Journal of Contemporary Applied Mathematics
V. 14, No 2, 2024, December
ISSN 2222-5498
https://doi.org/10.69624/2222-5498.2024.2.61

Complex Solutions in Magnetic Schrodinger Equations with Critical Nonlinear Terms

S. Jain, F. Hashemi, M. Alimohammady, C. Cesarano and P. Agarwal

Abstract. Through the application of minimization techniques, we demonstrate the existence of a complex solution to the magnetic Schrodinger equation

$$(-\hbar i\nabla + B(y))^2 w + W(y)w = |w|^{2^*-2}w \text{ in } \mathbb{R}^N,$$

Here $N \geq 3$, $B : \mathbb{R}^N \to \mathbb{R}^N$ represents the magnetic potential, and $W : \mathbb{R}^N \to \mathbb{R}$ denotes the bounded electric potential.

Key Words and Phrases: Magnetic Schrodinger equations, critical nonlinearities, minimization problem, concentration-compactness method, Lagrange

2010 Mathematics Subject Classifications: 65K10, 90C05

1. Introduction

This article aims to examine the magnetic Schrodinger equation

$$(-\hbar i\nabla + B(y))^2 w + W(y)w = |w|^{2^*-2}w \text{ in } \mathbb{R}^N, \tag{1}$$

In this context $w : \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, i is the imaginary unit, and $B = (B_1, ..., B_N) : \mathbb{R}^N \to \mathbb{R}^N$ represents the magnetic (or vector) potential. Here, it is important to mention that the many complex models were studied with the different types of Schrodinger equations [1, 9, 10, 16, 17, 18, 19], [4], [5], [11], [20], [2].

In this article, we adopt the following assumptions.

(B_1) $B \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and there exists a point $y_0 \in \mathbb{R}^N$ where B is continuous at y_0;

(B_2) $B(\frac{y}{b}) = bB(y)$ for all $b > 0$;

(B_3) $B \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$.

A common example of a function that meets conditions (B_1)-(B_3) is $B(y) = \frac{B}{|y|}$, where B is a constant vector.

Theorem 1.1. *Assuming $N \geq 3$ and that conditions (B_1)-(B_3) are satisfied, problem* (1) *admits a non-trivial complex solution.*

In 1989, Esteban and Lions [19] discovered solutions to

$$(-i\nabla + A(x))^2 u + \lambda u = |u|^4 u \quad \text{in} \quad \mathbb{R}^3$$

with $\lambda \in \mathbb{R}$ by addressing constrained minimization problems through Concentration-Compactness methods. In 2006, Barile, Cingolani, and Secchi [6] established existence results using abstract perturbation techniques to

$$(i\nabla + \varepsilon A(x))^2 w + \varepsilon^\alpha V(x) w = |u|^{2^*-2} w \quad \text{in} \quad \mathbb{R}^N,$$

where $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in [1, 2]$, $N > 4$ and the potentials A and V are bounded continuous and Lebesgue measurable. In 2011, Liang and Zhang [14] studied standing wave solutions $\psi(x,t) = e^{-\frac{iEt}{\hbar}} u(x)$, $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $N \geq 3$ to

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + iA(x))^2 \psi + W(x)\psi - h(x, |\psi|^2)\psi,$$

This establishes the presence of at least one solution, and for every $m \in \mathbb{N}$, there are at least m pairs of solutions under appropriate conditions. In 2014, Liang and Song [13] examined

$$-\varepsilon^2 (\nabla + iA(x))^2 u + V(x) u = |u|^{2^*-2} u + h(x, |u|^2) u \quad \text{in} \quad \mathbb{R}^N$$

Given that $N \geq 3$ and $V(y)$ is a nonnegative potential, we establish the existence of at least one solution and m pairs of solutions for every $m \in \mathbb{N}$ by employing Lions' second Concentration-Compactness method along with the Concentration-Compactness principle at infinity, provided that $\hbar > 0$ is sufficiently small, to achieve a (PS) condition of type c.

2. Notation and variational tools

To present the variational framework of the problem, we define

$$H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C}) = \{w \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_{\hbar,B} w \in L^2(\mathbb{R}^N, \mathbb{C})\}$$

with $\nabla_{\hbar,B} w = (\hbar\nabla + iB)w$. The space $H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$ is a Hilbert space equipped with the scalar product

$$(w, v)_{\hbar,B} := Re \int_{\mathbb{R}^N} (\nabla_{\hbar,B} w . \overline{\nabla_{\hbar,B} v} + W(y) w \bar{v}) dy \quad \text{for any } w, v \in H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$$

where **Re** denotes the real part of a complex number and the **bar** represents complex conjugation, respectively. The norm associated with this inner product is

$$\|w\|_{\hbar,B} = \int (|\nabla_{\hbar,B} w|^2 + W(y)|w|^2 dy)^{\frac{1}{2}} \quad \text{for} \quad w \in H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C}).$$

and $C_0^\infty(\mathbb{R}^N,\mathbb{C})$ is dense in $H^\alpha_{\hbar,B}(\mathbb{R}^N,\mathbb{C})$ with respect to the norm $\|.\|_{\hbar,B}$ (see [19] , section 2) and [7, theorem 7.22]. It is important to note that for every $w\in H^\alpha_{\hbar,B}(\mathbb{R}^N,\mathbb{C})$, it holds that

$$\int_{\mathbb{R}^N}|\nabla_{\hbar,B}w|^2=\hbar^2\int_{\mathbb{R}^N}|\nabla w|^2+\int_{\mathbb{R}^N}|B(x)|^2|w|^2-2Re\int\hbar\nabla w.iB(y)\bar{w},$$

given that there is no connection between $H^\alpha_{\hbar,B}(\mathbb{R}^N,\mathbb{C})$ and $H^\alpha(\mathbb{R}^N,\mathbb{C})$; that is, $H^\alpha_{\hbar,B}(\mathbb{R}^N,\mathbb{C})\nsubseteq H^\alpha(\mathbb{R}^N,\mathbb{R})$ and $H^\alpha(\mathbb{R}^N,\mathbb{R})\nsubseteq H^\alpha_{\hbar,B}(\mathbb{R}^N,\mathbb{C})$, we will often utilize the following diamagnetic inequality throughout this paper (refer to [15, theorem 7.21]).

$$\hbar|\nabla|w|(y)|\leq|\nabla_{\hbar,B}w(y)|\quad\text{for almost every}\quad y\in\mathbb{R}^N.$$

This indicates that, if $w\in H^\alpha_{\hbar,B}(\mathbb{R}^N,\mathbb{C})$ then $|w|\in H^1(\mathbb{R}^N,\mathbb{R})$. So, $w\in L^p(\mathbb{R}^N,\mathbb{C})$ for all $p\in[2,2^*]$.

Furthermore, we examine the space

$$D^{1,2}_{\hbar,B}(\mathbb{R}^N,\mathbb{C})=\{w\in L^{2^*}(\mathbb{R}^N,\mathbb{C}),\int_{\mathbb{R}^N}|\nabla_{\hbar,B}w|^2dy<+\infty\}$$

which represents the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$ in terms of the norm

$$\|w\|_{\varrho^{1,2}_{\hbar,B}}=(\int|\nabla_{\hbar,B}w|^2dy)^{\frac{1}{2}}\quad\text{for}\quad w\in D^{1,2}_{\hbar,B}(\mathbb{R}^N,\mathbb{C}),$$

associated with the inner product

$$(w,v)_{D^{1,2}_{\hbar,B}}=Re\int_{\mathbb{R}^N}\nabla_{\hbar,B}w.\overline{\nabla_{\hbar,B}v}dy\quad\text{for}\quad w,v\in D^{1,2}_{\hbar,B}(\mathbb{R}^N,\mathbb{C}).$$

It is important to remember that $D^{1,2}_{\hbar,B}(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{C})$. It is also advantageous to define

$$D^{1,2}(\mathbb{R}^N,\mathbb{R})=(\int_{\mathbb{R}^N}|\nabla w|^2dy)^{\frac{1}{2}}\quad\text{for}\quad w\in D^{1,2}(\mathbb{R}^N,\mathbb{R}),$$

related to the inner product

$$(w,v)_{D^{1,2}}=Re\int_{\mathbb{R}^N}\nabla w\nabla vdy\quad\text{for}\quad w,v\in D^{1,2}(\mathbb{R}^N,\mathbb{R}).$$

Keep in mind that $D^{1,2}(\mathbb{R}^N,\mathbb{R})\hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$ and we refer to as $C_\wp>0$ the best constant of Sobolev embedding $D^{1,2}(\mathbb{R}^N,\mathbb{R})\hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$; that is

$$C_\wp(\int_{\mathbb{R}^N}|w|^{2^*}dy)^{\frac{2}{2^*}}\leq\int_{\mathbb{R}^N}|\nabla w|^2dy\quad\text{for all}\quad w\in D^{1,2}(\mathbb{R}^N,\mathbb{R}).$$

If C denotes the best constant of the embedding $\varrho^{1,2}_{\hbar,B}(\mathbb{R}^N,\mathbb{C})\to L^{2^*}(\mathbb{R}^N,\mathbb{C})$,, that is

$$C=\inf_{w\in\varrho^{1,2}_{\hbar,B}}\frac{\int_{\mathbb{R}^N}|\nabla_{\hbar,B}w|^2dy}{(\int_{\mathbb{R}^N}|w|^{2^*}dy)^{\frac{2}{2^*}}}$$

we have that $C = C_{\wp}$; for details see [3, theorem 1.1].

The energy functional can be expressed as $\Gamma_{\hbar,B} : H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ associated with (1) is specified as

$$\Gamma_{\hbar,B}(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy + \frac{1}{2}\int_{\mathbb{R}^N} (W(y)|w|^2) dy - \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dy$$
$$\Gamma_{\hbar,B}(w) = \frac{1}{2}\|w\|^2_{\hbar,B} - \frac{1}{2^*}\|w\|^{2^*}_{2^*}, \quad \text{for} \quad w \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}).$$

We achieve $\Gamma_{\hbar,B} \in C^1(H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}), \mathbb{R})$ with Gâteaux differential

$$\Gamma'_{\hbar,B}(w)v = Re\int_{\mathbb{R}^N} (\nabla_{\hbar,B} w.\overline{\nabla_{\hbar,B} v} + W(y)w\bar{v})dy - \int |w|^{2^*-2} wv dy \quad \text{for all} \quad w, v \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}),$$

and its critical points are the weak solutions to (1). Let us consider

$$\varrho_{\hbar,B} = \inf\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy : \quad w \in \vartheta_{\hbar,B}\}$$

where

$$\vartheta_{\hbar,B} = \{w \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} : \quad J(w) = 1\} \quad \text{with} \quad N \geq 3,$$

with

$$J(w) = \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dy - \frac{1}{2}\int_{\mathbb{R}^N} (W(y)|w|^2) dy \quad \text{for} \quad w \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}).$$

From this point forward, we can denote by

$$\varrho_{\hbar,B} = \inf_{w \in \vartheta_{\hbar,B}} \tau_{\hbar,B}(w),$$

where, to simplify notation,

$$\tau_{\hbar,B}(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy \quad \text{for} \quad w \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C}).$$

Additionally, we define

$$J(w) = \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dy - \frac{1}{2}\int_{\mathbb{R}^N} (W(y)|w|^2) dy \quad \text{for} \quad w \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$$
$$J'(w)v = Re\int_{\mathbb{R}^N} (|w|^{2^*-1} - W(y)|w|)\bar{v} dy \quad \text{for} \quad w, v \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$$

3. Preliminary results

Here, we present some preliminary results that will be utilized in the proof of Theorem 1.1 in section 4.

Lemma 3.1. *Under the assumption that (B_1), the following statements are valid:*

(i) $\vartheta_{\hbar,B}$ *is not empty;*

(ii) $\vartheta_{\hbar,B}$ *is a* C^1*-manifold.*

Proof. (i) for $w \neq 0$ one has $J(w) < 0$ if $t > 0$ is small, and $J(tu) \to +\infty$ if $t \to +\infty$. Hence, $J(t_0) = 1$, for some $t_0 > 0$. since

$$\begin{aligned} J(tw) &= \frac{1}{2^*}\int |tw|^{2^*} - \frac{1}{2}\int W(y)|tw|^2 \\ &= \frac{t^{2^*}}{2^*}\int |w|^{2^*} - \frac{t^2}{2}\int W(y)|w|^2. \end{aligned}$$

(ii) According to the definition of J and J' provided in Section 2, for all $w \in \vartheta_{\hbar,B}$ we obtain

$$J'(w)(v) = \int_{\mathbb{R}^N} (|w|^{2^*} - W(y)|w|^2)dy$$

since $w \in \vartheta_{\hbar,B}$, therefore $J(w) = 1$, it can be concluded that

$$J(w) = \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} - \frac{1}{2}\int (W(y)|w|^2)dy = 1,$$

therefore

$$2^*\int_{\mathbb{R}^N} |w|^{2^*}dy - 2^*\int_{\mathbb{R}^N} (W(y)|w|^2)dy > 2\int_{\mathbb{R}^N} |w|^{2^*}dy - 2^*\int_{\mathbb{R}^N} (W(y)|w|^2)dy = 22^*,$$

therefore

$$\int_{\mathbb{R}^N} |w|^{2^*}dy - \int_{\mathbb{R}^N} (W(y)|w|^2)dy > 2 > 0 \quad \text{for all} \quad w \in \vartheta_{\hbar,B}.$$

Then $J'(w)v \neq 0$ for all $w \in \vartheta_{\hbar,B}$. □

Lemma 3.2. *Any minimizing sequence* $\{w_n\}$ *for* $\varrho_{\hbar,B}$ *is bounded in* $H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$.

Proof. Given $\{w_n\}$ a minimizing sequence for $\varrho_{\hbar,B}$ in $H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$. We need to demonstrate $\|w_n\|_{\hbar,B} \leq c$ for all $n \in \mathbb{N}$ and for some constant $c > 0$. It is known that

$$\|w_n\|_{\hbar,B} = \int (|\nabla_{\hbar,B} w_n|^2 + W(y)|w_n|^2 dy)^{\frac{1}{2}}, \quad w_n \in H^\alpha_{\hbar,B}(\mathbb{R}^N, \mathbb{C}).$$

First, we derive

$$\frac{1}{2}\int |\nabla_{\hbar,B} w_n|^2 dy \to \varrho_{\hbar,B} \quad \text{as} \quad n \to +\infty$$

and $J(w_n) = 1$, that is $\frac{1}{2^*}\int_{\mathbb{R}^N} |w_n|^{2^*}dy - \frac{1}{2}\int_{\mathbb{R}^N} (W(y)|w_n|^2)dy = 1$. As a result,

$$\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w_n|^2 dy \leq c$$

for all $n \in \mathbb{N}$ and for some $c > 0$ and

$$\int_{\mathbb{R}^N} (W(y)|w_n|^2)dy = \frac{2}{2^*}\int_{\mathbb{R}^N} |w_n|^{2^*} dy - 2$$
$$\leq \frac{2}{2^*}\int_{\mathbb{R}^N} |w_n|^{2^*} dy$$
$$\leq \frac{2}{2^* C^{\frac{2^*}{2}}}(\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w_n|^2 dy)^{\frac{2^*}{2}} \leq \bar{c}.$$

As a consequence,

$$\|w_n\|^2_{\hbar,B} = \int (|\nabla_{\hbar,B} w_n|^2 + W(y)|w_n|^2)dy \leq c + \bar{c}.$$

So

$$\|w_n\|^2_{\hbar,B} \leq c^*.$$

Therefore, $\{w_n\}$ is bounded in $H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$ □

Remark 3.3. *The weak limit w of any minimizing sequence $\{w_n\}$ associated with $D_{\hbar,B}$ is nontrivial. Since, if $|w| = 0$ then $J(w) = 0$ that is absurd given $J(w) = 1$ and $w \in \vartheta_{\hbar,B}$.*

4. Proof of Theorem

We demonstrate that $\varrho_{\hbar,B}$ is achieved by w is the nontrivial weak limit of the minimizing sequence $\{w_n\}$ to $\varrho_{\hbar,B}$.

Indeed, since $\{w_n\}$ is bounded, we have $w_n \rightharpoonup w$ in $H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$ and being the weak limit w is not trivial, we deduce that

$$\tau_{\hbar,B}(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w_n|^2 dy \leq \liminf_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy = \varrho_{\hbar,B}.$$

Now, it is necessary to demonstrate that $w \in \vartheta_{\hbar,B}$. Since

$$J(w) = \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dy - \frac{1}{2}\int_{\mathbb{R}^N} W(y)|w|^2 dy$$

and $w_n \rightharpoonup w$ in $H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$ and $\{w_n\}$ is bounded.

Therefore

$$J(w) \leq \liminf J(w_n), \quad as \quad n \to \infty.$$

Now, since $J(w_n) = 1$ then $J(w) \leq 1$. If we can also establish that

$$J(w) \geq 1, \tag{2}$$

then we obtain the result that $w \in \vartheta_{\hbar,B}$ and $\tau_{\hbar,B}(w) = \varrho_{\hbar,B}$, or equivalently

$$\tau_{\hbar,B}(u) = \varrho_{\hbar,B} = \min\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy : w \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})\backslash\{0\}, J(w) = 1\}.$$

To show (2), we abtain

$$
\begin{aligned}
1 = J(w_n) &= \frac{1}{2^*}\int_{\mathbb{R}^N} |w_n|^{2^*} dy - \frac{1}{2}\int_{\mathbb{R}^N} (W(y)|w_n|^2) dy \\
&= \frac{1}{2^*}\int_{B_R} |w_n|^{2^*} dy - \frac{1}{2}\int_{B_R} (W(y)|w_n|^2) dy \\
&+ \frac{1}{2^*}\int_{B_R^c} |w_n|^{2^*} dy - \frac{1}{2}\int_{B_R^c} (W(y)|w_n|^2) dy
\end{aligned} \tag{3}
$$

Now, since

$$
\begin{aligned}
\frac{1}{2^*}\int_{B_R} |w_n|^{2^*} dy &= 1 + \frac{1}{2}\int_{B_R} (W(y)|w_n|^2) dy \\
-\frac{1}{2^*}\int_{B_R^c} |w_n|^{2^*} dy &+ \frac{1}{2}\int_{B_R^c} (W(y)|w_n|^2) dy
\end{aligned} \tag{4}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{2^*}\int_{B_R} |w_n|^{2^*} dy - \frac{1}{2}\int_{B_R} (W(y)|w_n|^2) dy \\
&= 1 + \int_{B_R^c} (\frac{1}{2} W(y)|w_n|^2 - \frac{1}{2^*}|w_n|^{2^*}) dy
\end{aligned} \tag{5}
$$

Now, by limit inferior with respect to the sum of sequences, we obtain

$$
J(w) \geq 1. \tag{6}
$$

so, $\tau_{\hbar,B}(w) = D_{\hbar,B}$ and $w \in \mu_{\hbar,B}$; that is

$$
\varrho_{\hbar,B} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy, \quad J(w) = \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dy - \frac{1}{2}\int_{\mathbb{R}^N} (W(y)|w|^2) dy = 1.
$$

According to the Lagrange multipliers theorem, there exists a multiplier $\kappa \in \mathbb{R}$ such that

$$
\tau'_{\hbar,B}(w) = \kappa J'(w);
$$

specifically, for all $v \in H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$ we derive $\tau'_{\hbar,B}(w)v = \kappa J'(w)v$

$$
Re\int \nabla_{\hbar,B} w \overline{\nabla_{\hbar,B} v} dy = \kappa Re\int (|w|^{2^*-1} - W(y)|w|)\bar{v} dy. \tag{7}
$$

By modifying the arguments presented by Berestycki and Lions [7] , we can establish $\kappa > 0$. Indeed, the first remark is that $\kappa \neq 0$; if not, namely if $\kappa = 0$ are would have $T'_{\hbar,B}(w) = 0$ and in particular $\int_{\mathbb{R}^N} |\nabla_{\hbar,B} w|^2 dy = 0$. Therefore, $w = 0$ which is impossible.

Specifically, it follows that $\kappa > 0$. Indeed, Assume for the sake of contradiction that $\kappa < 0$.

Additionally, note that $J'(w) \neq 0$; otherwise,

$$J'(w)v = Re \int_{\mathbb{R}^N} (|w|^{2^*-1} - W(y)|w|)\bar{v} dy = 0,$$

then $\frac{1}{2^*}|w|^{2^*} - \frac{1}{2}W(y)|w|^2 = 0$ which results in a contradiction with $J(w) = 1$.

Now, if $\kappa < 0$ since $J'(w) \neq 0$ so there exists a test function w such that $J'(w)w \neq 0$ since $J(w + \hbar w) \cong J(w) + \hbar J'(w)w$ and $\tau_{\hbar,B}(w + \hbar w) \cong \tau_{\hbar,B}(w) + \hbar\kappa J'(w)w$ for $\hbar \to 0$ and $\kappa < 0$, it is possible to choose $\hbar > 0$ small enough so that $v = w + \hbar v$ satisfies $J(v) > J(w) = 1$ and $\tau_{\hbar,B}(v) < \tau_{\hbar,B}(w) = \varrho_{\hbar,B}$ which is absurd.

Then, $\kappa > 0$. Then w in $H^{\alpha}_{\hbar,B}(\mathbb{R}^N, \mathbb{C})$ satisfies (in the weak senses)

$$-\nabla_{\hbar,B} w = \kappa(|w|^{2^*-1} - W(y)|w|).$$

Now, we intend to demonstrate that by utilizing an appropriate change of variables, the mentioned w, say w_κ, satisfies

$$Re \int_{\mathbb{R}^N} \nabla_{\hbar,B} w_\kappa \overline{\nabla_{\hbar,B} v} dy = Re \int_{\mathbb{R}^N} (|w_\kappa|^{2^*-1} - W(y)|w_\kappa|)\bar{v} dy$$

specifically, w_κ satisfies (in the sense)

$$-\nabla_{\hbar,B} w_\kappa + W(y) w_\kappa = |w_\kappa|^{2^*-2} w_\kappa,$$

so that w_κ is a solution to (1). Indeed, since

$$\begin{aligned} Re \int_{\mathbb{R}^N} \nabla_{\hbar,B} w \overline{\nabla_{\hbar,B} v} dy &= Re \int_{\mathbb{R}^N} \nabla_\hbar w . \overline{\nabla_\hbar v} dy + Re \int_{\mathbb{R}^N} \nabla_\hbar w . \overline{iB(y) v} dy \\ &+ \int_{\mathbb{R}^N} iB(y) w . \overline{\nabla_\hbar v} dy + Re \int_{\mathbb{R}^N} iB(y) w . \overline{iB(y) v} dy, \end{aligned} \tag{8}$$

by replacing $w(y) = w_\kappa(\sqrt{\kappa} y)$ (that is, $w_\kappa(y) = w(\frac{y}{\sqrt{\kappa}})$) in (7) and by applying the change of variable $y = \sqrt{\kappa} y$, we obtain

$$\begin{aligned} &Re \int_{\mathbb{R}^N} \nabla_\hbar w(\frac{y}{\sqrt{\kappa}}) . \overline{\nabla v(\frac{y}{\sqrt{\kappa}})} \frac{1}{(\sqrt{\kappa})^N} dy \\ &+ Re \int_{\mathbb{R}^N} \nabla_\hbar w(\frac{y}{\sqrt{\kappa}}) . \overline{iB(\frac{y}{\sqrt{\kappa}}) v(\frac{y}{\sqrt{\kappa}})} \frac{1}{(\sqrt{\kappa})^N} dy \\ &+ Re \int_{\mathbb{R}^N} iB(\frac{y}{\sqrt{\kappa}}) w(\frac{y}{\sqrt{\kappa}}) . \overline{\nabla_\hbar v(\frac{y}{\sqrt{\kappa}})} \frac{1}{(\sqrt{\kappa})^N} dy \\ &+ Re \int_{\mathbb{R}^N} iB(\frac{y}{\sqrt{\kappa}}) w(\frac{y}{\sqrt{\kappa}}) . \overline{iB(\frac{y}{\sqrt{\kappa}}) v(\frac{y}{\sqrt{\kappa}})} \frac{1}{(\sqrt{\kappa})^N} dy \end{aligned} \tag{9}$$

So

$$
\begin{aligned}
&Re\int_{\mathbb{R}^N}\sqrt{\kappa}\nabla_{\hbar}w_\kappa.\overline{\sqrt{\kappa}\nabla_{\hbar}v(\frac{y}{\sqrt{\kappa}})}\frac{1}{(\sqrt{\kappa})^N}dy\\
&+Re\int_{\mathbb{R}^N}\sqrt{\kappa}\nabla_{\hbar}w_\kappa.\overline{iB(\frac{y}{\sqrt{\kappa}})v(\frac{y}{\sqrt{\kappa}})}\frac{1}{(\sqrt{\kappa})^N}dy\\
&+\sqrt{\kappa}Re\int_{\mathbb{R}^N}iB(\frac{y}{\sqrt{\kappa}})w_\kappa(y).\overline{\sqrt{\kappa}\nabla_{\hbar}v(\frac{y}{\sqrt{\kappa}})}\frac{1}{(\sqrt{\kappa})^N}dy\\
&+Re\int_{\mathbb{R}^N}iB(\frac{y}{\sqrt{\kappa}})w_\kappa(y).\overline{iB(\frac{y}{\sqrt{\kappa}})v(\frac{y}{\sqrt{\kappa}})}\frac{1}{(\sqrt{\kappa})^N}dy\\
&=\kappa(Re\int_{\mathbb{R}^N}(|w_\kappa(y)|^{2^*-1}-W(\frac{y}{\sqrt{\kappa}})|w_\kappa(y))|\overline{v(\frac{y}{\sqrt{\kappa}})})\frac{1}{(\sqrt{\kappa})^N}dy. \qquad (10)
\end{aligned}
$$

If we simplify $\frac{1}{(\sqrt{\kappa})^N}$ and in evidence κ, it follows that

$$
\begin{aligned}
&Re\int_{\mathbb{R}^N}(\nabla_{\hbar}+i\frac{1}{\sqrt{\kappa}}B(\frac{y}{\sqrt{\kappa}}))w_\kappa(y).\overline{(\nabla_{\hbar}+i\frac{1}{\sqrt{\kappa}}B(\frac{y}{\sqrt{\kappa}}))v(\frac{y}{\sqrt{\kappa}})}dy\\
=&Re\int_{\mathbb{R}^N}(|w_\kappa(y)|^{2^*-1}-W(\frac{y}{\sqrt{\kappa}})|w_\kappa(y)|)\overline{v(\frac{y}{\sqrt{\kappa}})}dy.
\end{aligned}
$$

By the given assumption (B_2), and since $\frac{1}{\sqrt{\kappa}}B(\frac{y}{\sqrt{\kappa}})=B(y)$ for all $y\in\mathbb{R}^N$, we have

$$
Re\int_{\mathbb{R}^N}\nabla_{\hbar,B}w_\kappa(y).\overline{\nabla_{\hbar,B}v(\frac{y}{\sqrt{\kappa}})}dy=Re\int_{\mathbb{R}^N}(|w_\kappa(y)|^{2^*-1}-W(\frac{y}{\sqrt{\kappa}})|w_\kappa(y)|)\overline{v(\frac{y}{\sqrt{\kappa}})}dy;
$$

therefore we can conclude w_κ satisfies (9).

References

[1] M.A. Abd El Salam, M.A. Ramadan, M.A. Nassar, P.Agarwal, Y.M. Chu; Matrix computational collocation approach based on rational Chebyshev functions for nonlinear differential equations, Adv Differ Equ , **2021** 331 (2021). https://doi.org/10.1186/s13662-021-03481-y

[2] S. Almuthaybiri, R. Ghanmi, T. Saanouni; On the nonlinear Schrödinger equation with critical source term: global well-posedness, scattering and finite time blowup, AIMS Mathematics,**9**(11) (2024), 30230–30262. doi: 10.3934/math.20241460

[3] G. Arioli, A. Szulkin; A semilinear Schrödinger equations in the presence of a magnetic field, Arch. Ration. Mech. Anal., **170** (2003), 277-295.

[4] M. Attary, P. Agarwal; On developing an optimal Jarratt-like class for solving nonlinear equations, Ital. J. Pure Appl. Math, **43**(2020), 523–530.

[5] Z. Avazzadeh, H. Hassani, E.A. Bayati, M. Ebadi, P. Agarwal; Optimal solution of nonlinear 2D variable-order fractional optimal control problems using generalized Bessel polynomials, Journal of Vibration and Control, (2024). doi:10.1177/10775463241227475

[6] S. Barile, S. Cingolani, S. Secchi; Single-peaks for a magnetic Schrödinger equation with critical growth, Adv. Differential Equations, **11** (10) (2006), 1135-1166.

[7] H. Berestycki, P. L. Lions; Nonlinear scalar field equations I and II, Arch. Rat. Mech. Anal. **82** (1983), 313-375.

[8] M. Esteban, P. L. Lions; Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, PDE and Calculus of Variations, Vol. I, 401-449, Progr. Nonlinear Differential Equations Appl. 1, Birkhäuser Boston, MA, 1989.

[9] A.R. Hadhoud, P. Agarwal, A.A.M. Rageh; Numerical treatments of the nonlinear coupled time-fractional Schrödinger equations,Mathematical Methods in the Applied Sciences, **45** (1) (2022), 7119-7143.

[10] H. Hassani, Z. Avazzadeh, P. Agarwal, et al.; Generalized Bernoulli-Laguerre Polynomials: Applications in Coupled Nonlinear System of Variable-Order Fractional PDEs. J Optim Theory Appl, **2200** (2024), 371-1¤73. https://doi.org/10.1007/s10957-023-02346-6

[11] C. Ji, V.D. Radulescu; Concentration phenomena for nonlinear magnetic Schrödinger equations with critical growth. Isr. J. Math. **241** (2021), 465-1¤70. https://doi.org/10.1007/s11856-021-2105-5

[12] H. Khalil, M. Khalil, I. Hashim, P. Agarwal, Extension of Operational Matrix Technique for the Solution of Nonlinear System of Caputo Fractional Differential Equations Subjected to Integral Type Boundary Constrains. Entropy, **2021** 23, 1154. https://doi.org/10.3390/e23091154

[13] S. Liang, Y. Song; Multiplicity of solutions of perturbed Schrödinger equation with electromagnetic fields and critical nonlinearity in $\mathbb{R}^N$, Bound. Value Probl. **240** (2014), 1-14.

[14] S. Liang, J. Zhang; Solutions of perturbed Schrödinger equations with electromagnetic fields and critical nonlinearity, Proc. Edinb. Math. Soc., **54** (1) (2011), 131-147.

[15] E. H. Lieb, M. Loss; Analysis, Graduate Studies in Mathematics, Vol. 14, Amer. Math. Soc., 1997.

[16] M.M. Mousa, P. Agarwal, F. Alsharari, S. Momani; Capturing of solitons collisions and reflections in nonlinear Schrödinger type equations by a conservative scheme based on MOL, Adv Differ Equ, **2021** 346 (2021). https://doi.org/10.1186/s13662-021-03505-7

[17] A.Rahmoune, D. Ouchenane, S. Boulaaras, P. Agarwal; Growth of solutions for a coupled nonlinear Klein-Gordon system with strong damping, source, and distributed delay terms. Adv Differ Equ, **2020**, 335 (2020). https://doi.org/10.1186/s13662-020-02801-y

[18] N.A. Shah, P. Agarwal, J.D. Chung, E.R. El-Zahar, Y.S. Hamed; Analysis of Optical Solitons for Nonlinear Schrödinger Equation with Detuning Term by Iterative Transform Method, Symmetry, **12** (2020), 1850. https://doi.org/10.3390/sym12111850

[19] M. Shams, N. Rafiq, N. Kausar, N. et al.; On iterative techniques for estimating all roots of nonlinear equation and its system with application in differential equation. Adv Differ Equ, **2021** 480 (2021). https://doi.org/10.1186/s13662-021-03636-x

[20] R. Wen, J. Yang; Multiple solutions to critical magnetic Schrödinger equations. Acta Math Sci **44** (2024), 1373-1¤793. https://doi.org/10.1007/s10473-024-0411-9

S. Jain
Department of Mathematics,
Poornima College of Engineering-302022, India
E-mail: shilpijain1310@gmail.com

F. Hashemi
University of Mazandran, Babolsar, 47416 Iran
E-mail: f.hashemi04@umail.umz.ac.ir

M. Alimohammady
University of Mazandran, Babolsar, 47416 Iran
*E-mail:*amohsen@umz.ac.ir

C. Cesarano
Universit'a Telematica Internazionale Uninettuno, Rome, 00186 Italy
E-mail: clemente.cesarano@uninettunouniversity.net

P. Agarwal
Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE,
and
Anand International College of Enginerring, Jaipur 303012, India
and
International Telematic University Uninettuno Corso Vittorio Emanuele II, 3900186 Roma, Italy.
E-mail: goyal.praveen2011@gmail.com

Received 02 August 2024
Accepted 24 November 2024