equiv f_{1,1,1}(x,y,z)+$$

$$+\int_x^{x_1}\int_y^{y_1}\int_z^{z_1} f_{1,1,1}(\alpha,\beta,\gamma)[A_{0,0,0}(\alpha,\beta,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{\gamma} K_{0,0,0}(\tau,\xi,\eta;\alpha,\beta,\gamma)d\tau d\xi d\eta]d\alpha d\beta d\gamma+$$

$$+\int_y^{y_1}\int_z^{z_1} f_{1,1,1}(x,\beta,\gamma)[A_{1,0,0}(x,\beta,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{\gamma} K_{1,0,0}(\tau,\xi,\eta;x,\beta,\gamma)d\tau d\xi d\eta]d\beta d\gamma+$$

$$+\int_x^{x_1}\int_z^{z_1} f_{1,1,1}(\alpha,y,\gamma)[A_{0,1,0}(\alpha,y,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{\gamma} K_{0,1,0}(\tau,\xi,\eta;\alpha,y,\gamma)d\tau d\xi d\eta]d\alpha d\gamma+$$

$$+\int_x^{x_1}\int_y^{y_1} f_{1,1,1}(\alpha,\beta,z)[A_{0,0,1}(\alpha,\beta,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{z} K_{0,0,1}(\tau,\xi,\eta;\alpha,\beta,z)d\tau d\xi d\eta]d\alpha d\beta+$$

$$+\int_z^{z_1} f_{1,1,1}(x,y,\gamma)[A_{1,1,0}(x,y,\gamma)+$$

$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{\gamma} K_{1,1,0}(\tau,\xi,\eta;x,y,\gamma)d\tau d\xi d\eta]d\gamma+$$

$$+\int_x^{x_1} f_{1,1,1}(\alpha,y,z)[A_{0,1,1}(\alpha,y,z)+$$

$$+\int_{\sqrt{x_0x_1}}^{\alpha}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} K_{0,1,1}(\tau,\xi,\eta;\alpha,y,z)d\tau d\xi d\eta]d\alpha+$$

$$+\int_y^{y_1} f_{1,1,1}(x,\beta,z)[A_{1,0,1}(x,\beta,z)+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{\beta}\int_{\sqrt{z_0z_1}}^{z} K_{1,0,1}(\tau,\xi,\eta;x,\beta,z)d\tau d\xi d\eta]d\beta. \tag{9}$$

Using the equality (8), i.e., the equality $f(Vu)=(V^*f)(u)$, as well as the general form of linear functionals in $W_p^{(1,1,1)}(G)$ we obtain that the operator V has a conjugate operator of the form:

$$V^*=(\omega_{0,0,0},\omega_{1,0,0},\omega_{0,1,0},\omega_{0,0,1},\omega_{1,1,0},\omega_{0,1,1},\omega_{1,0,1},\omega_{1,1,1}):\mathrm{E}_q^{(1,1,1)}\to\mathrm{E}_q^{(1,1,1)}.$$

5. Construction of the fundamental solution for the boundary value problem

Now take an arbitrary point$(x, y, z) \in \bar{G}$ and consider the system:

$$\begin{cases} \omega_{0,0,0} f = 1, \\ (\omega_{1,0,0} f)(\alpha) = \theta(x-\alpha), \alpha \in (x_0, x_1), \\ (\omega_{0,1,0} f)(\beta) = \theta(y-\beta), \beta \in (y_0, y_1), \\ (\omega_{0,0,1} f)(\gamma) = \theta(z-\gamma), \gamma \in (z_0, z_1), \\ (\omega_{1,1,0} f)(\alpha, \beta) = \theta(x-\alpha)\theta(y-\beta), (\alpha, \beta) \in G_1 \times G_2, \\ (\omega_{0,1,1} f)(\beta, \gamma) = \theta(y-\beta)\theta(z-\gamma), (\beta, \gamma) \in G_2 \times G_3, \\ (\omega_{1,0,1} f)(\alpha, \gamma) = \theta(x-\alpha)\theta(z-\gamma), (\alpha, \gamma) \in G_1 \times G_3, \\ (\omega_{1,1,1} f)(\alpha, \beta, \gamma) = \theta(x-\alpha)\theta(y-\beta)(z-\gamma), (\alpha, \beta, \gamma) \in G, \end{cases} \tag{10}$$

where, $\theta(z)$ is a Heaviside function on R , i.e., $\theta(z) = \begin{cases} 1, z > 0 \\ 0, z \leq 0 \end{cases}$.

Definition. If for each given point$(x, y, z) \in \bar{G}$ the system (10) has at least one solution

$$f(x,y,z) = (f_{0,0,0}(x,y,z), f_{1,0,0}(\alpha; x,y,z), f_{0,1,0}(\beta; x,y,z), f_{0,0,1}(\gamma; x,y,z),$$
$$f_{1,1,0}(\alpha,\beta; x,y,z), f_{0,1,1}(\beta,\gamma; x,y,z), f_{1,0,1}(\alpha,\gamma; x,y,z), f_{1,1,1}(\alpha,\beta,\gamma; x,y,z)) \in \mathrm{E}_q^{(1,1,1)},$$

then this solution will be called the fundamental (θ -fundamental) solution of the problem (1),(2).

The last component$f_{1,1,1}(\alpha, \beta, \gamma; x, y, z)$ can be considered as a new Riemann function for the boundary value problem set in the middle of the domain. It can be shown that if the coefficients$A_{i,j,k}(x, y, z)$ have continuous derivatives$D_x^i D_y^j D_z^k A_{i,j,k}(x, y, z)$ in the domain$\bar{G}$, and$K_{i,j,k}(\tau,\ \xi,\ \eta;\ x,\ y,\ z) \equiv 0$ then$f_{1,1,1}(\alpha, \beta, \gamma; x, y, z)$ is a Riemann function in the classical sense.

Theorem. Problem (1), (2) has a uniqueθ -fundamental solution$f(x, y, z)$. Thus the solution$u \in W_p^{(1,1,1)}(G)$ to problems (1), and (2) can be represented usingθ -fundamental solution in the form of

$$u(x,y,z) = \varphi_{0,0,0} f_{0,0,0}(x,y,z) + \int_{\sqrt{x_0 x_1}}^{x_1} \varphi_{1,0,0}(\alpha) f_{1,0,0}(\alpha; x,y,z) d\alpha +$$

$$+ \int_{\sqrt{y_0 y_1}}^{y_1} \varphi_{0,1,0}(\beta) f_{0,1,0}(\beta; x,y,z) d\beta + \int_{\sqrt{z_0 z_1}}^{z_1} \varphi_{0,0,1}(\gamma) f_{0,0,1}(\gamma; x,y,z) d\gamma +$$

$$+ \int_{\sqrt{z_0 z_1}}^{z_1} \varphi_{0,0,1}(\gamma) f_{0,0,1}(\gamma; x,y,z) d\gamma + \int_{\sqrt{x_0 x_1}}^{x_1} \int_{\sqrt{y_0 y_1}}^{y_1} \varphi_{1,1,0}(\alpha,\beta) f_{1,1,0}(\alpha,\beta; x,y,z) d\alpha d\beta +$$

$$+ \int_{\sqrt{y_0 y_1}}^{y_1} \int_{\sqrt{z_0 z_1}}^{z_1} \varphi_{0,1,1}(\beta,\gamma) f_{0,1,1}(\beta,\gamma; x,y,z) d\beta d\gamma +$$
$$+ \int_{\sqrt{x_0 x_1}}^{x_1} \int_{\sqrt{z_0 z_1}}^{z_1} \varphi_{1,0,1}(\alpha,\gamma) f_{1,0,1}(\alpha,\gamma; x,y,z) d\alpha d\gamma +$$

$$+ \int_{\sqrt{x_0 x_1}}^{x_1} \int_{\sqrt{y_0 y_1}}^{y_1} \int_{\sqrt{z_0 z_1}}^{z_1} \varphi_{1,1,1}(\alpha,\beta,\gamma) f_{1,1,1}(\alpha,\beta,\gamma; x,y,z) d\alpha d\beta d\gamma. \tag{11}$$

Proof. The existence of the uniqueness ofθ -the fundamental solution follows from system (10). To prove the validity of representation (11) we will compare the right parts of identities (7) and (8) on the solutions$u \in W_p^{(1,1,1)}(G)$ and$f \in \mathrm{E}_q^{(1,1,1)}$ of problem (1), (2) and system (10). Then we obtain:

$$\varphi_{0,0,0} f_{0,0,0}(x,y,z) + \int_{\sqrt{x_0x_1}}^{x_1} \varphi_{1,0,0}(\alpha) f_{1,0,0}(\alpha; x,y,z) d\alpha + \int_{\sqrt{y_0y_1}}^{y_1} \varphi_{0,1,0}(\beta) f_{0,1,0}(\beta; x,y,z) d\beta +$$
$$+ \int_{\sqrt{z_0z_1}}^{z_1} \varphi_{0,0,1}(\gamma) f_{0,0,1}(\gamma; x,y,z) d\gamma + \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{y_0y_1}}^{y_1} \varphi_{1,1,0}(\alpha,\beta) f_{1,1,0}(\alpha,\beta; x,y,z) d\alpha d\beta +$$
$$+ \int_{\sqrt{y_0y_1}}^{y_1} \int_{\sqrt{z_0z_1}}^{z_1} \varphi_{0,1,1}(\beta,\gamma) f_{0,1,1}(\beta,\gamma; x,y,z) d\beta d\gamma +$$
$$+ \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{z_0z_1}}^{z_1} \varphi_{1,0,1}(\alpha,\gamma) f_{1,0,1}(\alpha,\gamma; x,y,z) d\alpha d\gamma +$$
$$+ \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{y_0y_1}}^{y_1} \int_{\sqrt{z_0z_1}}^{z_1} \varphi_{1,1,1}(\alpha,\beta,\gamma) f_{1,1,1}(\alpha,\beta,\gamma; x,y,z) d\alpha d\beta d\gamma =$$
$$= u(\sqrt{x_0x_1}, \sqrt{y_0y_1}, \sqrt{z_0z_1}) +$$
$$+ \int_{\sqrt{x_0x_1}}^{x_1} u_x(\alpha, \sqrt{y_0y_1}, \sqrt{z_0z_1}) \theta(x-\alpha) d\alpha + \int_{\sqrt{y_0y_1}}^{y_1} u_y(\sqrt{x_0x_1}, \beta, \sqrt{z_0z_1}) \theta(y-\beta) d\beta +$$
$$+ \int_{\sqrt{z_0z_1}}^{z_1} u_z(\sqrt{x_0x_1}, \sqrt{y_0y_1}, \gamma) \theta(z-\gamma) d\gamma +$$
$$+ \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{y_0y_1}}^{y_1} u_{xy}(\alpha,\beta,z) \theta(x-\alpha)\theta(y-\beta) d\alpha d\beta +$$
$$+ \int_{\sqrt{y_0y_1}}^{y_1} \int_{\sqrt{z_0z_1}}^{z_1} u_{yz}(\sqrt{x_0x_1}, \beta, \gamma) \theta(y-\beta)\theta(z-\gamma) d\beta d\gamma +$$
$$+ \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{z_0z_1}}^{z_1} u_{xz}(\alpha, \sqrt{y_0y_1}, \gamma) \theta(x-\alpha)\theta(z-\gamma) d\alpha d\gamma + \quad (12)$$
$$+ \int_{\sqrt{x_0x_1}}^{x_1} \int_{\sqrt{y_0y_1}}^{y_1} \int_{\sqrt{z_0z_1}}^{z_1} u_{xyz}(\alpha,\beta,\gamma) \theta(x-\alpha)\theta(y-\beta)\theta(z-\gamma) d\alpha d\beta d\gamma$$

The integral representation (4) of the functions$u \in W_p^{(1,1,1)}(G)$ shows that the right-hand side of formula (12) coincides with the value $u(x,y,z)$ of the solution $u(\alpha,\beta,\gamma)$ at the point(x,y,z) . By this, the validity of the representation (11) is proved. The theorem is proved.

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